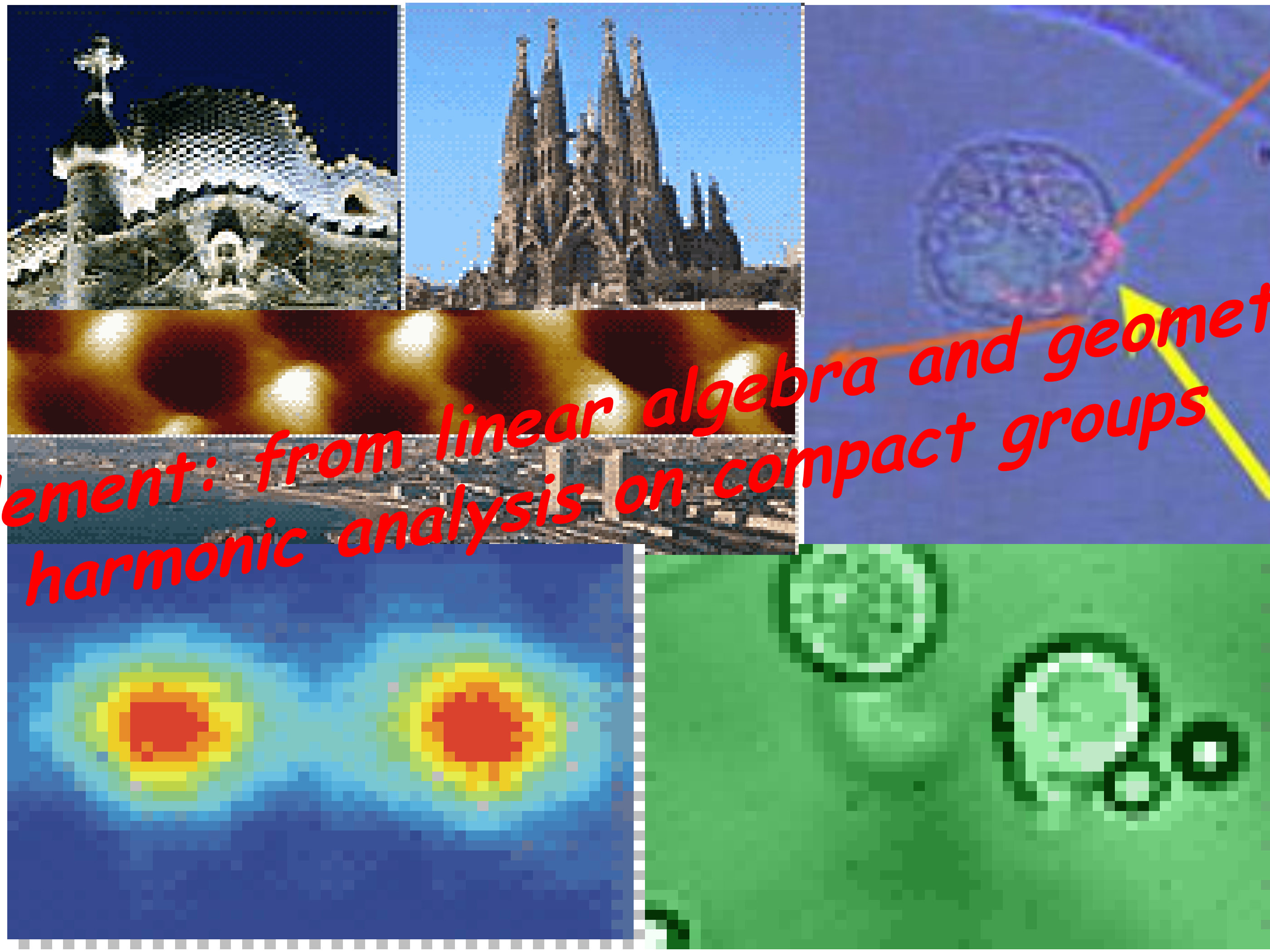
Entanglement: from linear algebra and geometry to harmonic analysis on compact groups



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QT Quantum Optical Information Technology

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Barcelona – Quantum Optics Theory

PhD ICFO:

Armand Niederberger,
Christian Trefzger,
Ania Kubasiak,
Sibylle Braungardt

Postdocs ICFO:

Ujjwal Sen,
Aditi Sen (De),
Chiara Menotti,
Jonas Larson,
Jarek Korbicz,
Geza Toth,
Mirta Rodriguez

Ex-Hannoveraner:

Anna Sanpera (ICREA full prof. UAB),
Dagmar Bruβ (C4, Düsseldorf)
L. Santos (W3, Hannover),
Veronica Ahufinger (ICREA junior, UAB),
J. Mompart (assoc. prof, UAB),
Carla Faria (lect. UC, London)
P. Öhberg (lect., Glasgow),
L. Sanchez-Palencia (CNRS, Orsay),
Z. Idziaszek (ten. track, Warsaw),
U.V. Poulsen (ten. track, Aarhus),
B. Damski (res. assoc., Los Alamos),
M. Baranov (res. assoc., Amsterdam),
P. Pedri (postdoc, Orsay),
O Gühne (postdoc, Innsbruck).
P. Hyllus (postdoc, Hannover)
A.Kantian (PhD, Innsbruck), T. Meyer (PhD, Düsseldorf),
A. Cojuhovschi (PhD, Hannover),

Barcelona – Quantum Optics Theory

Collaborations: Theory

MPI Garching – J. I. Cirac
UAB, Barcelona – A. Sanpera (G. Fis. Teor.),
Univ. Hannover – L. Santos, H-U. Everts (ITP),
Univ. Düsseldorf – D. Bruβ
Univ. Paris-Sud, Orsay – G. Shlyapnikov (LPT)
Univ. Innsbruck – P. Zoller, H. Briegel
NIST, Gaithesburg – P. Julienne
Oxford University – D. Jaksch
CFT, Warsaw – K. Rzążewski, M. Kuś,
Univ. Jagielloński, Cracow – J. Zakrzewski,
J. Dziarmaga, K. Sacha
Univ. Gdańsk – P. Horodecki
UAB, Barcelona - V. Ahufinger, J. Mompart, G. Morigi (G. Optica)
UB, Barcelona – J.I. Latorre, N. Barberàn, M. Guillemas, A. Polls
ICFO, Barcelona – A. Acín, Ll. Torner
Univ. Pavia – Ch. Macchiavelo
Univ. Arizona, Tuscon – J. Wehr
Harvard Univ. – R.J. Glauber

Collaborations: Experiments

Univ. Hannover - W. Ertmer, J. Arlt,
E. Tiemann (IQO)
Univ. Darmstadt - G. Birkl (Darmstadt),
Univ. Siegen - C. Wunderlich
Univ. Hamburg - K. Sengstock, K. Bongs
LENS, Firenze – Massimo Inguscio
Univ. Innsbruck - R. Blatt,
N. Bohr. Inst., Kopenhagen - E. Polzik
ICFO – J. Eschner, M. Mitchel, J. Biegert

Outline

- Motivations: Why entanglement?
  - ✓ Applications in Foundations of Quantum Mechanics (philosophy)
  - ✓ Applications in Quantum Information (technology)
  - ✓ Applications in Strongly Correlated Systems (physics)
  - ✓ Applications in Mathematics (positive maps)
- Introduction:
  - ✓ Definition of separability and separable states
  - ✓ Separability problem
  - ✓ Positive partial traspose criterion
  - ✓ Horodecki's necessary and sufficient criterion and positive maps
- Separability for positive definite functions on compact groups
  - ✓ Definitions and preliminary notions
  - ✓ Necessary and sufficient criterion
- Fourier transforms and "generating function" formalism
  - ✓ Fourier transforms of p.d. functions
  - ✓ Horodecki's theorem "recovered"

Introduction: Separability problem

Let $\mathcal{H}$ - a Hilbert space. ϱ is a state iff $\varrho \geq 0$, $\varrho = \varrho^{\dagger}$, $\operatorname{Tr} \varrho = 1$

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, R.F. Werner in 1989:

$\boxed{\text{Def. 1.1}}$ A state ϱ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called separable if it can be approximated in the trace norm by convex combinations of the form:

$$\varrho \simeq \sum_{m=1}^{K} p_m |x_m\rangle\langle x_m| \otimes |y_m\rangle\langle y_m| ,$$

where $|x_m\rangle \in \mathcal{H}_A$, $|y_m\rangle \in \mathcal{H}_B$, $p_m \geq 0$, $\sum_{m=1}^{K} p_m = 1$

Otherwise, ϱ is entangled.

