Control, Constraints and Quanta
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Coherent States Transforms & Representation Theory of Lie Groups

Application to Uniform Magnetic Field On Two-Dimensional Surfaces

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Coherent states indexed by elements of a group

Let G be a locally compact topological group with a left Haar measure $d\nu$.

Δ its modular function: $d\nu(gx) = \Delta(x)d\nu(g)$, g, x in G.

Def 1 : A Hilbert space $(\mathfrak{H}, \langle\ ,\ \rangle)$ has a system $\{\phi_g\}$ of coherent states, labeled by elements g of G if :

(i) there is a representation (Rep) $T: g \to T_g$ of G by the unitary operators T_g on $\mathfrak{H}$,

(ii) there is a vector ϕ_0 in $\mathfrak{H}$ such that $\phi_g = T_g[\phi_0]$ and for arbitrary ψ in $\mathfrak{H}$ we have that

$$\boxed{\langle \psi, \psi \rangle = \int_G \left| \left\langle \psi, \phi_g \right\rangle \right|^2 d\nu(g)} \qquad (1)$$

A polarization of (1) gives the equality

$$\boxed{\langle \psi_1, \psi_2 \rangle = \int_G \left\langle \psi_1, \phi_g \right\rangle \left\langle \psi_2, \phi_g \right\rangle d\nu(g),\ \psi_1, \psi_2 \text{ in } \mathfrak{H}.} \qquad (2)$$

Remark : The condition (2) is equivalent to the the **square integrability** of the representation.

Square Integrable Representation

Def 2 : Let T be unitary Rep of G in a Hilbert space $(\mathfrak{H}, <,>)$. The coefficients of T are complex-valued functions on G of the form:

$$\boxed{G \ni g \mapsto C_{u,v}(g) := < u, T(g)v >_{\mathfrak{H}}}$$

u, v are arbitrary fixed vectors in $\mathfrak{H}$.

Def 3 : An irreducible Rep. $(T, \mathfrak{H})$ of G is called square integrable if it has a nonzero square integrable coefficient.

Theorem 1 (*Duflo – Moore*, JFA, 1976)

Let $(T, \mathfrak{H})$ be a square integrable Rep of G. Then, $\exists!$ operator K on $\mathfrak{H}$, self-adjoint positive, semi-invariant with weight Δ^{-1}, satisfying

$$\boxed{< C_{u,v}, C_{u',v'} >_{L^2(G,d\nu)} = < u, u' >< K^{-1/2}v', K^{-1/2}v >}$$

for all u, u' in $\mathfrak{H}$, and all v, v' in $Dom\ K^{-1/2}$).

Remark: ▲ If G is unimodular ($\Delta = 1$), K is a multiplication by scalar.

▲ If G is not unimodular, then K not bounded.

The Coherent State transform

Def 4 : The coherent state transform is the map

$$\boxed{W : \mathfrak{H} \to L^2(G, d\nu)}$$

defined by

$$\boxed{\phi \mapsto W[\phi](g) := \left\langle \phi, \Phi_g \right\rangle, g \text{ in } G}$$

This transformation is an **isometry** (up the a multiplicative constant).

We can write its inversion on its range as

$$\boxed{\phi = \frac{1}{c_{\phi_0}} \int_G d\nu(g) W[\phi](g) \Phi_g}$$

$c_{\phi_0} > 0$ is a constant.

Coherent States indexed by elements of a Quotient Space

- Let K be a closed subgroup of G.
- Let $X = G/K$: homogeneous space with a measure ν invariant under the action of G.
- Denote $\sigma : X \to G$ a Borel section (in the principal bundle $G \to G/K$).
- Let T_g be a (continous) unitary Rep of G in a Hilbert space $(\mathfrak{H}, \langle .,. \rangle)$.

Def 5 : A vector ϕ_0 in $\mathfrak{H}$ will be treated as **reference state** if for all u in K

$$\boxed{T_u[\phi_0] = \chi(u)\phi_0.}$$

The mapping $u \mapsto \chi(u)$ define a **character** of the subgroup K.

