# Non-integrability of the optimal control problem for $n$-level quantum systems

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## Aim

- To study integrability of the hamiltonian equation (adjoint equation) for the optimal control problem for $n$-level quantum systems
- To show usefulness of the Morales-Ramis theory in proving non-integrability
- More precisely, to prove:

  **Theorem 1** *The adjoint equation for the optimal control problem for $n$-level quantum systems is not integrable for $n \geq 4$*

## Plan

- Schrödinger equation for $n$-level systems
- Formulation and simplifications of the optimal problem;
- Sub-Riemannian formulation of the problem
- Optimal controls: Pontryagin Maximum Principle
- Main result: nonintegrability for $n \geq 4$
- Morales-Ramis theorem and differential Galois group
- Classification of integrable homogeneous sub-Riemannian problems in dimension 3

## Introduction

- Consider a quantum system with a finite number of (distinct) levels in interaction with a time dependent external field.
- The energies of the system state appearing on the diagonal, we put $\mathcal{H}_0 = \operatorname{diag}(E_1, \ldots, E_n)$.
- The time-functions $\Omega_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{C}$, for $1 \leq j \leq n-1$ have their supports in $[t_0, t_1]$. They couple the states by pairs.
- The hamiltonian $\mathcal{H}$ is given by:

$$\mathcal{H} = \begin{pmatrix} E_1 & \Omega_1(t) & 0 & \dots & 0 \\ \Omega_1^*(t) & E_2 & \Omega_2(t) & \ddots & \vdots \\ 0 & \Omega_2^*(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & E_{n-1} & \Omega_{n-1}(t) \\ 0 & \cdots & 0 & \Omega_{n-1}^*(t) & E_n \end{pmatrix}$$

$$= \mathcal{H}_0 + \begin{pmatrix} 0 & \Omega_1(t) & 0 & \dots & 0 \\ \Omega_1^*(t) & 0 & \Omega_2(t) & \ddots & \vdots \\ 0 & \Omega_2^*(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \Omega_{n-1(t)} \\ 0 & \cdots & 0 & \Omega_{n-1}^*(t) & 0 \end{pmatrix}$$

## Schrödinger equation

- The state vector $\psi(\cdot) : \mathbb{R} \longrightarrow \mathbb{C}^n$ satisfies the Schrödinger equation

$$i\frac{d\psi(t)}{dt} = \mathcal{H}\psi = (\mathcal{H}_0 + \sum_{j=1}^{n-1} \Omega_j(t)\mathcal{H}_j)\psi$$

  (we have assumed coupling of neighboring levels only).

- We represent

$$\psi(t) = \psi_1(t)e_1 + \psi_2(t)e_2 + \cdots \psi_n(t)e_n,$$

  where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{C}^n$

- We have $\mid \psi_1(t) \mid^2 + \mid \psi_2(t) \mid^2 + \cdots + \mid \psi_n(t) \mid^2 = 1$.

- For $t < t_0$ and $t > t_1$, $\mid \psi_j(t) \mid^2$ is the probability of measuring the energy $E_j$. Notice that $\frac{d}{dt} \mid \psi_j(t) \mid^2 = 0$, for $t < t_0$ and $t > t_1$.

# Optimal problem

**Problem :**

Assuming that

$$| \psi_1(t) |^2 = 1, \quad \text{for} \;\; t < t_0$$

find suitable interaction functions $\Omega_j(t)$, $1 \leq j \leq n-1$, such that

$$| \psi_i(t) |^2 = 1, \quad \text{for} \;\; t > t_1$$

for some chosen $i \in \{2, \ldots, n\}$, say $i = n$, and such that the cost

$$E = \frac{1}{2} \int_{t_o}^{t_1} \sum_{j=1}^{n-1} | \, \Omega_j(t) \, |^2 \, dt \longrightarrow min.$$

(minimize the energy of the transfer pulses).

## Resonant case

Optimal interaction functions $\Omega_j$ correspond to lasers that are in resonance (*real resonant case*, Brockett, Khaneja, Glaser, and Boscain, Charlot, Gauthier):

$$\Omega_j(t) = u_j(t)e^{i\omega_j t}, \quad \omega_j = E_{j+1} - E_j,$$

for $1 \leq j \leq n-1$, where $u_j(\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ are real controls. The cost function becomes

$$E = \frac{1}{2}\int_{t_o}^{t_1} \sum_{j=1}^{n-1} u_j^2(t)dt.$$

## Eliminating the drift

We will show how to eliminate the drift $\mathcal{H}_0 = \mathrm{diag}\,(E_1, \ldots, E_n)$ using a unitary change of coordinates. Assume $\psi(t)$ satisfies the Schrödinger equation. Choose $U(t) = \mathrm{diag}\,\left(e^{-iE_1 t}, e^{-iE_2 t}, \ldots, e^{-iE_n t}\right)$, a unitary time dependent matrix, and put

$$\psi(t) = U(t)\tilde{\psi}(t).$$

Then $\tilde{\psi}(t)$ satisfies the Schrödinger equation

$$i\frac{d\tilde{\psi}(t)}{dt} = \tilde{\mathcal{H}}\tilde{\psi},$$

where the new hamiltonian

$$\tilde{\mathcal{H}} = U^{-1}\mathcal{H}U - iU^{-1}\frac{dU}{dt}$$

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & \Omega_1(t)e^{-i\omega_1 t} & 0 & 0 \\ \Omega_1^*(t)e^{i\omega_1 t} & 0 & \ddots & \vdots \\ 0 & \Omega_2^*(t)e^{i\omega_2 t} & \ddots & 0 \\ \vdots & \ddots & 0 & \Omega_{n-1}(t)e^{-i\omega_{n-1} t} \\ 0 & \cdots & \Omega_{n-1}^*(t)e^{i\omega_{n-1} t} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & u_1(t) & 0 & \dots & 0 \\ u_1(t) & 0 & u_2(t) & \ddots & \vdots \\ 0 & u_2(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & u_{n-1}(t) \\ 0 & \cdots & 0 & u_{n-1}(t) & 0 \end{pmatrix}.$$

