The XXZ model and applications to quantum computing

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Joint work with Nachtergaele, Sims and Starr

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Introduction

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- Basic requirement is that the experimenter has access to two states effectively decoupled from the environment, and transitions between these states controlled to simulate gates.
- We investigate the possibility of using excited states at the interfaces called 'kinks' of one dimensional magnetic systems such as the ferromagnetic XXZ Heisenberg model.
- For spins of magnitude $J \geq 3/2$ and with suitable anisotropy such as the ferromagentic XXZ model isolated excitations are possible. If one could build one-dimensional spin J systems like the XXZ model, it would be a good starting point to encode qubits and unitary gates.

Qubit and quantum gates

- A qubit is a(ny) quantum system with a state space $\cong \mathbb{C}^2$. The state of a qubit is some basis $\{|0\rangle, |1\rangle\}$ is a normalized vector $\psi = a|0\rangle + b|1\rangle$ $a, b \in \mathbb{C}$ $|a|^2 + |b|^2 = 1$
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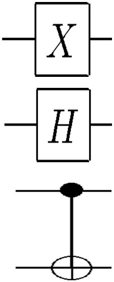
$$\alpha|0\rangle + \beta|1\rangle \quad \xrightarrow{X} \quad \beta|0\rangle + \alpha|1\rangle$$

$$\alpha|0\rangle + \beta|1\rangle \quad \xrightarrow{H} \quad \alpha\frac{|0\rangle+|1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$|ab\rangle \quad \rightarrow \quad |a, a \oplus b\rangle$$



The XXZ Model

- The XXZ model, a mathematical model for magnetism. Atoms or ions are fixed at sites of a subset of a discrete lattice ($\mathbb{Z}$).
- State space for a single site α is $\mathcal{H}_\alpha = \mathbb{C}_\alpha^{2J+1}$. For the entire system $\mathcal{H}(\Lambda, \mathrm{J}) = \otimes_{\alpha\in\Lambda}\mathcal{H}_\alpha \qquad \Lambda \subset \mathbb{Z} \qquad |\Lambda| :=$ number of sites.
- Hamiltonian of the XXZ model with $\Lambda = [-L, L]$ is given by

$$H_L^{\mathrm{k}}(\Delta^{-1}) = \sum_{\alpha=-L}^{L-1} \Big[(\mathrm{J}^2 - S_\alpha^3 S_{\alpha+1}^3) - \Delta^{-1}(S_\alpha^1 S_{\alpha+1}^1 + S_\alpha^2 S_{\alpha+1}^2)\Big] + \mathrm{J}\sqrt{1-\Delta^{-2}}(S_{-L}^3 - S_L^3)$$

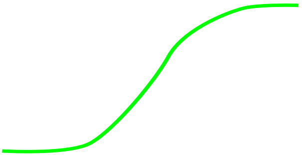
The Hamiltonian

- Hamiltonian is non negative.

- The XXZ kink Hamiltonian commutes with the operator $S^3_{tot} = \sum_{\alpha=-L}^{L} S^3_\alpha$. We define $\mathcal{H}_M$ to be the eigenspace of S^3_{tot} with eigenvalue $M \in \{-J(2L+1), \ldots, J(2L+1)\}$. These subspaces are called "sectors", and they are invariant subspaces for $H_L^{\mathrm{k}}(\Delta^{-1})$.

Ground States and spectral gap

- It was discovered (Alcaraz, Salinas, Wrezinski 1995) that under suitable boundary conditions the XXZ model posseses a family of ground states (kink and antikink). Ground state vectors are known in closed form and are unique in each sector.
- The magetization profile of the ground state vectors shows a sharp transition in the magnetization from a fully polarized down at the left to a fully polarized up at the right.



- Koma, Nachtergaele and Starr [2000] proved that there is a non vanishing spectral gap above all the kink ground states for all values of J.

Main theorem

Theorem

For suitably large anisotropy Δ the first excited state in each sector of the spin J XXZ model for $J \geq 3/2$ is an isolated eigenvalue i.e. there is a nonvanishing gap that is uniform in L with the rest of the spectrum.