The set of vectors $\{\phi_x\}$ defined by

$$\boxed{\phi_x := T_{\sigma(x)}[\phi_0],\ x \text{ in } X}$$

form a family of coherent states if

$$\boxed{\int_X d\nu(x) \langle \phi_0, \phi_x \rangle \langle \phi_x, \phi_0 \rangle = \langle \phi_0, \phi_0 \rangle.}$$

Representation square integrable modulo subgroup & section

Def 6 : *(J.P.Antoine)* The Rep T is square integrable modulo ($\mathfrak{H}$,σ) if there exists a non zero vector ϕ_0 in $\mathfrak{H}$ such that

$$\boxed{0 < \int_X d\nu(x) \langle T(\sigma(x)\phi_0), \phi_0 \rangle \langle \phi_0, T(\sigma(x)\phi_0) \rangle = \langle \phi, A_\sigma \phi \rangle < \infty, \ \forall \phi \text{ in } \mathfrak{H}}$$

where A_σ is a bounded, positive, invertible operator with a densely defined (possibly unbounded) inverse and the *integral converges weakly*.

Def 7 : Coherent states $\{\phi_x\}$, indexed by points x in X, are defined as the orbite of ϕ_0 under G, through the representation T and the section σ :

$$\boxed{\phi_x := T_{\sigma(x)}[\phi_0] = |\phi_x >}$$

Remark : ▲ The square integrability condition can be written as a **resolution of the identity**

$$\boxed{\int_X d\nu(x) |\phi_x >< \phi_x| = A_\sigma}$$

▲If A_σ^{-1} is bounded, the set $\{\phi_x\}$ is called a frame. If $A_\sigma = c\mathbf{1}$, $c > 0$, $\{\phi_x\}$ is called a tight frame.

The reduced Coherent State Transform

Def 8 : The reduced Coherent State Transform is the unitary map

$$\boxed{\mathfrak{H} \to L^2(X, d\nu)}$$

defined by

$$\boxed{\mathcal{W}_{red} : \phi \mapsto \mathcal{W}_{red}[\phi](x) := \langle \phi, \phi_x \rangle_{\mathfrak{H}}.}$$

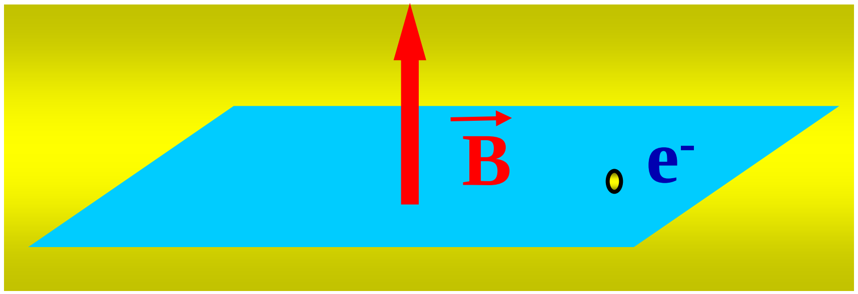
We can write its inversion on its range $\mathcal{W}_{red}(\mathfrak{H})$ as:

$$\boxed{\mathcal{W}_{red}^{-1}(f) = \int_X \overline{f(x)} A_\sigma^{-1}[\phi_x] d\nu(x), \qquad f \text{ in } \mathcal{W}_{red}(\mathfrak{H})}$$

Remark : These formula acquires a simpler form when A_σ is a multiple of the identity.

The Landau Problem

1930



$$H_B := \frac{1}{2}\left[\left(i\frac{\partial}{\partial x} - \frac{B}{2}y\right)^2 + \left(i\frac{\partial}{\partial y} + \frac{B}{2}x\right)^2\right]$$

Schrödinger operator

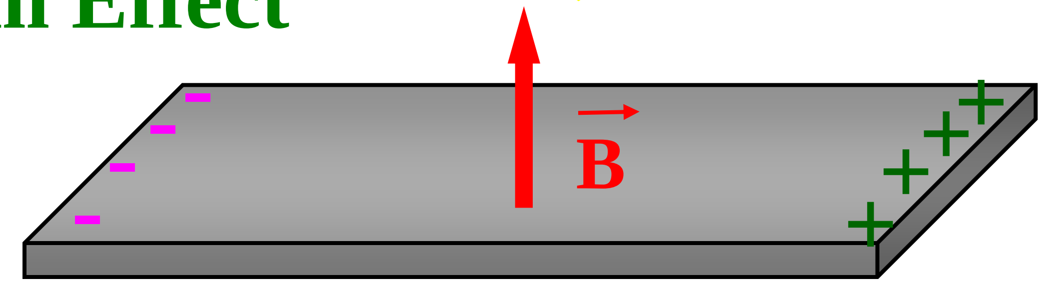
Energy levels
Em= (m+1/2)B, m=0,1,2….
(Landau Levels)