Introduction:

Separability problem:

Given a state ϱ decide if it is separable or not?

It has been proven that it belongs to NP class.

Theorem 1.1 [A. Peres, Horodecey]

If a state ϱ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is separable then the partially transposed operator:

$\varrho^{T_B} := (\mathbb{1}_A \otimes T)\varrho$ is positive, where T is a transposition map, and $\mathbb{1}_A$ is identity map on $\mathcal{H}_A$

PPT criterion ↑ is necessary and sufficient in $\dim \mathcal{H}_A \cdot \dim \mathcal{H}_B \leq 6$

Introduction:

Theorem 1.2 [Horodecki]
Let $\mathcal{L}(\mathcal{H})$ denote the space of linear operators on $\mathcal{H}$, and let $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a state on a finite dimensional Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The state ρ is separable iff for all linear maps: $\Phi : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$ preserving positive operators (so called positive maps),

$$(\mathbb{1}_A \otimes \Phi)\rho \geq 0 \quad \text{as an operator}$$

on $\mathcal{H}_A \otimes \mathcal{H}_A$.

Preliminary notions:

$\boxed{\text{non-commutative Fourier analysis on a compact group:}}$

$A \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$

i) identify $\mathcal{H}_A, \mathcal{H}_B$ as representation spaces of unitary, irreducible representations π_α, π_β of some compact groups G_1, G_2

ii) pass from A to a function $\varphi_A : G_1 \times G_2 \to \mathbb{C}$, the non-commutative Fourier transform of A,

$$A \mapsto \varphi_A(g_1, g_2) := \mathrm{tr}\left[A\, \pi_\alpha(g_1) \otimes \pi_\beta(g_2)\right]$$

Apart from the trivial case $\dim \mathcal{H}_A = \dim \mathcal{H}_B = 1$, the groups G_1, G_2 are non-abelian!

Preliminary notions:

Def. 2.1 A contineous complex function φ on a group G with a Haar measure dg is called positive definite if it is bounded and satisfies:

$$\iint dg\,dh\, \overline{f(g)}\varphi(g^{-1}h)f(h) \geq 0$$

for any function f, contineous and with compact support

Theorem 2.1 [Gel'fand, Naimark, Segal]
With every $\varphi \in \mathcal{P}(G)$ we can associate a Hilbert space $\mathcal{H}_\varphi$, an unitary representation π_φ of G in $\mathcal{H}_\varphi$, and a vector v_φ, cyclic for π_φ, such that

$$\varphi(g) = \langle v_\varphi | \pi_\varphi(g) v_\varphi \rangle$$

π_φ - is unique up to a unitary equivalence

$\mathcal{P}(G)$ — set of p.d. functions, $\mathcal{P}_1(G)$ ~ set of normalized p.d.f., $\varphi(e) = 1$

Preliminary notions:

$\mathcal{P}(G)$ ~ is a closed convex cone in $C(G)$, the space of contineous complex-valued functions on G with the topology of uniform convergence on compact sets, called compact convergence.

$\boxed{\text{Def. 2.2}}$ We define Sep_0 as the set of all functions $\varphi \in \mathcal{P}(G_1 \times G_2)$ which can be represented as finite convex combinations

$$\varphi(g_1, g_2) = \sum_{m=1}^{K} p_m \varepsilon_m(g_1) \eta_m(g_2), \text{ where } \begin{array}{l} \varepsilon_m \in \mathcal{E}_1(G_1) \\ \eta_m \in \mathcal{E}_2(G_2) \end{array}$$

where $\mathcal{E}_i(G_i)$ - the sets of pure normalized functions i.e. such that π_φ from GNS construction is irreducible. $\mathcal{E}_1(G_1)$ are extreme points of $\mathcal{P}_1(G)$.