## Invariance under the unitary transformation

- Notice that, if we write $\tilde{\psi}(t) = \tilde{\psi}_1(t)e_1 + \tilde{\psi}(t)_2(t)e_2 + \cdots + \tilde{\psi}_n(t)e_n$, then $\mid \tilde{\psi}_j(t) \mid^2 = \mid \psi_j(t) \mid^2$ implying that $\mathcal{H}$ and $\tilde{\mathcal{H}}$ have the same population distribution.
- Moreover, the cost

$$E = \frac{1}{2}\int_{t_o}^{t_1} \sum_{j=1}^{n-1} \mid \Omega_j(t) \mid^2 dt = \frac{1}{2}\int_{t_o}^{t_1} \sum_{j=1}^{n-1} u_j^2(t)dt$$

does not change since we have applied a change of coordinates in the state-space only.

## From $\mathbb{C}^n$ to $\mathbb{R}^n$

Consider the control system in $\mathbb{C}^n$

$$\dot{\psi}_1 = -iu_1(t)\psi_2$$

$$\dot{\psi}_j = -i(u_{j-1}(t)\psi_{j-1} + u_j\psi_{j+1}) \quad \text{for } 2 \leq j \leq n-1$$

$$\dot{\psi}_n = -iu_{n-1}(t)\psi_{n-1}.$$

- Denote $\psi_j = v_j + iw_j$, $1 \leq j \leq n$ and consider the real and the imaginary part of the above equation.
- Set $x_1 = v_1$, $x_2 = w_2$ and, in general, $x_j = v_j$, if $j = 2k-1$, and $x_j = w_j$ if $j = 2k$.
- Replace $u_j$ by $-u_j$, for $j = 2k$ (does not change the cost) to get

$$\dot{x} = \mathcal{H}_{\mathbb{R}}x, \quad x \in \mathbb{R}^n,$$

where

$$\mathcal{H}_{\mathbb{R}} = \begin{pmatrix} 0 & u_1(t) & 0 & \cdots & 0 \\ -u_1(t) & 0 & u_2(t) & \ddots & \vdots \\ 0 & -u_2(t) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & u_{n-1(t)} \\ 0 & \cdots & 0 & -u_{n-1}(t) & 0 \end{pmatrix}.$$

Introduce the vector fields (infinitesimal generators of rotation in the $(x_i, x_j)$-space)

$$f_{i,j} = x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}, \quad 1 \leq i, j \leq n$$

## Optimal problem in $\mathbb{R}^n$

The problem is now: find real controls $u_1(t), \ldots, u_{n-1}(t)$ such that the corresponding trajectory of

$$\dot{x} = \mathcal{H}_{\mathbb{R}} x = \sum_{j=1}^{n-1} u_j f_{j,j+1}(x), \quad x \in \mathbb{R}^n,$$

joins given $x_0$ and $x_T$ and

$$E = \frac{1}{2} \int_{t_o}^{t_1} \sum_{j=1}^{n-1} u_j^2(t) dt \longrightarrow min.$$

## Lifting the problem to SO(n)

- The Lie algebra

$$\{f_{1,2}, \ldots, f_{n-1,n}\}_{LA} = \mathrm{vect}_{\mathbb{R}}\,\{f_{i,k}, \quad 1 \leq i < k \leq n\} = \mathfrak{so}(n)$$

- Let $F_{i,k}$ stand for the left invariant vector fields on SO(n) that satisfy exactly the same commutation relations as $f_{i,k}$.
- We lift our optimal control problem to the following left invariant on G=SO($n$): find controls $u_j(t)$ that minimize the energy $E$ of the curve $X(t) \in G$ =SO($n$) (time evolution operator) satisfying

$$\dot{X} = \sum_{j=1}^{n-1} u_j F_{j,j+1}, \quad E = \frac{1}{2}\int_{t_o}^{t_1} \sum_{j=1}^{n-1} u_j^2(t)dt \longrightarrow min.$$

- It is a sub-Riemannian problem!!!

## Sub-Riemannian manifold

A *sub-Riemannian manifold* is a triple $(M, \mathcal{D}, B)$, where

- $M$ is a smooth manifold,
- $\mathcal{D}$ is a smooth distribution of rank $m$ on $M$
- $B$ a smoothly varying positive definite bilinear form on $\mathcal{D}$, that is, an Euclidean product on $\mathcal{D}$.

## Controllability: Rashevsky and Chow

Put $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_{s+1} = \mathcal{D}_s + [\mathcal{D}, \mathcal{D}_s]$. If for each point $q \in M$, there exists an integer $r(q)$ (called the nonholonomy degree at $q$) such that $\mathcal{D}_{r(q)}(q) = T_qM$, then any two points in $M$ can be joined by a curve that is almost everywhere tangent to $\mathcal{D}$, called a *horizontal curve.*

## Sub-Riemannian metric

Put $\|v\| = (B(v,v))^{1/2}$, for any $v \in \mathcal{D}(q) \subset T_qM$, and let $\gamma : I \to M$ be a horizontal curve. We define the length $l(\gamma)$ of $\gamma$ as

$$l(\gamma) = \int_I \|\dot{\gamma}(t)\| dt.$$

We can thus endow $M$ with a metric $d$: the sub-Riemannian distance $d(q_1, q_2)$ between two pints $q_1$ and $q_2$ is the infimum of $l(\gamma)$ over all horizontal curves joining $q_1$ and $q_2$.

- Sub-Riemannian geometry problem: find horizontal curves minimizing the length $l(\gamma)$, i.e. find sub-Riemannian geodesics.