Bound states

Interest on Landau Problem

Quantum Hall Effect

- **Hall (1880)**



Classical Hall effect

- **Klitzing et al., Nobel Price, 1985**

Quantification of the
Conductance of Hall courant
(IQHE)

Works of Laughlin, Kohomoto, Nijs, Nightinagale, and Thouless

Quantization of the Conductance: has a geometric Nature

Mathematicians are Interested with study Quantum Hall Effect by different ways

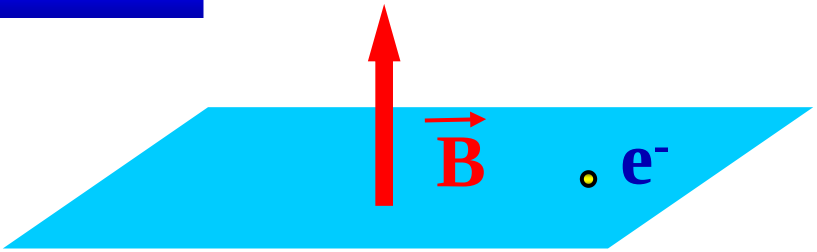
Objectifs of the talk

- To construct coherent states (CS) attached to Landau levels in *three type of geometry.*

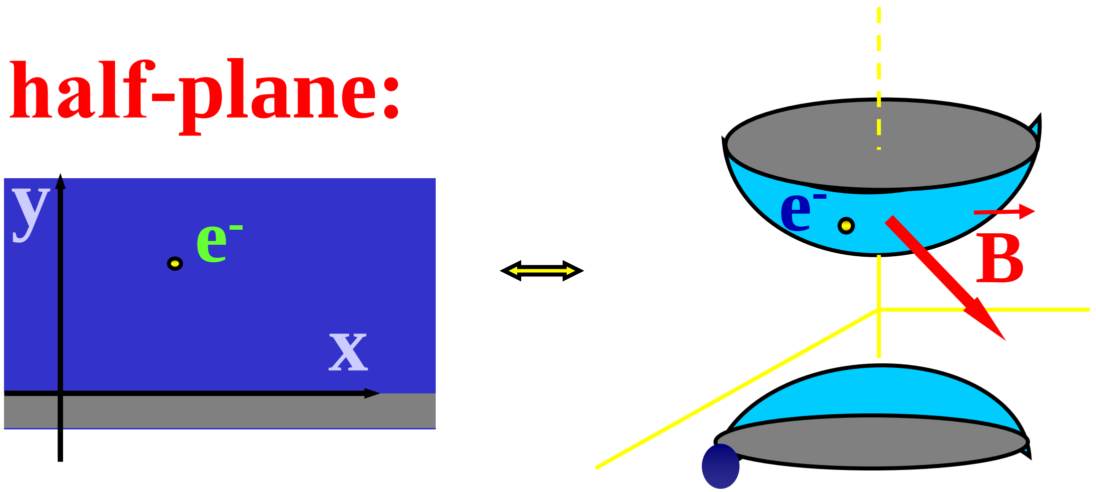
- To characterize bound states (*L^2 -eigenfunctions*) of the particle by mean of coherent state transforms (CST).