Preliminary notions:

Def. 2.2 (cont.)

A function $\varphi \in \mathcal{B}_1(G_1 \times G_2)$ is called separable if it is a uniform limit of elements of Sep.. The set of separable functions is denoted as $\overline{\text{Sep}}$. Functions, which are not separable are entangled.

Separability problem:

Given $\varphi \in \mathcal{B}_1(G_1 \times G_2)$, decide if it is separable, or entangled?

Separability of p.d.f on compact groups — — Quantum Mechanics meet Harmonic Analysis!

Necessary and sufficient criterion:

$\boxed{\text{Theorem 3.1}}$ (group theoretical PPT criterion)

If $\varphi \in \mathrm{Sep}$, then the function $(g_1, g_2) \mapsto \varphi(g_1, g_2^{-1})$ is positive definite.

$\boxed{\text{Theorem 3.2}}$

A function $\varphi \in \mathcal{B}_1(G_1 \times G_2)$ is separable iff for every bounded linear map $\Lambda : C(G_2) \to C(G_1)$, such that $\Lambda \mathcal{B}(G_2) \subset \mathcal{B}(G_1)$, the function

$$(id \otimes \Lambda)\varphi \quad \text{is positive definite on } G_1 \times G_1$$

Fourier transforms and "generating function" formalism

$$\widehat{G_1 \times G_2} \simeq \hat{G}_1 \times \hat{G}_2$$ (equiv. classes of irreducible strongly continuous unitary representations of $G_1 \times G_2$)

Matrix elements

$\pi^\alpha_{ij}(g_1) = \langle e_i | \pi_\alpha(g_1) e_j \rangle$, $\quad \tau^\beta_{kl}(g_2) = \langle \tilde{e}_k | \tau_\beta(g_2) \tilde{e}_l \rangle$

$\{e_i\}$ - orthonormal basis in $\mathcal{H}_\alpha$, $\{\tilde{e}_k\}$ - in $\tilde{\mathcal{H}}_\beta$

Formally

$$f = \sum_{\alpha,\beta} \sum_{i,\dots,l} f^{ijkl}_{\alpha\beta}\, \pi^\alpha_{ij} \otimes \tau^\beta_{kl} := n_\alpha m_\beta \int_{G_1}\int_{G_2} dg_1\, dg_2\, \overline{\pi^\alpha_{ij}(g_1)}\, \overline{\tau^\beta_{kl}(g_2)}\, f(g_1, g_2)$$

for $f \in C(G_1 \times G_2)$. Fourier series converges only in the L^2 norm and not uniformly. $n_\alpha = \dim \mathcal{H}_\alpha$, $m_\beta = \dim \tilde{\mathcal{H}}_\beta$

$$f * \Psi_u = \sum_{\alpha,\beta} \sum_{i,\dots,l} c^{\alpha\beta}_u f^{ijkl}_{\alpha\beta}\, \pi^\alpha_{ij} \otimes \tau^\beta_{kl},$$ where

$$c^{\alpha\beta}_u = \int_{G_1}\int_{G_2} dg_1\, dg_2\, \Psi_u(g_1, g_2)\, \overline{\chi_\alpha(g_1)}\, \overline{\chi_\beta(g_2)}$$

characters

(... ... $\tau_{\alpha\beta}$ ... $\mathcal{H}_\alpha \otimes \mathcal{H}_\beta$)

$$\hat{f}_{\alpha\beta} = \sum_{i,j,k,l} f^{ijkl}_{\alpha\beta} |e_i\rangle\langle e_j| \otimes |\tilde{e}_k\rangle\langle\tilde{e}_l| = n_\alpha m_\beta \iint_{G_1 G_2} dg_1\, dg_2\, f(g_1, g_2) \pi_\alpha^\dagger(g_1) \otimes \tau_\beta^\dagger(g_2)$$

$$f = \sum_{\alpha,\beta} tr\left[\hat{f}_{\alpha\beta} \pi_\alpha \otimes \tau_\beta\right] \quad , \quad f * \chi_u = \sum_{\alpha,\beta} c^{\alpha\beta}_u \, tr\left[\hat{f}_{\alpha\beta} \pi_\alpha \otimes \tau_\beta\right]$$

