Invariant observer and parameter estimation of quantum systems[1]

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Outline

Nonlinear asymptotic observers and symmetries

An invariant asymptotic observer for a 2-level system

Semi-local convergence proof

Possible extensions

Quantum systems with weak measures

Observer-based parameter estimation

Take $\frac{d}{dt}x = f(x, u(t), p)$ with output $y(t) = h(x)$ and control $u(t)$. The goal is to estimate p (and x) from noisy measures of y.
State-parameter asymptotic observer. Can we find g_1 and g_2 such that the solution $(\hat{x}(t), \hat{p}(t))$ of

$$
\begin{aligned}
\frac{d}{dt}\hat{x}(t) &= f(\hat{x}, u(t), \hat{p}) + g_1(\hat{x}, u(t), \hat{p}, y(t)) \\
\frac{d}{dt}\hat{p}(t) &= g_2(\hat{x}, u(t), \hat{p}, y(t))
\end{aligned}
$$

with an arbitrary initial state $(\hat{x}_0, \hat{p}_0)$ converges towards $(x(t), p)$ as $t \to \infty$?
Intrinsic or symmetry-preserving asymptotic observers (Thesis of Nasradine Aghanann (2003) and of Silvère Bonnabel (2007)).

Intrinsic observers for mechanical systems (IEEE-AC, 2003)

The locally convergent observer for

$$\frac{d}{dt}q = v, \quad \nabla_{\frac{d}{dt}q} v = S(q,t), \qquad y = q$$

is given by ($\alpha, \beta > 0$ are arbitrary scalar gains)

$$\frac{d}{dt}\hat{q} = \hat{v} - \alpha \, \mathrm{grad}_{\hat{q}} F(\hat{q}, q)$$
$$\nabla_{\frac{d}{dt}\hat{q}} \hat{v} = \mathscr{T}_{//q \to \hat{q}} S(q,t) - \beta \, \mathrm{grad}_{\hat{q}} F(\hat{q}, q) + R(\hat{v}, \mathrm{grad}_{\hat{q}} F(\hat{q}, q)) \hat{v}$$

where

- F is the square of the geodesic distance between q and $\hat{q}$
- R is the curvature tensor
- $\mathscr{T}_{//q \to \hat{q}}$ is the parallel transport along the geodesic from q to $\hat{q}$.

Intrinsic: independent of coordinates on the configuration manifold.

Invariant systems with equivariant output

- Consider

$$\frac{d}{dt}x = f(x,u)$$
$$y = h(x,u)$$

  with $x \in \mathscr{X} \subset \mathbb{R}^n$, $u \in \mathscr{U} \subset \mathbb{R}^m$ and $y \in \mathscr{Y} \subset \mathbb{R}^p$, $p \leq n$.
  Here $u(t)$ stands for known inputs (constant parameters, measured perturbations, controlled input,...)
- Take G a Lie group acting separately on $\mathscr{X}$, on $\mathscr{U}$, and on $\mathscr{Y}$: for each $g \in G$
  - φ_g diffeomorphism on $\mathscr{X}$
  - ψ_g diffeomorphism on $\mathscr{U}$
  - $\rho_g$diffeomorphism on $\mathscr{Y}$
- with $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 \cdot g_2}$, $\psi_{g_1} \circ \psi_{g_2} = \psi_{g_1 \cdot g_2}$, $\rho_{g_1} \circ \rho_{g_2} = \rho_{g_1 \cdot g_2}$

Invariant systems with equivariant output (end)

$$\frac{d}{dt}x = f(x,u), \qquad y = h(x,u)$$

- *G*-invariant system: for all $g \in G$,

$$\frac{d}{dt}X = f(X,U)$$

  where $(X,U) = (\varphi_g(x), \psi_g(u))$.
- *G*-equivariant output: for all $g \in G$,

$$Y = h(X,U)$$

  where $(X,U,Y) = (\varphi_g(x), \psi_g(u), \rho_g(y))$.