3 type of geometry

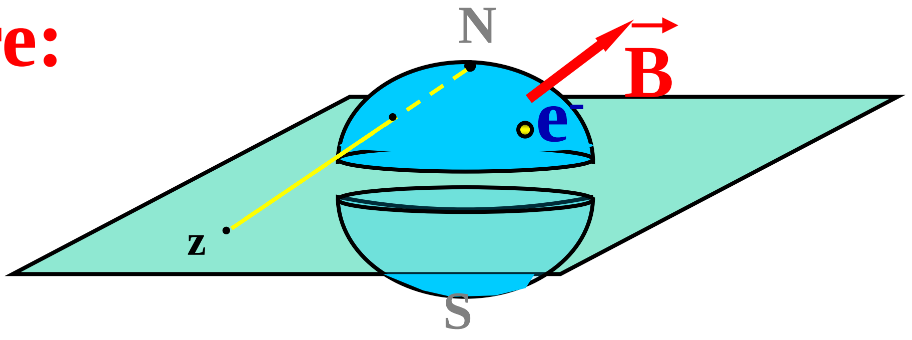
1. Euclidean plane:



2. Poincaré upper half-plane:



3. Riemann sphere:



Coherent states via Schrödinger Rep of the Heisenberg group

■ Coherent states attached to Landau levels are constructed via a Rep of the Heisenberg group $G = \mathbb{H}_1$ on the Hilbert space $L^2(\mathbb{R}, d\xi)$:

$$\boxed{\rho_B(x,y,t)[\psi](\xi) := \exp i\left(Bt - \sqrt{B}y\xi + \frac{B}{2}xy\right)\,\psi\left(\xi - \sqrt{B}x\right)}$$

(x,y,t) in $\mathbb{H}_1, B > 0$, ψ in $L^2(\mathbb{R}, d\xi)$ and ξ in $\mathbb{R}$, called **Schrödinger representation**.

■ The Unit Irr Rep ρ_B is square integrable modulo: the center $\mathbb{R}$ of $\mathbb{H}_1$ and the section σ_0 defined from $\mathbb{H}_1/\mathbb{R} = \mathbb{R}^2$ into $\mathbb{H}_1$ by $\sigma_0(x,y) = (x,y,0)$. By Theorem [DM] $\exists$ operator K in $L^2(\mathbb{R}, d\xi)$ self-adjoint, positive and semi-invariant such that

$$\boxed{\int_{\mathbf{R}^2} \left\langle \psi_1, \rho_B(x,y,0)[\phi_1] \right\rangle \left\langle \rho_B(x,y,0)[\phi_2], \psi_2 \right\rangle d\mu(x,y) = \langle \psi_1, \psi_2 \rangle \left\langle K^{\frac{1}{2}}\phi_1, K^{\frac{1}{2}}\phi_2 \right\rangle}$$

for all ψ_1, ψ_2 in $L^2(\mathbb{R}, d\xi)$ and ϕ_1, ϕ_2 in $Dom(K^{\frac{1}{2}})$. Here, $\mathbb{H}_1$ is unimodular and K **is the identity**.

■ As reference state, we choose the function Φ_m of $L^2(\mathbb{R}, d\xi)$ defined by

$$\boxed{\Phi_m(\xi) := \left(\sqrt{\pi}\,2^m m!\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\xi^2\right) H_m(\xi),\; m = 0, 1, \ldots}$$

where ξ in $\mathbb{R}$ and $H_m(.)$ is the mth Hermite polynomial.

■The coherent states attached to the mth Landau level, labelled by elements (x,y) of the coset space $\mathbb{R}^2$ and $m = 0, 1, \ldots,$ are defined by

$$\boxed{\Phi_{(x,y),B,m} := \rho_B(\sigma_0(x,y))[\Phi_m].}$$

Wave functions of these CS are given by

$$\boxed{\Phi_{(x,y),B,m}(\xi) = \left(\sqrt{\pi}\,2^m m!\right)^{-\frac{1}{2}} \exp\left(-i\sqrt{B}\xi y + i\frac{B}{2}xy - \frac{1}{2}\left(\xi - \sqrt{B}x\right)^2\right) H_m\left(\xi - \sqrt{B}x\right).}$$

■The square integrability of the Unit Irr Rep ρ_B modulo the subgroup $\mathbb{R}$ reformulated as a resolution of the identity. (*Denote* $|(x,y),B,m> := \Phi_{(x,y),B,m}$) :

$$\boxed{\mathbf{1}_{L^2(\mathbf{R},d\xi)} = \int_{\mathbf{R}^2} d\mu(x,y)|(x,y),B,m><m,B,(x,y)|}$$