Th. 4.1 $\varphi \in P(G_1 \times G_2)$ iff $\hat{\varphi}_{\alpha\beta} \geq 0$ for all $[\pi_\alpha]$, $[\tau_\beta]$

Lemma 4.1 $\varphi \in P(G_1 \times G_2)$ is separable iff all $\hat{\varphi}_{\alpha\beta} \in \mathcal{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ are separable for all $[\pi_\alpha]$, $[\tau_\beta]$ $(\in \hat{G}_1, \in \hat{G}_2)$

Note: $\rho \in \mathcal{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$, $\varphi_\rho = Tr(\rho \pi_\alpha(\cdot) \otimes \tau_\beta(\cdot))$, then

$$(\hat{\varphi}_\rho)_{\gamma\nu} = \delta_{\alpha\gamma} \delta_{\beta\nu} \rho$$

Corollary 4.1 ρ is separable iff $\varphi_\rho \in Sep$

Lemma 4.1 A bounded linear map $\Lambda : C(G_2) \to C(G_1)$ is positive definite, i.e. $\Lambda P(G_2) \subset P(G_1)$ iff all maps

$$\hat{\Lambda}^\beta_\alpha : \mathcal{L}(\mathcal{H}_\beta) \longrightarrow \mathcal{L}(\mathcal{H}_\alpha)$$ are positive for all $[\pi_\alpha]$, $[\tau_\beta]$

Corollary 4.2 A map $\Phi : \mathcal{L}(\mathcal{H}_\beta) \to \mathcal{L}(\mathcal{H}_\alpha)$ is positive iff the map $\Lambda_\Phi : C(G_2) \to C(G_1)$ is positive definite where $\Lambda_\Phi f(g_1) := \mathrm{tr}\left[(\Phi \hat{f}_\beta)\overline{\pi}_\alpha(g_1)\right]$ for every $f \in C(G_2)$.

Lemma 4.3 A bounded linear map $\Lambda : C(G_2) \to C(G_1)$ is G_2-positive definite i.e. $(id \otimes \Lambda)\mathcal{P}(G_2 \times G_2) \subset \mathcal{P}(G_2 \times G_1)$ iff all maps $\hat{\Lambda}^\beta_\alpha : \mathcal{L}(\overline{\mathcal{H}}_\beta) \to \mathcal{L}(\mathcal{H}_\alpha)$ are completely positive for all $[\pi_\alpha]$, $[\tau_\beta]$

Corollary 4.3 A map $\Lambda : C(G_2) \to C(G_1)$ is completely positive iff it is G_2-positive definite.

Lemma 4.4 A bounded linear map $\Lambda : C(G_2) \to C(G_1)$ is completely positive definite iff all maps $\hat{\Lambda}^\beta_\alpha :$ $\mathcal{L}(\overline{\mathcal{H}}_\beta) \to \mathcal{L}(\mathcal{H}_\alpha)$ are completely positive.

Theorem 4.2 A state $\rho \in \mathcal{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\beta)$ is separable iff for all $[\pi_\gamma] \in \hat{G}_1$, and all positive maps $\Phi_\gamma : \mathcal{L}(\overline{\mathcal{H}}_\beta) \to \mathcal{L}(\mathcal{H}_\gamma)$

$$(1_\alpha \otimes \Phi_\gamma)\rho \geq 0 \quad (\in \mathcal{L}(\mathcal{H}_\alpha \otimes \mathcal{H}_\gamma))$$

CONCLUSIONS (The Tragedy of Hamlet, by Shakespeare):

- *There are more thing in heaven and earth, Horatio, than are dreamt of in your philosophy.*

Wow!!!

- **See:**

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Entanglement of positive definite functions on compact groups, J. K. Korbicz, J. Wehr, and M. Lewenstein, arXiv:0705.2965, "in print" in Comm. Math. Phys.

Remark on a group-theoretical formalism for quantum mechanics and the quantum-to-classical transition, J. K. Korbicz, M. Lewenstein Found. Phys. **37**, 879-895 (2007)