Invariant observer

The asymptotic observer

$$\frac{d}{dt}\hat{x} = F(\hat{x}, u, y)$$

is called invariant iff, for all $g, \hat{x}, u, y$

- $F\big(x, u, h(x, u)\big) = f(x, u)$.
- The transformation

$$(\hat{X}, U, Y) = (\varphi_g(\hat{x}), \psi_g(u), \rho_g(y))$$

  leaves the observer equations unchanged :

$$\frac{d}{dt}\hat{X} = F(\hat{X}, U, Y)$$

Symmetry preserving pre-observers

$$\frac{d}{dt}\hat{x} = F(\hat{x}, u, y)$$

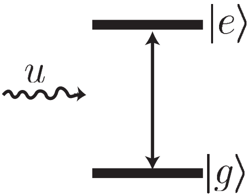
is a G-invariant pre-observer for the G-invariant system $\frac{d}{dt}x = f(x, u)$ with G-equivariant output $y = h(x, u)$ if and only if

$$F(\hat{x}, u, y) = f(\hat{x}, u) + \sum_{i=1}^{n} \mathscr{L}_i\big(I(\hat{x}, u), E(\hat{x}, u, y)\big) w_i(\hat{x})$$

where

- $(w_1, ..., w_n)$ is an invariant frame.
- $E(\hat{x}, u, y)$ is composed of invariant output errors (scalars): $E(x, u, h(x, u)) \equiv 0$.
- $I(\hat{x}, u)$ is composed of invariant scalar functions.
- for each i, $\mathscr{L}_i\big(I(\hat{x}, u), 0\big) \equiv 0$

The estimation problem for a two-level system



$$\frac{d}{dt}\rho = -\imath \left[\frac{\Delta}{2}\sigma_z + \frac{u\mu}{2}\sigma_x, \rho \right]$$
$$y = \mathrm{Tr}\left(\sigma_z \rho\right),$$

where:

- ρ is the density matrix: a 2×2 symmetric ≥ 0 matrix with $\mathrm{Tr}\left(\rho\right) = 1$ and $\mathrm{Tr}\left(\rho^2\right) = 1$ (here a projector).
- the Pauli matrices satisfy $\sigma_x^2 = 1$, $\sigma_x \sigma_y = \imath \sigma_z$, ... with

  $$\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|, \ \sigma_y = -\imath |e\rangle\langle g| + \imath |g\rangle\langle e|, \ \sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$$

- the two real parameters are Δ (the difference between the atomic frequency (transition $|g\rangle \leftrightarrow |e\rangle$) and the laser frequency of amplitude u) and $\mu > 0$ the laser/atom coupling strength.

Invariance versus $SU(2)$ action

For any $U \in SU(2)$, the transformation ((u, y, Δ, μ) unchanged)

$$\rho \mapsto \varpi = U\rho U^\dagger, \qquad \sigma_x \mapsto \varsigma_x = U\sigma_x U^\dagger, \ldots$$

leaves

$$\frac{d}{dt}\rho = -\imath \left[\frac{\Delta}{2}\sigma_z + \frac{u(t)\mu}{2}\sigma_x, \rho\right], \quad y = \mathrm{Tr}\left(\sigma_z\rho\right)$$

unchanged:

$$\frac{d}{dt}\varpi = -\imath \left[\frac{\Delta}{2}\varsigma_z + \frac{u(t)\mu}{2}\varsigma_x, \varpi\right], \quad y = \mathrm{Tr}\left(\varsigma_z\varpi\right),$$

and ς_x,ς_y,ς_z are new Pauli matrices.

The non-linear asymptotic observer

$$\frac{d}{dt}\hat{\rho} = -\imath \left[\frac{\hat{\Delta}}{2}\sigma_z + \frac{u\hat{\mu}}{2}\sigma_x, \hat{\rho} \right] - K_\rho(\hat{y} - y) \overbrace{\left(\sigma_z\hat{\rho} + \hat{\rho}\sigma_z - 2\text{Tr}\left(\sigma_z\hat{\rho}\right)\hat{\rho}\right)}^{\text{Tr}(\sigma_x\hat{\rho})\ \imath[\sigma_y,\hat{\rho}]-\text{Tr}\left(\sigma_y\hat{\rho}\right)\ \imath[\sigma_x,\hat{\rho}]}$$

$$\frac{d}{dt}\hat{\mu} = -uK_\mu \text{Tr}\left(\sigma_y\hat{\rho}\right)\ (\hat{y} - y)$$

$$\frac{d}{dt}\hat{\Delta} = -uK_\Delta \text{Tr}(\sigma_x\hat{\rho})\ (\hat{y} - y)$$

with positive gains K_ρ, K_μ and K_Δ. Preservation of $\text{Tr}(\hat{\rho}) = 1$ and $\text{Tr}\left(\hat{\rho}^2\right) = 1$.