Remark: For $m = 0$ and $B = 2$, these CS reduce to

$$\boxed{\Phi_{(x,y),2,0}(\xi) = \Phi_\alpha(\xi) = \pi^{-\frac{1}{4}} \exp\left(-\frac{1}{2}\xi^2 + \sqrt{2}\,\xi\alpha - \frac{1}{2}\alpha^2 - \frac{1}{2}|\alpha|^2\right), \quad \alpha := x - iy \text{ in } \mathbb{C}.}$$

These CS coincide with wave functions of the "**nonspreading wave packets**" of the harmonic oscillator, firstly constructed by Schrödinger 1926.

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Mathematical and General

Characterization of two-dimensional Euclidean Landau states by coherent state transforms

Z Mouayn 2004 *J. Phys. A: Math. Gen.* **37** 4813-4819 doi:10.1088/0305-4470/37/17/011

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Abstract. We construct a family of generalized coherent states attached to Landau levels of a charged particle moving in the two-dimensional Euclidean plane under a perpendicular uniform magnetic field. We prove that the ranges of the corresponding coherent state transforms coincide with spaces of bound states of the particle. This provides us with a new characterization of two-dimensional Euclidean Landau states.

Hyperbolic Landau problem

1987 : Contet : the Landau problem has been generalized to the hyperbolic plane .

1969 : Aslaken and Klauder : CS where constructed in connection with the affine group.

1992 : Daubechies : role of affine coherent states in continuous wavelet transformation

2002 : Sakaguchi and Hayachi : a kind of annihilation creation operators for affine coherent states

- \+ other authors

- The Hamiltonian on the Poincaré upper-half plane $\mathbf{H}^2 = \{\zeta = x + iy \text{ in } \mathbb{C},\ \operatorname{Im}\zeta > 0\}$:

$$\boxed{H_B := y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 2iBy\frac{\partial}{\partial x}.}$$

acting in the Hilbert space $L^2(\mathbf{H}^2, y^{-2}dxdy)$.

- The spectrum of H_B in the Hilbert space $L^2(\mathbf{H}^2, y^{-2}dxdy)$ consists of two parts: (1) an absolutely continuous spectrum in $(-\infty, 0]$ (2) **a finite number of infinitely degenerate eigenvalues**:

$$\boxed{E_m^B := (B-m)(B-m-1), \quad 0 \le m < B - \frac{1}{2} \quad \text{when } 2B > 1.}$$

- Spaces of bound states :

$$\boxed{\mathcal{E}_m^B(\mathbf{H}^2) = \left\{\varphi \text{ in } L^2(\mathbf{H}^2, y^{-2}dxdy), \quad H_B\varphi = E_m^B\varphi\right\}}$$

Coherent states via a representation the affine group

- The affine group is the set $A_f(\mathbb{R}) := \mathbb{R} \times \mathbb{R}_+^*$. The group law is given by

$$\boxed{(x,y).(x',y') = (x+yx',yy').}$$

$A_f(\mathbb{R})$ is endowed with the left Haar measure

$$\boxed{d\nu(x,y) = (2\pi y)^{-2}dxdy.}$$

$A_f(\mathbb{R})$ is nonunimodular with modular function $\Delta(x,y) = y^{-1}$

- We focus on the Rep π_+ of $A_f(\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)$

$$\boxed{\pi_+(x,y)[\phi](\xi) = \exp(2i\pi x\xi)\phi(y\xi),\ \phi \text{ in } L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)}$$

The Rep π_+ is square integrable. By the Duflo-Moore theorem $\exists K$ operator acting in $L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)$ self-adjoint, positive and semi-invariant with weight Δ^{-1} such that:

$$\boxed{\int_{A_f(\mathbb{R})} \left\langle \varphi_1, \pi_+(x,y)\psi_1 \right\rangle \left\langle \pi_+(x,y)\psi_2, \varphi_2 \right\rangle d\nu(x,y) = \langle \varphi_1, \varphi_2 \rangle \left\langle K^{\frac{1}{2}}\psi_1, K^{\frac{1}{2}}\psi_2 \right\rangle}$$

for all ψ_1, ψ_2 in $Dom(K^{\frac{1}{2}})$ and all φ_1, φ_2 in $L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)$ and $K^{\frac{1}{2}}$ is the operator given by

$$\boxed{K^{\frac{1}{2}}[\psi](\xi) = \xi^{-\frac{1}{2}}\psi(\xi),\ \psi \text{ in } Dom(K^{\frac{1}{2}}).}$$