Spectrum

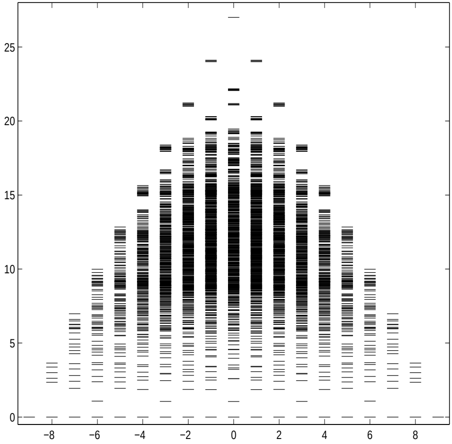


Figure: The Spectrum of the XXZ Model for J = 3/2, chain of length 6

Spectrum

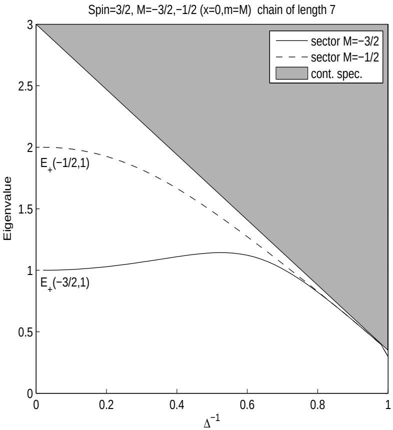


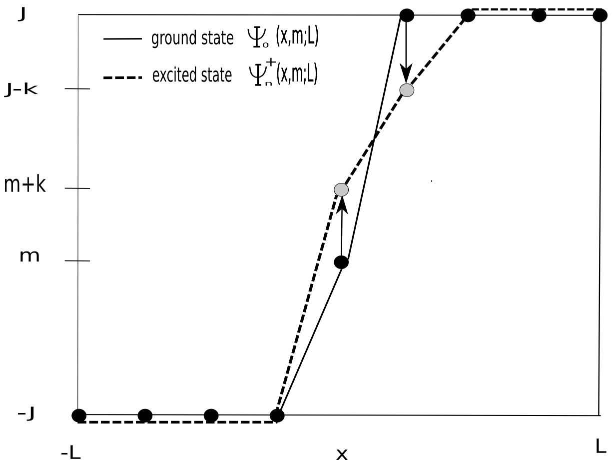
Figure: The Spectrum of the XXZ Model for $J = 3/2$ chain of length 7

XXZ Hamiltonian as a perturbation

$$H_L^{\mathrm{k}}(\Delta^{-1}) = H_L^{\mathrm{k}}(0) + \Delta^{-1} H_L^{(1)} + \left(1 - \sqrt{1 - \Delta^{-2}}\right) H_L^{(2)}$$

$$\begin{aligned} H_L^{\mathrm{k}}(0) &\equiv \text{Diagonal Hamiltonian (Ising limit)} \\ H_L^{(1)} &\equiv \text{Tridiagonal} \\ H_L^{(2)} &\equiv \text{Boundary term} \end{aligned}$$

Ising Excitations



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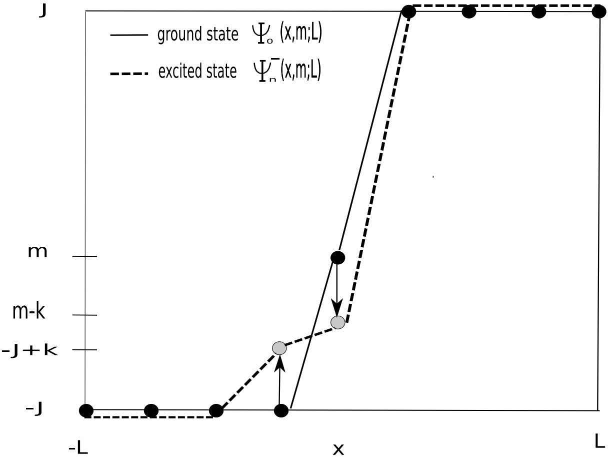
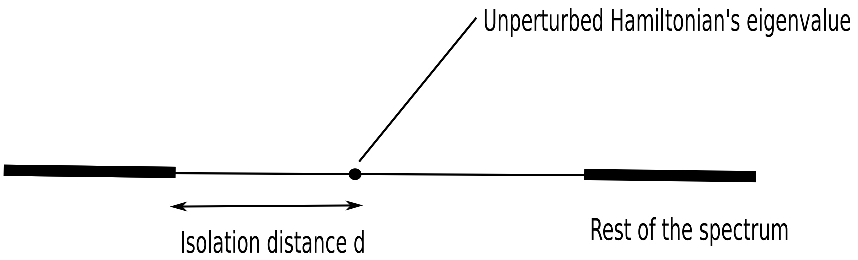