## Minimizing: energy versus length

- The energy $E(\gamma)$ of a curve $\gamma$ is defined as

$$E(\gamma) = \frac{1}{2}\int_I \|\dot{\gamma}(t)\|^2 dt.$$

- Analytically it is more convenient to minimize the energy $E(\gamma)$ rather than the length $l(\gamma)$.

- As in Riemannian geometry, due to Cauchy-Schwartz inequality, the minimizers of both problems coincide. Namely, a horizontal curve $\gamma$ minimizes the energy $E$ among all horizontal curves joining $q_1$ and $q_2$ in time $T$ if and only if it minimizes the length $l$ among all horizontal curves joining $q_1$ and $q_2$ and is parameterized to have constant speed $c = d(q_1, q_2)/T$.

## Formulating an optimal control problem

- For a given framing $\mathcal{D} = \langle f_1, ..., f_m \rangle$ by $m$ orthonormal vector fields, any integral curve $x(t)$ of $\mathcal{D}$ satisfies

$$\Sigma : \quad \dot{x}(t) = \sum_{i=1}^{m} f_i(x(t))u_i(t),$$

  where $u_i(t)$, for $1 \leq i \leq m$, are controls.

- A geodesic is a trajectory of $\Sigma$ that minimizes the energy

$$E = \frac{1}{2} \int_I \sum_{i=1}^{m} u_i^2(t)dt.$$

- The geometric problem of minimizing the subriemannian distance is the optimal control problem of minimizing the energy $E$ for the control-linear system $\Sigma$ (for example, for our quantum system).

## Pontryagin Maximum Principle (PMP)

- To solve this optimal control problem, we will apply the Pontryagin Maximum Principle (PMP) to the problem of minimization of $E$.
- Define the hamiltonian of the optimal control problem

$$\widehat{h} : T^*\mathbb{R}^n \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}, \;\; \widehat{h}(x,p,u) = \sum_{j=1}^{n-1} (< p, u_j f_{j,j+1}(x) > -\frac{1}{2} u_j^2).$$

- Define the maximized hamiltonian $h$ (solve $\frac{\partial \widehat{h}}{\partial u} = 0$ which gives $u_j =< p, f_{j,j+1} >$) by

$$h(x,p) = \max_u \widehat{h}(x,p,u) = \frac{1}{2} \sum_{j=1}^{n-1} (< p, f_{j,j+1}(x) >)^2$$

(a quadratic function on fibres).

## Pontryagin Maximum Principle - statement

**Theorem 2** *If a control $u(t)$ and the corresponding normal trajectory $x(t)$ minimize the cost $E$, then there exits a curve $p(t) \in T^*_{x(t)}\mathbb{R}^n$ in the cotangent bundle such that $\lambda(t) = (x(t), p(t))$ satisfies the following hamiltonian equation $\dot{\lambda}(t) = \overrightarrow{h}(\lambda(t))$ on $T^*\mathbb{R}^n$:*

$$\dot{x} = \frac{\partial h}{\partial p}(x(t), p(t))$$

$$\dot{p} = -\frac{\partial h}{\partial x}(x(t), p(t)),$$

where $h$ is the maximized hamiltonian, and $u_j(t) =< p(t), f_{j,j+1}(x(t)) >$ are optimal controls.

## Integrability of the geodesic equation

- Our main problem: study integrability of the geodesic equation.
- Brockett and Dai started a systematic study of integrability of the geodesic equation (in terms of elliptic functions).
- 3-dimensional nilpotent cases are integrable: Heisenberg (in terms of trigonometric functions) and Martinet (in terms of elliptic functions, Bonnard, Chyba, Trelat); and the tangent case?
- Jurdjevic has shown integrability (in terms of elliptic functions) of several invariant SR-problems on Lie groups.
- There exist nonintegrable sub-Rimennian geodesic equations in nilpotent cases (a 6-dim. example of Montgomery-Shapiro).
- Complete list of integrable cases for 3-dim. homogenous SR-spaces

# Pontryagin Maximum Principle on SO($n$)

Using the PMP we conclude that if $X(t)$ is a minimizing curve in $G$=SO(n), then there exits a curve $P(t) \in T^*_{X(t)}G$ such that $(X(t), P(t))$ satisfies the hamiltonian system

$$\dot{X} = \frac{\partial H}{\partial P}(X(t), P(t))$$
$$\dot{P} = -\frac{\partial H}{\partial X}(X(t), P(t)),$$

where $H : T^*G \longrightarrow \mathbb{R}$ is given by

$$H(X, P) = \frac{1}{2}\sum_{j=1}^{n-1}(< P, F_{j,j+1} >)^2.$$

## Poisson structure on $\mathfrak{g}^*$

- Upon the identification of the space of left invariant vector fields on $G$=SO(n) with the Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$ of $G$, the hamiltonian $H(X,P) = \frac{1}{2}\sum_{j=1}^{n-1}(< P, F_{j,j+1} >)^2$ becomes identified with a quadratic function on $\mathfrak{g}^*$.
- The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ carries a Poisson bracket defined, for any smooth functions $\varphi_1$ and $\varphi_2$ on $\mathfrak{g}^*$, by

$$\{\varphi_1, \varphi_2\}(\eta) = \langle \eta, [d\varphi_1, d\varphi_2](\eta)\rangle, \quad \text{for each } \eta \in \mathfrak{g}^*.$$

## Adjoint equation

To the hamiltonian $H$ on $\mathfrak{g}^*$ (considered as a Poisson manifold) we associate the *Hamiltonian vector field* $\overrightarrow{H}$ on $\mathfrak{g}^*$ defined by

$$\overrightarrow{H}(\varphi) = \{\varphi, H\}, \qquad \text{for each } \varphi \in C^\infty(\mathfrak{g}^*).$$