■ As "vacuum vector" we choose the function $\Psi_{B,m}$ of $L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)$ defined by

$$\boxed{\Psi_{B,m}(\xi) := \left(\frac{\Gamma(2B-m)}{m!}\right)^{-\frac{1}{2}} \xi^{B-m} e^{-\frac{1}{2}\xi} L_m^{2(B-m)-1}(\xi)}$$

where ξ in $\mathbb{R}_+^*$ and $L_p^{(\nu)}(\,)$ Laguerre polynomial.

■ The coherent states labelled by elements (x,y) of the group $A_f(\mathbb{R})$, $B > 0$ and $m = 0, 1, \ldots,$ are defined by : $\Phi_{(x,y),B,m} := \pi_+(x,y)[\Psi_{B,m}]$.

Wave functions of these CS are of the form

$$\boxed{\Phi_{(x,y),B,m}(\xi) = \left(\frac{\Gamma(2B-m)}{m!}\right)^{-\frac{1}{2}} \left(\xi y\right)^{B-m} e^{-\frac{1}{2}\xi y} \exp\left(\frac{1}{2}i\xi x\right) L_m^{2(B-m)-1}(\xi y)}$$

■The square integrability of the UIR π_+ can be reformulated as a resolution of the identity (denote $|(x,y),B,m> := \Phi_{(x,y),B,m}$). *The unity of* $L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)$ *is solved as*

$$\boxed{\mathbf{1}_{L^2(\mathbf{R}_+^*,\xi^{-1}d\xi)} = c_{B,m}^{-1} \int_{\mathbf{A}_f(\mathbf{R})} d\mu(x,y) |(x,y),B,m >< (x,y),B,m|}$$

where $c_{B,m}$ *equals the norm square in* $L^2(\mathbb{R}_+^*, \xi^{-1}d\xi)$ *of the state* $K^{\frac{1}{2}}\Psi_{B,m}$.

Remark : For $m = 0$, $|(x,y),B,0>$ coincides with affine CS constructed by Aslaken and Klauder.

Coherent state transforms and characterization theorem

- We identify $A_f(\mathbb{R})$ with $\mathbf{H}^2 = \{\zeta = x + iy \text{ in } \mathbb{C},\ \mathrm{Im}\,\zeta > 0\}$ by setting $\zeta = x + iy = (x, y)$.
- The coherent state transform :

$$\boxed{\mathcal{W}_{B,m} : L^2(\mathbb{R}_+^*, \xi^{-1} d\xi) \to L^2(\mathbf{H}^2)}$$

is defined by

$$\boxed{\mathcal{W}_{B,m}[\varphi](\zeta) := c_{B,m}^{-\frac{1}{2}} \int_{\mathbf{R}_+^*} \Phi_{(x,y),B,m}(\xi)\overline{\varphi(\xi)}\xi^{-1} d\xi, \qquad \zeta \text{ in } \mathbf{H}^2.}$$

Thanks to the square integrability of the UIR π_+, $\mathcal{W}_{B,m}$ is an isometrical embedding.

- Characterization theorem for space of bound states:

Theorem: For $2B \geq 1$, m in $\mathbb{Z}_+$ and $m \leq B - 1/2$, we have that

$$\boxed{\mathcal{W}_{B,m}\left[L^2(\mathbb{R}_+^*, \xi^{-1} d\xi)\right] = \mathcal{E}_m^B(\mathbf{H}^2).}$$

Characterization of hyperbolic Landau states by coherent state transforms

Z Mouayn 2003 *J. Phys. A: Math. Gen.* **36** 8071-8076 doi:10.1088/0305-4470/36/29/311

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Abstract. We deal with a family of generalized coherent states obtained by means of operators of an unitary irreducible representation of the group of affine transformations of the real line. We prove that the ranges of the corresponding coherent state transforms coincide with spaces of bound states of the Landau Hamiltonian in the hyperbolic plane. This provides us with a new characterization of hyperbolic Landau states.