Convergence results from averaging arguments (RWA) under the following assumptions and gains design:

- slowly varying u versus Rabi pulsation $|u\mu|$: $\left|\frac{d}{dt}u\right| \ll u^2\mu$.
- Small detuning $|\Delta|, |\hat{\Delta}| \ll |u|\mu$ and $|\hat{\mu}_{t=0} - \mu| \ll \mu$.
- Small gains: $K_\rho \ll |u|\mu$, $\sqrt{K_\mu} \ll \mu$, $K_\Delta \ll K_\mu\mu$.

$SU(2)$ invariance of the non-linear observer

For any $U \in SU(2)$, the transformation $((\hat{\Delta}, \hat{\mu})$ unchanged)

$$\hat{\rho} \mapsto \hat{\varpi} = \mapsto U \hat{\rho} U^{\dagger}, \sigma_x \mapsto \varsigma_x = U \sigma_x U^{\dagger}, \ldots$$

leaves the asymptotic observer

$$\frac{d}{dt}\hat{\rho} = -\imath \left[\frac{\hat{\Delta}}{2}\sigma_z + \frac{u\hat{\mu}}{2}\sigma_x, \hat{\rho} \right]$$

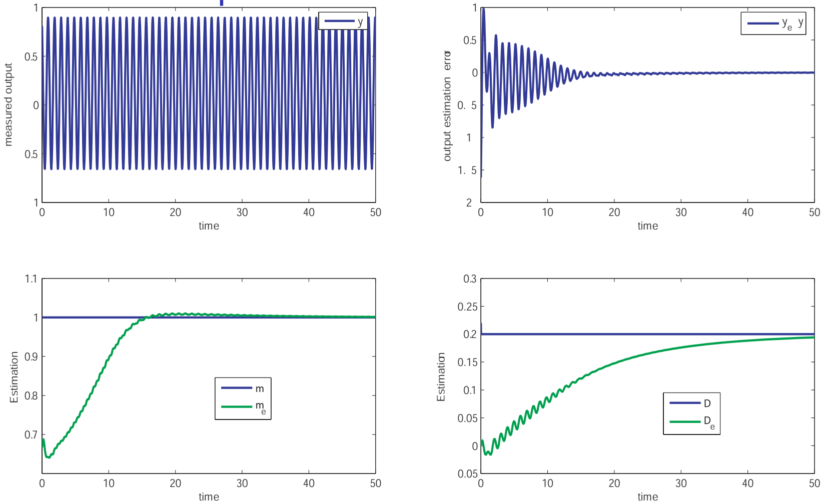
$$- K_\rho (\mathrm{Tr}(\sigma_z \hat{\rho}) - y) \; (\sigma_z \hat{\rho} + \hat{\rho} \sigma_z - 2\mathrm{Tr}(\sigma_x \hat{\rho}) \hat{\rho})$$

$$\frac{d}{dt}\hat{\mu} = -u K_\mu \mathrm{Tr}\left(\sigma_y \hat{\rho}\right) \; (\mathrm{Tr}(\sigma_z \hat{\rho}) - y)$$

$$\frac{d}{dt}\hat{\Delta} = -u K_\Delta \mathrm{Tr}(\sigma_x \hat{\rho}) \; (\mathrm{Tr}(\sigma_z \hat{\rho}) - y)$$

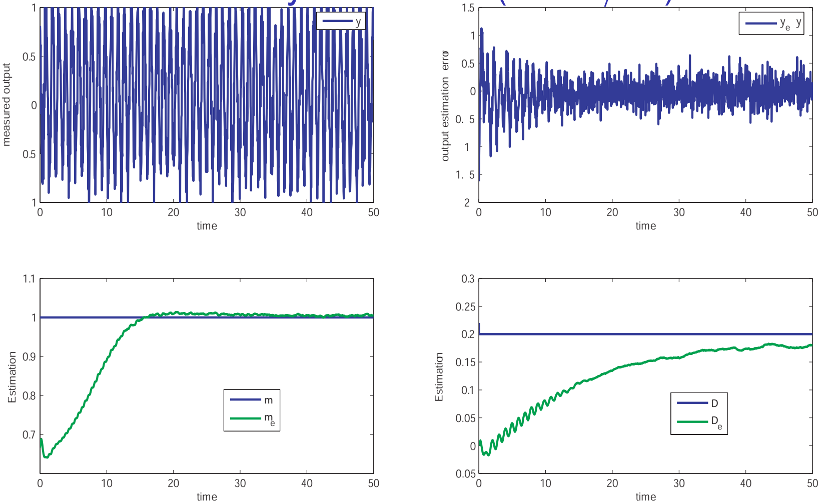
unchanged.