We will call the differential equation

$$\dot{\eta}(t) = \overrightarrow{H}(\eta(t)), \quad \eta(t) \in \mathfrak{g}^*,$$

defined on $\mathfrak{g}^*$ by the Hamiltonian vector field $\overrightarrow{H}$ associated to $H$, the *adjoint equation* of the hamiltonian system

$$\begin{aligned} \dot{X} &= \frac{\partial H}{\partial P}(X(t), P(t)) \\ \dot{P} &= -\frac{\partial H}{\partial X}(X(t), P(t)) \quad \left(\dot{\eta}(t) = \overrightarrow{H}(\eta(t))\right). \end{aligned}$$

Recall the functions

$$H_{i,k} =< P, F_{i,k} >,$$

for $1 \le i < k \le n$, which allow to rewrite the hamiltonian as

$$H = \frac{1}{2} \sum_{j=1}^{n-1} H_{j,j+1}^2,$$

the optimal controls as

$$u_j(t) = H_{j,j+1}(t) =< P(t), F_{j,j+1}(X(t)) >,$$

and the corresponding hamiltonian system as

$$\dot{X} = \sum_{j=1}^{n-1} H_{j,j+1} F_{j,j+1}$$

$$\dot{H}_{i,k} = \{H, H_{i,k}\}, \quad 1 \le i < k \le n, \quad \Big( \dot{\eta}(t) = \overrightarrow{H}(\eta(t)) \Big).$$

## Integrability

- The adjoint equation is a Lie-Poisson equation defined by a Poisson structure on $\mathfrak{g}^*$ whose structure constants $C_{i,k\ j,l}^{q,s}$ are those defining the Lie algebra $\mathfrak{g} = \mathfrak{so}(n)$.
- This Poisson structure is degenerated and of rank, say, $2r$.
- Since $\dim \mathfrak{g}^* = \frac{n(n-1)}{2} = N$, the Poisson structure admits $k = N - 2r$ Casimir functions $C_1, \ldots, C_{N-2r}$ whose common constant level sets $M_c = \{\eta \in \mathfrak{g}^* \ : \ C_1(\eta) = c_1, \ldots, C_{N-2r}(\eta) = c_{N-2r}\}$ are $2r$-dimensional submanifolds of $\mathfrak{g}^*$ equipped with a symplectic structure defined by the restriction of the Poisson structure to $M_c$.
- The adjoint equation restricted to $M_c$ is a hamiltonian equation.

## Integrability - definition

- If a Lie-Poisson equation possesses $k + r$ functionally independent first integrals belonging to a category $\mathcal{C}$ such that the first $k$ integrals are Casimir functions and the remaining $r$ ones commute, then we will say that this equation is integrable in the category $\mathcal{C}$.

## 3-level system

Easy to integrate (Brockett, Boscain et al. for the quantum system)

The adjoint equation takes the form

$$\dot{H}_{1,2} = H_{1,3}H_{2,3}$$

$$\dot{H}_{2,3} = -H_{1,3}H_{1,2}$$

$$\dot{H}_{1,3} = 0$$

We get $H_{1,3}(t) = \text{const.} = a$ and

$$u_1(t) = H_{1,2}(t) = r\cos(at+\varphi)$$
$$u_2(t) = H_{1,2}(t) = -r\sin(at+\varphi).$$

$H_{1,3}$ is a Casimir function; we integrate the system on its constant level sets.

Now it suffices to integrate the linear time-varying system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = u_1 \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ -x_3 \\ x_2 \end{pmatrix} \tag{1}$$

which has the first integral:

$$h = x_1^2 + x_2^2 + x_3^2. \tag{2}$$

Three linearly independent solutions can be taken as:

The first solution is

$$x_1(t) = u_2(t), \quad x_2(t) = a, \quad x_3(t) = u_1(t). \tag{3}$$

The second solution is given by

$$\begin{aligned} x_1(t) &= -\omega u_1 \sin \omega t - a u_2 \cos \omega t, \\ x_2(t) &= r^2 \cos \omega t, \\ x_3(t) &= \omega u_2 \sin \omega t - a u_1 \cos \omega t, \end{aligned} \tag{4}$$

where $\omega = \sqrt{r^2 + a^2}$. The third solution is given by

$$\begin{aligned} x_1(t) &= \omega u_1 \cos \omega t - a u_2 \sin \omega t, \\ x_2(t) &= r^2 \sin \omega t, \\ x_3(t) &= -\omega u_2 \cos \omega t - a u_1 \sin \omega t, \end{aligned} \tag{5}$$

## Main result

**Theorem 3** *For the $n$-level system, $n \geq 4$, the complexification of the adjoint equation on $\mathfrak{so}(n)^*$ is not integrable in the meromorphic category. More precisely, restricted to the leaves $M_c$ of the symplectic foliation on $\mathfrak{so}(n)^*$, does not possess any meromorphic first integral independent of the hamiltonian, i.e. is not Liouville integrable on $M_c$.*

## 4-level system: Adjoint equation on $\mathfrak{so}(4)^*$

- By restricting the $AE$ to $\{H_{i,k} = 0\}$, where $i \geq 5$ or $k \geq 5$, the nonintegrability problem of the general $n$-level system reduces to that of the 4-level system.
- We will consider the complexification $AE_{\mathbb{C}}$ of $AE$ on $\mathfrak{so}(4)^*$ by taking $x_i \in \mathbb{C}$ and $t \in \mathbb{C}$, where $x_1 = H_{1,2}$, $x_2 = H_{2,3}$, $x_3 = H_{1,3}$, $x_4 = H_{3,4}$, $x_5 = H_{1,4}$, and $x_6 = H_{4,2}$.
- The complexified $AE_{\mathbb{C}}$ reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}x = J(x)\nabla H(x), \qquad x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{C}^6, \quad t \in \mathbb{C}$$

where

$$H = H(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_4^2),$$

and

$$J(x) = \begin{bmatrix} 0 & x_3 & -x_2 & 0 & x_6 & -x_5 \\ -x_3 & 0 & x_1 & -x_6 & 0 & x_4 \\ x_2 & -x_1 & 0 & x_5 & -x_4 & 0 \\ 0 & x_6 & -x_5 & 0 & x_3 & -x_2 \\ -x_6 & 0 & x_4 & -x_3 & 0 & x_1 \\ x_5 & -x_4 & 0 & x_2 & -x_1 & 0 \end{bmatrix},$$

It is a Lie-Poisson system: $\operatorname{rank} J(x) = 4$ so $J(x)$ defines a Poisson structure (a "degenerated symplectic structure").