Coherent states attached to Landau levels on the Poincaré disc

Z Mouayn 2005 *J. Phys. A: Math. Gen.* **38** 9309-9316 doi:10.1088/0305-4470/38/42/010

Full text PDF (107 KB) | References

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Abstract. We construct a family of generalized coherent states attached to Landau levels of a charged particle moving in the Poincaré disc under a perpendicular uniform magnetic field. The corresponding coherent state transforms enable us to connect, by an integral transform, spaces of bound states of the particle with the space of square integrable functions on the real line. The established connection provides us with a new way to obtain hyperbolic Landau states.

Spherical Landau problem

- 1931 : Dirac : the earlier works on the motion of a charged particle on a sphere under influence of stationary normal magnetic field.
- 1948 : Harish-Chandra : existence of bound states of an electron according to Dirac monopole
- 1974 : Wu and Yang : defined a vector potential on '*sub-spaces*' , identified eigenfunctions of the Hamiltonian corresponding to their vector potential as sections.
- 1971, 1972 : Radcliffe and Perelomov : coherent states labelled by points of sphere, called spin CS, have been introduced within the group theoretical formalism.
- 1979 : Klauder : The path integral using these spin coherent states has been introduced as integral over the two-dimensional sphere.
- 1999 : Aschler and Grabert : Coherent state path integral for spin $\frac{1}{2}$ in an arbitrary magnetic field
- \+ other auhtors

■ The Hamiltonian on the Riemann sphere $\mathbb{C}$ **union** $\{\infty\} \simeq S^2$ we consider :

$$\boxed{H_B = -\left(1+|z|^2\right)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - Bz\left(1+|z|^2\right)\frac{\partial}{\partial z} + B\bar{z}\left(1+|z|^2\right)\frac{\partial}{\partial \bar{z}} + B^2\left(1+|z|^2\right) - B^2}$$

acting in the Hilbert space $L^2(S^2) := L^2\left(S^2, \left(1+|z|^2\right)^{-2} d\mu(z)\right)$,

$d\mu(z) = \pi^{-1} dx dy$ being the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

■ The spectrum of H_B consists on an **infinite number of eigenvalues** of the form

$$\boxed{E_m^B := (2m+1)B + m(m+1),\ m = 0,1,2,..}$$

with **finite degeneracy** = $2B + 2m + 1$.

■ Spaces of bound states :

$$\boxed{\mathcal{E}_m^B(S^2) = \left\{\varphi \text{ in } L^2(S^2), \quad H_B\varphi = E_m^B\varphi\right\}}$$

Coherent states via representation of $SU(2)$

- The $SU(2)$ is the unitary group of the second order with unity determinant.

- Every Unit Irr Rep of $SU(2)$ is equivalent to one of the representation $T_j,\ j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ acting in the Hilbert space $\mathcal{P}_j$ of polynomials of $2j$ degree :

$$\boxed{T_j(g)[\phi](\xi) := (\beta\xi + \bar{\alpha})^{2j}\phi\left(\frac{\alpha\xi - \bar{\beta}}{\beta\xi + \bar{\alpha}}\right),\ \phi \text{ in } \mathcal{P}_j}$$

where

$$\boxed{SU(2) \ni g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},\ |\alpha|^2 + |\beta|^2 = 1.}$$

The scalar product in the Hilbert space $\mathcal{P}_j$ is given by

$$\boxed{\langle \phi, \psi \rangle_{\mathcal{P}_j} = (2j+1)\int_{\mathbb{C}} \left(1 + |\xi|^2\right)^{-2j-2} \phi(\xi)\overline{\psi(\xi)}d\mu(\xi),}$$

- The quotient space $SU(2)/U(1)$ = sphere S^2. The point of this sphere is determined by the unit vector $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^2 = 1$. Let $g_{\boldsymbol{\sigma}}$ be the element of $SU(2)$ which transforms the north pole vector $(0,0,1)$ into the vector $\boldsymbol{\sigma} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