Simulation with perfect measures



Initial conditions: $\rho_0 = \frac{1+\cos(\frac{\pi}{5})\sigma_x+\sin(\frac{\pi}{5})\cos(\frac{\pi}{1.4})\sigma_y+\sin(\frac{\pi}{5})\sin(\frac{\pi}{1.4})\sigma_z}{2}$,
$\mu = 1, \Delta = \frac{1}{5}$, $\hat{\rho}_0 = \sigma_x \rho_0 \sigma_x$
Control/gains: $u = 1$, $K_\rho = 2\varepsilon|u|\mu$, $K_\mu = 2\varepsilon^2\mu^2$ and
$K_\Delta = 2\varepsilon^2|u|\mu^2$ with $\varepsilon = \frac{1}{5}$.

Simulation with noisy measures ($\sigma = 2/10$)



Initial conditions: $\rho_0 = \frac{1+\cos(\frac{\pi}{5})\sigma_x+\sin(\frac{\pi}{5})\cos(\frac{\pi}{1.4})\sigma_y+\sin(\frac{\pi}{5})\sin(\frac{\pi}{1.4})\sigma_z}{2}$,
$\mu = 1, \Delta = \frac{1}{5}$, $\hat{\rho}_0 = \sigma_x \rho_0 \sigma_x$
Control/gains: $u = 1$, $K_\rho = 2\varepsilon|u|\mu$, $K_\mu = 2\varepsilon^2\mu^2$ and $K_\Delta = 2\varepsilon^2|u|\mu^2$ with $\varepsilon = \frac{1}{5}$.

Assumptions

In

$$\frac{d}{dt}\hat{\rho} = -\imath \left[\frac{\hat{\Delta}}{2}\sigma_z + \frac{u\hat{\mu}}{2}\sigma_x, \hat{\rho} \right] \\ - K_\rho (\mathrm{Tr}(\sigma_z\hat{\rho}) - y) \left(\sigma_z\hat{\rho} + \hat{\rho}\sigma_z - 2\mathrm{Tr}(\sigma_z\hat{\rho})\hat{\rho}\right)$$

$$\frac{d}{dt}\hat{\mu} = -uK_\mu \mathrm{Tr}\left(\sigma_y\hat{\rho}\right) \left(\mathrm{Tr}(\sigma_z\hat{\rho}) - y\right)$$

$$\frac{d}{dt}\hat{\Delta} = -uK_\Delta \mathrm{Tr}(\sigma_x\hat{\rho}) \left(\mathrm{Tr}(\sigma_z\hat{\rho}) - y\right)$$

we assume that u is constant and that

$$\hat{\Delta} = \varepsilon\hat{d}, \quad K_\rho = 4k_\rho\varepsilon|u|\mu, \quad K_\mu = 2k_\mu\varepsilon^2\mu^2, \quad K_\Delta = 2k_\Delta\varepsilon^2|u|\mu^2$$

for $\varepsilon > 0$ small $\varepsilon \ll 1$, $k_\rho, k_\mu, k_\Delta \sim 1$.
Convergence based on perturbation techniques (Rotating Wave Approximation (RWA)) but up to order 2 in ε.