- Besides the Hamiltonian $H$, $AE_{\mathbb{C}}$ admits two additional first integrals

$$C_1 = \sum_{i=1}^{6} x_i^2, \qquad C_2 = x_1 x_4 + x_2 x_5 + x_3 x_6,$$

which are actually the Casimir function of the Poisson structure defined by $J(x)$; the first integrability requirement is satisfied.

- Each level set

$$\mathcal{M}_{a,b} := \{x \in \mathbb{C}^6 \,|\, C_1(x) = a, \quad C_2(x) = b\},$$

is a 4-dimensional symplectic manifold on which $AE_{\mathbb{C}}$ is hamiltonian with Hamiltonian function $H_{|\mathcal{M}_{a,b}}$. We need one more first integral!

## Morales-Ramis theory

Consider a complex analytic hamiltonian differential equation

$$\frac{dx}{dt} = v(x), \quad t \in \mathbb{C},$$

on an analytic symplectic manifold M (say, $\mathbb{C}^n$). Let $\varphi(t)$ be its non-stationary solution and $\Gamma$ its maximal analytic prolongation (Riemann surface). Take the linearization (variational equation) along $\Gamma$

$$\frac{d\xi}{dt} = \frac{\partial v}{\partial x}(\varphi(t))\xi$$

**Theorem 4** (Morales-Ramis) *If the hamiltonian system on $M$ ($\mathbb{C}^n$) is Liouville integrable in the meromorphic category, then the identity component of the differential Galois group of the (normal) variational equation along $\Gamma$ is abelian.*

## Differential Galois group

Consider a homogeneous ordinary linear differential equation in $\mathbb{C}^n$, over the field $F = \mathbb{C}(z)$ of rational functions of $z \in \mathbb{C}$

$$L(Y) = \frac{\mathrm{d}}{\mathrm{d}z}Y - A(z)Y = 0, \quad Y \in \mathbb{C}^n,$$

where $A_i^j \in \mathbb{C}(z)$

- Where do the solutions live?

**Theorem 5** *There exits a unique (up to isomorphism) $PV_L \supset \mathbb{C}(z)$, the smallest differential field extension containing $n$ linearly independent, over $\mathbb{C}$, solutions of $L(Y) = 0$ (Picard-Vessiot extension).*

We have $(PV_L, D) \supset (\mathbb{C}(z), \frac{\mathrm{d}}{\mathrm{d}z})$, where the derivation $D$ restricted to $\mathbb{C}(z)$ is $\frac{\mathrm{d}}{\mathrm{d}z}$.

## Differential Galois group - continuation

The space of solutions $V = \{Y \in PV_L \mid L(Y) = 0\}$ is a linear space over $\mathbb{C}$.

**Definition 1** Differential Galois group of $L$ is the group of differential automorphisms of $PV_L$ (i.e., commuting with the derivation $D$) preserving all elements of $\mathbb{C}(z)$.

The differential Galois group, denoted $Gal(PV_L \backslash \mathbb{C}(z))$

- preserves solutions
- preserves polynomial relations among them
- is an algebraic subgroup of $\mathrm{SL}(n, \mathbb{C})$(in the hamiltonian case of $\mathrm{Sp}(n, \mathbb{C})$.

$AE_{\mathbb{C}}$ admits the invariant space

$$\mathcal{M}^3 = \{x \in \mathbb{C}^6 \,|\, x_4 = x_5 = x_6 = 0\},$$

foliated by the phase curves $\Gamma_{h,f} = \mathbb{S}^1_{\mathbb{C}}$, complex circles, given by

$$x_1^2 + x_2^2 = h, \qquad x_3 = f$$

The normal variational equations along $\Gamma_{h,f}$ reduces to the form

$$w'' = r(z)w, \qquad r(z) = \frac{\alpha_0}{z^2} + \frac{\alpha_h}{(z-h)^2} + \frac{\beta_0}{z} + \frac{\beta_h}{z-h}$$

Singular points at $z = 0$ and $z = h$ are regular but at $\infty$ is irregular. Indeed, we have

**Lemma 1** *The differential Galois group of* $w'' = r(z)w$ *is* $\mathrm{SL}(2, \mathbb{C})$.

$\mathrm{SL}^0(2, \mathbb{C})$ is non-abelian, hence the adjoint equation is not integrable.

## How to calculate $Gal(PV_L \backslash \mathbb{C}(z))$, for $n = 2$?

**Lemma 2** *Let $\mathcal{G}$ be the differential Galois group of the equation $w'' = r(z)w$. Then one and only on of four cases can occur:*

(i) *$\mathcal{G}$ is conjugated to a triangular group; in this case equation $w'' = r(z)w$ has an exponential solution,*

(ii) *$\mathcal{G}$ is conjugated to a diagonal - antidiagonal group; in this case the equation $w'' = r(z)w$ has a solution of the form $w = \exp \int \omega$, where $\omega$ is algebraic over $\mathbb{C}(z)$ of degree 2,*

(iii) *$\mathcal{G}$ is finite; in this case all solutions of $w'' = r(z)w$ are algebraic,*

(iv) *$\mathcal{G} = \mathrm{SL}(2, \mathbb{C})$ and $w'' = r(z)w$ has no Liouvillian solution.*

**Lemma 3** *For equation $w'' = r(z)w$ we have:*

(i) *If the case (i) of Lemma 2 occurs, then* $\operatorname{ord}(z_i) = 1$ *or* $\operatorname{ord}(z_i) = 2k$ *for all* $z_i \in Sing$*, the set of singular points of* $r(z)$*, and* $\operatorname{ord}(\infty) = 2k$ *or* $\operatorname{ord}(\infty) > 2$

(ii) *If the case (ii) of Lemma 2 occurs, then* $Sing \neq \emptyset$ *and* $\operatorname{ord}(z_i) = 2k + 1 > 2$ *or* $\operatorname{ord}(z_i) = 2$*, for all* $z_i \in Sing$*.*

(iii) *If the case (iii) of Lemma 2 occurs, then* $\operatorname{ord}(z_i) \leq 2$ *for all* $z_i \in Sing$ *and* $\operatorname{ord}(\infty) \geq 2$

From the above Lemma it follows that for the equation $w'' = r(z)w$ the cases (i) and (iii) of Lemma 2 cannot occur. In order to show that the case (ii) of Lemma 2 does not occur either, we apply the Kovacic algorithm.