- Perelomov's coherent states $\Psi_{\boldsymbol{\sigma},\gamma}$ labeled by elements of the sphere and defined by:

$$\boxed{\Psi_{\boldsymbol{\sigma},\gamma} := T_j(g_{\boldsymbol{\sigma}})\left[\Phi_\gamma^j\right]}$$

wave functions of these CS can be written as

$$\boxed{\Psi_{\boldsymbol{\sigma},\gamma}(\xi) = \sqrt{\frac{(2j)!}{(j+\gamma)!(j-\gamma)!}}(\beta\xi + \bar{\alpha})^{j-\gamma}(\alpha\xi - \bar{\beta})^{j+\gamma}, \qquad \alpha = \cos\frac{\theta}{2}e^{i\varphi/2},\ \beta = \sin\frac{\theta}{2}e^{i\varphi/2}}$$

We identify the sphere S^2 with the extended complex plane by using the stereographic projection

$$\boxed{z = -\tan\frac{\theta}{2}e^{i\varphi}.}$$

One obtains the CS $\Psi_{z,\gamma}$ labeled by the stereographic coordinate z as:

$$\boxed{\Psi_{z,\gamma} := (-1)^{j+\gamma}e^{i\gamma\varphi}\Psi_{\boldsymbol{\sigma},\gamma}.}$$

■The CS attached to the spherical Landau level E_m^B, labeled by points z of $\mathbb{C}$ union$\{\infty\}$ are obtained by taking $j = B+m$ and $\gamma = B$. Their Wave functions are :

$$\boxed{\Psi_{z,B,m}(\xi) := c_{B,m}\left(\frac{(1+\xi z)^2}{1+|z|^2}\right)^B\left(\frac{(\xi-\bar{z})(1+\xi z)}{1+|z|^2}\right)^m}$$

where

$$\boxed{c_{B,m} := \sqrt{\frac{(2(B+m))!}{(2B+m)!\,m!}}.}$$

■These CS are justified by the square integrability of the UIR T_{B+m} modulo the subgroup $U(1)$.

■This square integrability can be rephrased as a resolution of the identity of Hilbert space $\mathcal{P}_{B+m}$ (denote $|z,B,m> := \Psi_{z,B,m}$) as:

$$\boxed{\mathbf{1}_{\mathcal{P}_{B+m}} = (2(B+m)+1)\int_{\mathbb{C}} d\mu(z)\left(1+|z|^2\right)^{-2}|z,B,m><m,B,z|}$$

Remark : For $m = 0$, $|z,B,0>$ coincides with the spin coherent state constructed by Perelomov.

Coherent state transforms and characterization theorem

- The coherent state transform :

$$\boxed{\mathcal{W}_{B,m} : \mathcal{P}_{B+m} \to L^2(\mathbf{S}^2)}$$

is defined by

$$\boxed{\mathcal{W}_{B,m}[\phi](z) := \left(2(B+m)+1\right)^{\frac{1}{2}} < \Psi_{z,B,m}, \phi >_{\mathcal{P}_{B+m}}}$$

is an isometrical embedding. Explicitly,

$$\boxed{\mathcal{W}_{B,m}[\phi](z) = \kappa_{B,m} \int_{\mathbb{C}} \left(\frac{(1+\bar{\xi} z)^2}{1+|z|^2} \right)^B \left(\frac{(\xi - \bar{z})(1+\bar{\xi} z)}{1+|z|^2} \right)^m \overline{\phi(\xi)} \left(1+|\xi|^2\right)^{-2(B+m)-2} d\mu(\xi)}$$

with $\kappa_{B,m} = c_{B,m}\left(2(B+m)+1\right)^{\frac{1}{2}}$.

- Characterization theorem for space of bound states:

Theorem : For B, m in $\mathbb{Z}_+$, we have that

$$\boxed{\mathcal{W}_{B,m}\left[\mathcal{P}_{B+m}\right] = \mathcal{E}_m^B(\mathbf{S}^2).}$$

Reports on Mathematical Physics
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Coherent states attached to landau levels on the riemann sphere

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We construct a family of generalized coherent states attached to Landau levels of a charged particle moving in the Riemann sphere under a perpendicular uniform magnetic field. Bound states of the particle are then characterized as coherent state transforms of polynomial functions. This provides us with a new characterization of spherical Landau states.

Thank you very much