In the interaction frame

Consider the following time-varying transformation

$$\rho = e^{-\imath\frac{u\mu t\sigma_x}{2}}\xi e^{\imath\frac{u\mu t\sigma_x}{2}}, \qquad \hat{\rho} = e^{-\imath\frac{u\mu t\sigma_x}{2}}\hat{\xi} e^{\imath\frac{u\mu t\sigma_x}{2}}.$$

The dynamics reads:

$$\frac{d}{dt}\xi = -\imath\left[\frac{\Delta}{2}e^{\imath u\mu t\sigma_x}\sigma_z, \xi\right]$$

$$\frac{d}{dt}\hat{\xi} = -\imath\left[\frac{\hat{\Delta}}{2}e^{\imath u\mu t\sigma_x}\sigma_z + \frac{u(\hat{\mu}-\mu)}{2}\sigma_x, \hat{\xi}\right] - K_\rho \mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z(\hat{\xi}-\xi)\right)$$

$$\ldots\left(e^{\imath u\mu t\sigma_x}\sigma_z\hat{\xi} + \hat{\xi}e^{\imath u\mu t\sigma_x}\sigma_z - 2\mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z\hat{\xi}\right)\hat{\xi}\right)$$

$$\frac{d}{dt}\hat{\mu} = -uK_\mu \mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_y\hat{\xi}\right)\mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z(\hat{\xi}-\xi)\right)$$

$$\frac{d}{dt}\hat{\Delta} = -uK_\Delta \mathrm{Tr}\left(\sigma_x\hat{\xi}\right)\mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z(\hat{\xi}-\xi)\right).$$

First order secular approximation

By assumption the frequency $u\mu$ is large and thus we neglect terms rotating at $u\mu$ and also $2u\mu$ (first order in ε):

$$
\begin{aligned}
\frac{d}{dt}\xi &= 0 \\
\frac{d}{dt}\hat{\xi} &= -\imath \left[\frac{u(\hat{\mu}-\mu)}{2}\sigma_x, \hat{\xi} \right] \\
&\qquad - \frac{K_\rho}{2}\mathrm{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)\left(\sigma_y\hat{\xi}+\hat{\xi}\sigma_y - 2\mathrm{Tr}\left(\sigma_y\hat{\xi}\right)\hat{\xi}\right) \\
&\qquad - \frac{K_\rho}{2}\mathrm{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right)\left(\sigma_z\hat{\xi}+\hat{\xi}\sigma_z - 2\mathrm{Tr}\left(\sigma_z\hat{\xi}\right)\hat{\xi}\right) \\
\frac{d}{dt}\hat{\mu} &= -\frac{uK_\mu}{2}\left(\mathrm{Tr}\left(\sigma_y\hat{\xi}\right)\mathrm{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right) - \mathrm{Tr}\left(\sigma_z\hat{\xi}\right)\mathrm{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)\right) \\
\frac{d}{dt}\hat{\Delta} &= 0.
\end{aligned}
$$

Convergence of $\hat{\xi}$ and $\hat{\mu}$

Up to second order terms, $\hat{\xi}$ and $\hat{\mu}$ obey an autonomous differential system where ξ is a parameter:

$$\frac{d}{dt}\hat{\xi} = -\imath \left[\frac{u(\hat{\mu}-\mu)}{2}\sigma_x, \hat{\xi}\right] - \frac{K_\rho}{2}\mathrm{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)\left(\sigma_y\hat{\xi}+\hat{\xi}\sigma_y - 2\mathrm{Tr}\left(\sigma_y\hat{\xi}\right)\hat{\xi}\right) - \frac{K_\rho}{2}\mathrm{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right)\left(\sigma_z\hat{\xi}+\hat{\xi}\sigma_z - 2\mathrm{Tr}\left(\sigma_z\hat{\xi}\right)\hat{\xi}\right)$$

$$\frac{d}{dt}\hat{\mu} = -\frac{uK_\mu}{2}\left(\mathrm{Tr}\left(\sigma_y\hat{\xi}\right)\mathrm{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right) - \mathrm{Tr}\left(\sigma_z\hat{\xi}\right)\mathrm{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)\right)$$

Local exponential convergence for any ξ (excepted some isolated values) and for any $K_\rho, K_\mu > 0$ via the Lyapounov function:

$$\frac{1}{2}\mathrm{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)^2 + \frac{1}{2}\mathrm{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right)^2 + \frac{1}{K_\mu}(\hat{\mu}-\mu)^2.$$

Second order secular approximation

We use Kapitsa short-cut method to compute these second order terms particularly important for ξ and $\hat{\Delta}$ since the first order secular terms vanish.