Proof sketch



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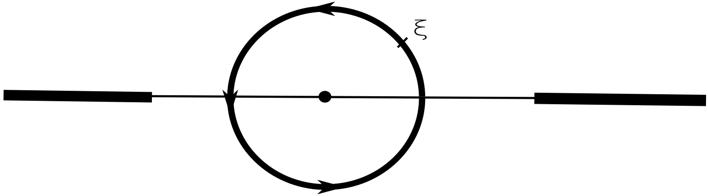

A resolvent formula

$$(H_L^{\mathrm{k}}(\Delta^{-1})-\xi)^{-1} = R(\xi)[1+\left(\Delta^{-1}H_L^{(1)}+(1-\sqrt{1-\Delta^{-2}})H_L^{(2)}\right)R(\xi)]^{-1}$$

where $R(\xi) = \left(H_L^{\mathrm{k}}(0)-\xi\right)^{-1}$

Proof sketch

$$\left\| \left(\Delta^{-1} H_L^{(1)} + \left(1 - \sqrt{1 - \Delta^{-2}} \right) H_L^{(2)} \right) R(\xi) \right\| < 1.$$

Proving an estimate of this form for Δ large enough, uniformly with respect to $\xi \in \Gamma$ is sufficient to guarantee analyticity of the spectral projections.

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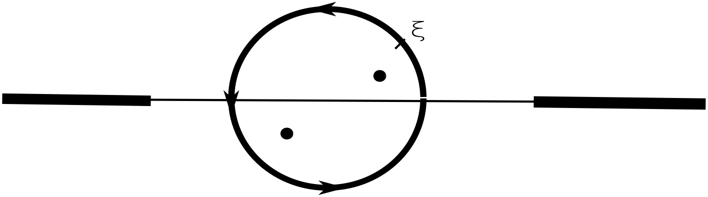
XXZ Hamitonian is a relatively bounded perturbation of the Ising limit

$$\left\| H_L^{(1)} \psi \right\| \leq \sqrt{J^2 + 2J^3} \, \| H_L^{\mathrm{k}}(0) \psi \| + 2J^2 \sqrt{J^2 + 2J^3} \, \| \psi \|$$

$$\left\| H_L^{(2)} \psi \right\| \leq 2J^2 \| \psi \|$$

Proof sketch



Relative boundedness $\Rightarrow$ existence of resolvent uniformly for all ξ in Γ $\Rightarrow$ analyticity of spectral projections$\Rightarrow$ Isolation of eigenvalues

Optimal transfer

We have been able get good theoretical and numerical approximations to our excitations. We are interested to find an optimal transfer from the ground state ψ_0 to the first excited state ψ_1 using our controls $v_j's$. The natural canditates for our control parameters would be components of a magnetic field.

$$H = H_L^{\mathrm{k}}(\Delta^{-1}) + \sum_{j=1}^{m} v_j H_j$$

$$\psi_0 \xrightarrow{\text{optimal transf.}} \psi_1$$

A qubit and a quantum NOT gate

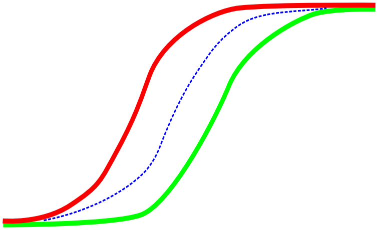
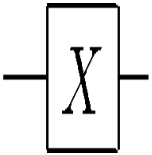


Figure: Magnetization profiles in the z-direction of the ground and of an excited state

$\alpha|0\rangle + \beta|1\rangle$



$\beta|0\rangle + \alpha|1\rangle$

Thanks for your attention!

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