$\boxed{n\textbf{-level quantum system}}$

$$\Downarrow$$

$\boxed{\textbf{optimal control problem: Pontryagin Maximum Principle}}$

$$\Downarrow$$

$\boxed{\textbf{sub-Riemannian problem on SO}(n)}$

$$\Downarrow$$

$\boxed{\textbf{nonintegrability of a hamiltonian system}}$

$$\Downarrow$$

$\boxed{\textbf{Differential Galois group and complex analysis}}$

## 3-dimensional sub-Riemannian spaces

- Classify all cases of integrable adjoint geodesic equation for homogeneous spaces
- Study integrability of the nilpotent tangent case.

## Homogenous and symmetric SR-spaces

- An sub-Riemannian *isometry* between SR-manifolds $(M, \mathcal{D}, B)$ and $(\tilde{M}, \tilde{\mathcal{D}}, \tilde{B})$ is a diffeomorphism $\psi : M \to \tilde{M}$ such that $\psi_*(\mathcal{D}) = \tilde{\mathcal{D}}$ and $B = \psi^*(\tilde{B})$.
- A *homogeneous sub-Riemannian space*, shortly, a *sub-homogeneous space*, is a sub-Riemannian manifold for which the group of its sub-Riemannian isometries is a Lie group that acts smoothly and transitively on the manifold.
- A sub-homogeneous space is said to be *symmetric*, shortly, *sub-symmetric*, if for each point $q \in M$ there exists an isometry $\psi$ such that $\psi(q) = q$ and $\psi_*|_{\mathcal{D}(q)} = -\mathrm{Id}$.

## 3-dimensional homogeneous sub-Riemannian spaces

**Lemma 4** *(Falbel-Gorodski) To any 3-dimensional SR-homogenous space* $(M, \mathcal{D}, B)$ *there corresponds a Lie group* $G$ *that acts simply and transitively on* $M$

The Lie algebra $\mathfrak{g}$ of $G$ has a decomposition $\mathfrak{g} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$, where for a chosen base point $q \in M$ we identify $\mathfrak{g}$ with $T_qM$, the subspace $\mathfrak{p}$ of $\mathfrak{g}$ with $\mathcal{D}(q)$, and the quadratic form $\mathfrak{b}$ defined on $\mathfrak{p}$ with $B$. The triple $(\mathfrak{g}, \mathfrak{p}, \mathfrak{b})$ will be called a *sub-Riemannian Lie algebra* (does not depend on the chosen base point $q$).

$$\mathfrak{g} = \operatorname{span}\{X_1, X_2, X_3\}$$

The Lie algebra of an orthonormal frame can be brought in the SR-symmetric case to the following normal form (sub-symmetric Lie algebras):

$$\begin{aligned}
[X_1, X_2] &= X_3, \\
[X_1, X_3] &= aX_2, \\
[X_2, X_3] &= bX_1,
\end{aligned}$$

where $(a, b) \in \mathbb{R}^2$.

# Integrability of the SR-symmetric case

**Theorem 6** *For a given sub-Riemannian homogeneous space, the following conditions are equivalent:*

(i) *The sub-Riemannian space is symmetric.*

(ii) *The adjoint equation has two functionally independent quadratic first integrals;*

(iii) *The optimal controls are elliptic functions;*

(iv) *All solutions of the complexified adjoint equation are single-valued functions of the complex time;*

## Nonintegrability of the SR-non symmetric

The Lie algebra of an orthonormal frame can be brought in the SR-symmetric case to the following normal form

$$
\begin{aligned}
[X_1, X_2] &= X_3, \\
[X_1, X_3] &= aX_2 + bX_3, \\
[X_2, X_3] &= 0,
\end{aligned}
$$

where $(a, b) \in \mathbb{R}^2$ and $ab \neq 0$. When $a = 0$ or $b = 0$ the underlying space is isometric to a sub-symmetric space. By a proper rescaling we can assume $b = 1$.

We distinguish two subsets of the classification parameter:

- $a \in \Lambda_p \subset \mathbb{R}$ if and only if there exist positive integers $m$ and $n$ such that $a = mn/(m-n)^2$

- $a \in \Lambda_r \subset \mathbb{R}$ if and only if there exist integers $m$ and $n$ such that $a = mn/(m-n)^2$ and $a \neq -1/4$.

**Theorem 7** *We have the following integrability properties of the adjoint equation of given non symmetric sub-homogeneous space defined by the parameter $a$.*

(i) *The adjoint equation admits a polynomial fist integral independent with the hamiltonian $H$ if and only if $a \in \Lambda_p$;*

(ii) *The adjoint equation admits a rational fist integral independent with the hamiltonian $H$ if and only if $a \in \Lambda_r$;*

(iii) *If $a \in \mathbb{R} \setminus \Lambda_r$ then the adjoint equation does not admit any real-meromorphic first integral independent with the hamiltonian $H$.*

## Lie algebra of the system

Consider the system

$$\dot{\xi} = \sum_{i=1}^{m} X_i(\xi) u_i.$$

on a manifold $M$. Denote $\mathcal{D} = \text{span}\,\{X_1, ..., X_m\}$.