We can decompose $\xi = \bar{\xi} + \delta\xi$: $\bar{\xi}$ is the no-oscillatory part, whereas $\delta\xi$ is the oscillatory one with zero mean and small amplitude $\|\delta\xi\| \ll \|\bar{\xi}\|$. Since $\frac{d}{dt}\xi = -\imath \left[\frac{\Delta}{2} e^{\imath u \mu t \sigma_x} \sigma_z, \xi\right]$ we have approximatively:

$$\delta\xi = \frac{\imath\Delta}{2u\mu}\left[\frac{\Delta}{2} e^{\imath u \mu t \sigma_x} \sigma_y, \bar{\xi}\right] + \dots$$

Plugging this relation into the true dynamics of ξ and taking the secular terms yields up to the third order:

$$\frac{d}{dt}\xi = -\imath \frac{\Delta^2}{2u\mu}[\sigma_x, \xi] + \dots$$

the term $\frac{\Delta^2}{2u\mu}$ corresponds exactly to Bloch-Siegert frequency shift.

Second order secular approximation (continued)

Since $\frac{d}{dt}\hat{\Delta} = -uK_\Delta \mathrm{Tr}\left(\sigma_x \hat{\xi}\right)\ \mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z(\hat{\xi}-\xi)\right)$ the secular effect can only comes from the part of $\delta\xi$ and $\delta\hat{\xi}$ with frequency $u\mu$: terms of double frequency $2u\mu$ have no secular effect. In the $\hat{\xi}$ dynamics

$$\frac{d}{dt}\hat{\xi} = -\imath\ \left[\frac{\hat{\Delta}}{2}e^{\imath u\mu t\sigma_x}\sigma_z + \frac{u(\hat{\mu}-\mu)}{2}\sigma_x, \hat{\xi}\right] - K_\rho \mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z(\hat{\xi}-\xi)\right)$$
$$\dots\left(e^{\imath u\mu t\sigma_x}\sigma_z\hat{\xi} + \hat{\xi}e^{\imath u\mu t\sigma_x}\sigma_z - 2\mathrm{Tr}\left(e^{\imath u\mu t\sigma_x}\sigma_z\hat{\xi}\right)\hat{\xi}\right)$$

the $u\mu$ frequency oscillatory term, denoted by $\delta_1\hat{\xi}$, comes only from $-\imath\ \left[\frac{\hat{\Delta}}{2}e^{\imath u\mu t\sigma_x}\sigma_z, \hat{\xi}\right]$. Thus

$$\delta_1\hat{\xi} = \frac{\imath\hat{\Delta}}{2u\mu}\left[e^{\imath u\mu t\sigma_x}\sigma_y, \hat{\xi}\right] \quad \text{and} \quad \delta\xi = \delta_1\xi = \frac{\imath\Delta}{2u\mu}\left[e^{\imath u\mu t\sigma_x}\sigma_y, \xi\right]$$

Second order secular approximation (end)

We have the following triangular and locally convergent dynamics:

$$\frac{d}{dt}\hat{\xi} \overset{\text{order 1}}{=} -\imath\left[\frac{u(\hat{\mu}-\mu)}{2}\sigma_x, \hat{\xi}\right]$$
$$-\frac{K_\rho}{2}\text{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)\left(\sigma_y\hat{\xi}+\hat{\xi}\sigma_y-2\text{Tr}\left(\sigma_y\hat{\xi}\right)\hat{\xi}\right)$$
$$-\frac{K_\rho}{2}\text{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right)\left(\sigma_z\hat{\xi}+\hat{\xi}\sigma_z-2\text{Tr}\left(\sigma_z\hat{\xi}\right)\hat{\xi}\right)$$

$$\frac{d}{dt}\hat{\mu} \overset{\text{order 1}}{=} -\frac{uK_\mu}{2}\left(\text{Tr}\left(\sigma_y\hat{\xi}\right)\text{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right)-\text{Tr}\left(\sigma_z\hat{\xi}\right)\text{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)\right)$$

$$\frac{d}{dt}\xi \overset{\text{order 2}}{=} -\imath\frac{\Delta^2}{2u\mu}[\sigma_x, \xi]$$

$$\frac{d}{dt}\hat{\Delta} \overset{\text{order 2}}{=} -\frac{K_\Delta}{\mu}\left(\text{Tr}\left(\sigma_x\hat{\xi}\right)^2\hat{\Delta}-\text{Tr}\left(\sigma_x\hat{\xi}\right)\text{Tr}\left(\sigma_x\xi\right)\Delta\right)$$
$$+\frac{K_\Delta\hat{\Delta}}{2\mu}\left(\text{Tr}\left(\sigma_y\hat{\xi}\right)\text{Tr}\left(\sigma_y(\hat{\xi}-\xi)\right)-\text{Tr}\left(\sigma_z\hat{\xi}\right)\text{Tr}\left(\sigma_z(\hat{\xi}-\xi)\right)\right).$$

Gain design via linear tangent approximation

With

$$\hat{\xi} - \xi = \frac{1 + \tilde{x}\sigma_x + \tilde{y}\sigma_y + \tilde{z}\sigma_z}{2}, \quad \tilde{\mu} = \hat{\mu} - \mu, \quad \tilde{\Delta} = \hat{\Delta} - \Delta$$

we have, around $\rho = \frac{1-\sigma_z}{2}$;

$$\frac{d}{dt}\tilde{y} = -u\tilde{\mu} - K_\rho \tilde{y}, \qquad \frac{d}{dt}\tilde{\mu} = uK_\mu \tilde{y}/2$$

and around $\rho = \frac{1-\sigma_x}{2}$

$$\frac{d}{dt}\tilde{\Delta} = -\frac{K_\Delta}{\mu}\tilde{\Delta}.$$

To respect the scaling, choose $0 < \varepsilon \ll 1$ and set

$$K_\rho = 2k_\rho \varepsilon |u| \mu, \quad K_\mu = 2\varepsilon^2 \mu^2, \quad K_\Delta = k_\Delta \varepsilon^2 |u| \mu^2$$

with k_ρ, k_Δ around 1.

Complex laser amplitude $u + \imath v$

The system is

$$\frac{d}{dt}\rho = -\imath \left[\frac{\Delta}{2}\sigma_z + \frac{\mu}{2}(u\sigma_x + v\sigma_y), \rho \right], \quad y = \mathrm{Tr}(\sigma_z \rho)$$

and the asymptotic observer reads:

$$\begin{aligned}
\frac{d}{dt}\hat{\rho} &= -\imath \left[\frac{\hat{\Delta}}{2}\sigma_z + \frac{\hat{\mu}}{2}(u\sigma_x + v\sigma_y), \hat{\rho} \right] \\
&\qquad - K_\rho(\mathrm{Tr}(\sigma_z \hat{\rho}) - y)\ (\sigma_z \hat{\rho} + \hat{\rho}\sigma_z - 2\mathrm{Tr}(\sigma_z \hat{\rho})\hat{\rho}) \\
\frac{d}{dt}\hat{\mu} &= -K_\mu \mathrm{Tr}\left((u\sigma_y - v\sigma_x)\hat{\rho}\right)\ (\mathrm{Tr}(\sigma_z \hat{\rho}) - y) \\
\frac{d}{dt}\hat{\Delta} &= -K_\Delta \mathrm{Tr}\left((u\sigma_x + v\sigma_y)\hat{\rho}\right)\ (\mathrm{Tr}(\sigma_z \hat{\rho}) - y)
\end{aligned}$$

N-level system

The system is ($\Delta^{kl} = 0$, no laser de-tuning here)

$$\frac{d}{dt}\rho = -\imath \left[\sum_{kl} \frac{u^{kl}\mu^{kl}}{2}\sigma_x^{kl}, \rho\right], \quad y_k = \mathrm{Tr}\left(P_k\rho\right)$$

with $P_k = |k\rangle\langle k|$ and its asymptotic observer reads:

$$\frac{d}{dt}\hat{\rho} = -\imath \left[\sum_{kl} \frac{u^{kl}\hat{\mu}^{kl}}{2}\sigma_x^{kl}, \hat{\rho}\right] - \sum_{k} K_\rho^k (\mathrm{Tr}\left(P_k\hat{\rho}\right) - y_k) \left(P_k\hat{\rho} + \hat{\rho}P_k - 2\mathrm{Tr}\left(P_k\hat{\rho}\right)\hat{\rho}\right)$$