- Let $\mathcal{L}_1 = \text{span}_{\mathbb{R}}\{X_1, \ldots, X_m\}$.
- Define inductively

$$\mathcal{L}_s = \mathcal{L}_{s-1} + [\mathcal{L}_{s-1}, \mathcal{L}_1] \quad \text{for } s \geq 2.$$

- Clearly $\mathcal{L}_s(p) = \mathcal{D}_s(p)$ and the sum

$$\mathcal{L}(X_1, \ldots, X_m) = \mathcal{L} = \sum_{s \geq 1} \mathcal{L}_s,$$

is the Lie algebra of the system.

## Weights

- For $q \in M$, put $L_s(q) = \{X(q) : \ X \in \mathcal{L}_s\}$
- Denote $n_s(q) = \dim\, L_s(q)$. For a completely nonholonomic system we have
$$1 \leq n_1(q) \leq n_2(q) \leq \cdots \leq n_{r(q)}(q) = n$$
and we will call $(n_1(q), n_2(q), \ldots, n_{r(q)}(q))$ the *growth vector* of the system (we will omit indicating the point if it is not confusing).
- Define weights $w_1 \leq \cdots \leq w_n$ by putting $w_j = s$ if $n_{s-1} < j \leq n_s$, with $n_0 = 0$.

## Privileged coordinates

- We will call $X_1\varphi, \ldots X_m\varphi$ the nonholonomic partial derivatives of order 1 of a function $\varphi$
- $X_{i_1}X_{i_2}\varphi$ nonholonomic derivatives of order two of $\varphi$ etc.
- If all the nonholonomic derivatives of order $\leq s-1$ of $\varphi$ vanish at $q$, we say that $\varphi$ is of order $\geq s$ at $q$. A function $\varphi$ is of *order* $s$ at $q$ if it is of order $\geq s$ but not of order $\geq s+1$.
- Local coordinates $(\xi_1, \ldots, \xi_n)$ are *privileged coordinates* at $q$ if the order of $\xi_i$ is $w_i$ for $1 \leq i \leq n$.
- The integers $(w_1, \ldots, w_n)$ are the *weights* of the privileged coordinates $(\xi_1, \ldots, \xi_n)$. Homogeneity is considered with respect to them.

# Nilpotent approximations

- Using privileged coordinates we can rewrite the system as

$$\dot{\xi}_j = \sum_{i=1}^{m} X_{ij}(\xi_1, \ldots, \xi_{j-1}) u_i + O(\|\xi\|^{w_j})$$

  for $1 \leq j \leq n$, where the components $X_{ij}$ are homogeneous polynomials of weighted degree $w_j - 1$.

- By dropping the terms $O(\|\xi\|^{w_j})$, we get

$$\dot{\xi} = \sum_{i=1}^{m} \widehat{X}_i(\xi) u_i, \quad \text{where} \quad \widehat{X}_i = \sum_{j=1}^{n} X_{ij}(\xi_1, \ldots, \xi_{j-1}) \frac{\partial}{\partial \xi_j},$$

  called the *nilpotent approximation* of the system. The Lie algebra $\mathcal{L}(\widehat{X}_1, \ldots, \widehat{X}_m)$ is nilpotent.

## 3-dimensional sub-Riemannian manifolds

Consider a 3-dimensional sub-Riemannian manifold $(M, \mathcal{D}, B)$, where

- $M$ is a 3-dimensional manifold,
- $\mathcal{D}$ is a rank 2 smooth distribution on $M$
- $B$ is a smoothly varying positive definite quadratic form on $\mathcal{D}$.
- Represent locally the sub-Riemannian structure $(M, \mathcal{D}, B)$ by the control system
$$\dot{\xi} = X_1(\xi)u_1 + X_2(\xi)u_2,$$
where the smooth vector fields $X_1$ and $X_2$ form an orthonormal frame of $\mathcal{D}$.

## Normal form

An *isometry* between two sub-Riemannian manifolds $(M, \mathcal{D}, B)$ and $(\tilde{M}, \tilde{\mathcal{D}}, \tilde{B})$ is a diffeomorphism $\phi : M \to \tilde{M}$ such that $\phi_*(\mathcal{D}) = \tilde{\mathcal{D}}$ and $B = \phi^*(\tilde{B})$. Agrachev et al have shown that there exists a sub-Riemannian isometry transforming the orthonormal frame $\langle X_1, X_2\rangle$ into an orthonormal frame, which in local coordinates $(x, y, z)$ takes the following normal form around $0 \in \mathbb{R}^3$:

$$X_1(x,y,z) = \left(1 + y^2\beta(x,y,z)\right)\frac{\partial}{\partial x} - xy\beta(x,y,z)\frac{\partial}{\partial y} + \frac{y}{2}\gamma(x,y,z)\frac{\partial}{\partial z}$$

$$X_2(x,y,z) = -xy\beta(x,y,z)\frac{\partial}{\partial x} + \left(1 + x^2\beta(x,y,z)\right)\frac{\partial}{\partial y} - \frac{x}{2}\gamma(x,y,z)\frac{\partial}{\partial z}.$$

## Contact case

- If $\gamma(0,0,0) \neq 0$, then we are in the contact case.
- The growth vector in the contact case is $(2,3)$ and the variables $x, y, z$ have weights 1, 1, and 2, respectively.
- The normal form for the nilpotent approximation is

$$\begin{aligned} \widehat{X}_1(x,y,z) &= \frac{\partial}{\partial x} + c\frac{y}{2}\frac{\partial}{\partial z} \\ \widehat{X}_2(x,y,z) &= \frac{\partial}{\partial y} - c\frac{x}{2}\frac{\partial}{\partial z}. \end{aligned}$$

- All cases are isometric to the Heisenberg case $c = 1$.
- The Heisenberg case is integrable in trigonometric functions.
- The general contact case (non nilpotent) has been completely analyzed by Agrachev, Gauthier, Kupka, and Chakir.