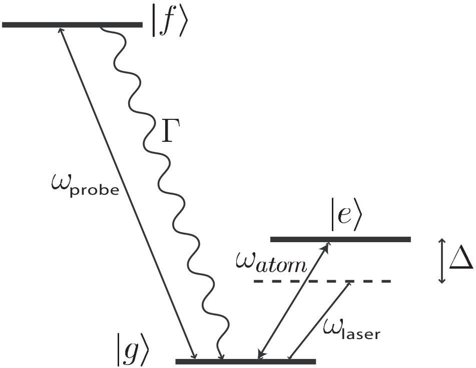
$$\frac{d}{dt}\hat{\mu}^{kl} = -K_\mu^{kl}\mathrm{Tr}\left(u\sigma_y^{kl}\hat{\rho}\right) \left(\mathrm{Tr}\left(\sigma_z^{kl}\hat{\rho}\right) - y_k + y_l\right)$$

where $\sigma_x^{kl} = |k\rangle\langle l| + |l\rangle\langle k|, \ldots$
Such extensions are possible since we start with an invariant observer for the 2-level system, i.e. we exploit the geometry.

Previous works and some references

- Parameter estimation for quantum systems: see, e.g., the works of H. Rabitz, H. Mabuchi and their collaborators.
- Identifiability for quantum systems: see, e.g., C. Lebris et al (COCV) where it is shown that resonant controls play a crucial role.
- Asymptotic observers and symmetries: few references (Aghannan, Bonnabel, Martin, R., Dayawansa and coworkers). See the preprint on Invariant asymptotic observers: `http://arxiv.org/abs/math.OC/0612193`
- A recent excellent book on (open) quantum systems : S. Haroche, J-M Raimond. Exploring the quantum: atoms, cavities and photons. Oxford University Press (Graduate texts), 2006.

An open 3-level quantum system



- ground state $|g\rangle$,
- excited state $|e\rangle$ of long life-time; $\Omega_{atom} = \frac{E_e - E_g}{\hbar}$.
- excited state $|f\rangle$ of short life-time $1/\Gamma$.
- probe laser: $\omega_{probe} = \frac{E_f - E_g}{\hbar}$.

Control laser frequency ω_{laser} and $\Delta = \omega_{atom} - \omega_{laser}$.
Measure: the fluorescence photons emitted by the unstable state $|f\rangle$.

The 3-level model (slow/fast)

Basic model of open quantum system (decoherence) (set of identical 3-level atoms) based on a master equation for the density matrix ρ (3×3, $\rho^\dagger = \rho$, $\rho \geq 0$, $\mathrm{Tr}(\rho) = 1$, $\mathrm{Tr}\left(\rho^2\right) \leq 1$):

$$\frac{d}{dt}\rho = \overbrace{-\frac{\imath}{\hbar}[H,\rho]}^{\text{Schrödinger}} + \overbrace{\frac{\Gamma}{2}\left(2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L\right)}^{\text{decoherence: Lindblad}}$$

with

$$\frac{1}{\hbar}H = \frac{\Delta}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{u\mu}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$$

and $L = |g\rangle\langle f|$ where $\Gamma \gg \Delta, u\mu$.
The flux of fluorescence photons (measure) is given by

$$y = \Gamma \mathrm{Tr}\left(L^\dagger L\rho\right).$$

The reduced 2-level model (slow) (CDC06 Mirrahimi-R)

Non commutative computations of the slow approximated model from

$$\frac{d}{dt}\rho = -\frac{\imath}{\hbar}[H,\rho] + \frac{\Gamma}{2}\left(2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L\right).$$

With $P = L^\dagger L = |e\rangle\langle e|$ (projector on $|e\rangle$) set

$$\rho_f = P\rho + \rho P - P\rho P, \quad \rho_s = (1-P)\rho(1-P) + L\rho L^\dagger.$$

then $\rho = \rho_s + \rho_f - LP\rho_f PL^\dagger$ and the slow dynamics reads (center manifold technique)

$$\frac{d}{dt}\rho_s = -\frac{\imath}{\hbar}[H_s,\rho_s] + \frac{2}{\Gamma}\left(2L_s\rho_s L_s^\dagger - L_s^\dagger L_s\rho_s - \rho_s L_s^\dagger L_s\right)$$

with output $y = \frac{4}{\Gamma}\mathrm{Tr}\left(L_s^\dagger L_s \rho_s\right)$ where $H_s = (1-P)H(1-P)$ and $L_s = (1-P)L\left(\frac{H}{\hbar}\right)(1-P)$.