## Martinet case

- If $\gamma$ is of order 1 with respect to $(x, y)$, then we are in the Martinet case
- The growth vector at $0 \in \mathbb{R}^3$ in the Martinet case is $(2, 2, 3)$ and the weights of the variables $x, y, z$ are 1, 1, and 3, respectively.
- the set of points, at which the growth vector is $(2, 2, 3)$, is a smooth surface (called *Martinet surface*) and the distribution $\mathcal{D}$ spanned by $X_1$ and $X_2$ is transversal to the Martinet surface.
- The normal form for the nilpotent approximation is

$$\widehat{X}_1(x, y, z) = \frac{\partial}{\partial x} + \frac{y}{2}(ax + by)\frac{\partial}{\partial z}$$
$$\widehat{X}_2(x, y, z) = \frac{\partial}{\partial y} - \frac{x}{2}(ax + by)\frac{\partial}{\partial z}.$$

## Martinet case - cont.

- All nilpotent Martinet cases are integrable in terms of elliptic functions.
- sub-Riemannian geometry in the general (non nilpotent) case has been intensively studied by Bonnard, Chyba, and Trélat.

## Tangent case

- The next degeneration, *tangent case*, occurs at points at which the distribution $\mathcal{D}$ is tangent to the Martinet surface.
- Generically, the growth vector at such a tangency point is $(2, 2, 2, 3)$ and the variables $x, y, z$ are of weights 1, 1, and 4, respectively.
- $\gamma$ is of order 2 with respect to $(x, y)$.
- The normal form of the nilpotent approximation of the tangent case is

$$\widehat{X}_1(x, y, z) = \frac{\partial}{\partial x} + \frac{y}{2}(ax^2 + by^2)\frac{\partial}{\partial z}$$
$$\widehat{X}_2(x, y, z) = \frac{\partial}{\partial y} - \frac{x}{2}(ax^2 + by^2)\frac{\partial}{\partial z}.$$

We can assume that $a = 1$ (by normalizing $z$).

## Tangent case: geodesic equation

The geodesic equation in the nilpotent tangent case is:

$$
(GE) \qquad
\begin{aligned}
\dot{x} &= p + \frac{ry}{2}(x^2 + by^2), \\
\dot{y} &= q - \frac{rx}{2}(x^2 + by^2), \\
\dot{z} &= \frac{1}{2}(x^2 + by^2)(yp - xq) + \frac{r}{4}(x^2 + y^2)(x^2 + by^2)^2, \\
\dot{p} &= -rxyu_1 + \frac{r}{2}(3x^2 + by^2)u_2, \\
\dot{q} &= -\frac{r}{2}(x^2 + 3by^2)u_1 + brxyu_2. \\
\dot{r} &= 0
\end{aligned}
$$

where $u_1 = p + \dfrac{ry}{2}(x^2 + by^2)$ and $u_2 = q - \dfrac{rx}{2}(x^2 + by^2)$.

# Integrability problem

- The hamiltonian $H$ and $H_1 = r$ are first integrals.
- **Integrability problem:** find a third first integral $H_2$, commuting with $H$ and $H_1$, and functionally independent with $H$ and $H_1$ (Liouville integrability).
- We will distinguish the *elliptic nilpotent tangent case*, for which $a = 1$ and $b > 0$ and the *hyperbolic nilpotent tangent case*, for which $a = 1$ and $b < 0$.

## Tangent case: integrable cases

- M. Pelletier proved that if $b = 1$ (symmetric elliptic case), then the Hamiltonian (GE) is integrable in the Liouville sense with an additional first integral given by

$$H_2 = xq - yp.$$

- Geometric reason: if $b = 1$, then the rotation in the $(x, y)$ space is a sub-Riemannian isometry.

- For $b = 0$, the geodesic equation (GE) is also integrable. In this case the third first integral has the form

$$H_2 = 6q + rx^3.$$

- Both cases are integrable in terms of elliptic functions.

## Main result

**Theorem 8** *The complexified geodesic equation for the 3-dimensional nilpotent tangent case is not meromorphically integrable in the Liouville sense, except for* $b = 1$ *and* $b = 0$*, that is, for* $b \in \mathbb{R} \setminus \{0, 1\}$ *the complexified system (GE) does not possess a meromorphic first integral, commuting with* $H$ *and* $H_1$ *and functionally independent with* $H$ *and* $H_1$.

# Proof

Our proof is based on the following:

**Theorem 9 (Morales-Ruiz and Ramis)** *Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighborhood of a phase curve* $\Gamma$*. Then the identity component of the differential Galois group of the normal variational equation associated with* $\Gamma$ *is Abelian.*

The $(x, y, p, q))$-part of the geodesic equation can be transformed to

$$
\begin{aligned}
\dot{z}_1 &= z_3, \\
\dot{z}_2 &= z_4, \\
\dot{z}_3 &= r\gamma z_1 z_2[(z_4 - z_3) - b(z_3 + z_4)], \\
\dot{z}_4 &= r\gamma z_1 z_2[(z_4 - z_3) + b(z_3 + z_4)].
\end{aligned}
$$

It is obvious that $z(t) = (0, ct, 0, c)$ with $c \neq 0$ is a solution of the above equations.

The normal variational equation can be represented as

$$\ddot{\xi}_1 = (1 - b)\gamma r c^2 t \xi_1.$$

where $(1-b)\gamma r c^2 \neq 0$, which gives the Airy equation. It is known that the differential Galois group of this equation is Sl(2,$\mathbb{C}$) and thus non Abelian.

## Conclusions

- We discussed (non)integrability of the geodesic equation (adjoint equation) for various Sub-Riemannian problems
- We show usefulness of the Morales-Ramis theory in proving non-integrability
- open problems: homogenous 4-dimensional SR-problems, general contact and quasi-contact SR-problems,...