Symmetries and cut points in optimal control problems on Lie groups

Yuri L. Sachkov

Russian Academy of Sciences
sachkov@sys.botik.ru

Control, Constraints and Quanta

International Research Conference
Bedlewo, Poland, 10–16 October 2007

Summary of the talk

1. Cut points and symmetries
2. Euler's elastic problem
3. Nilpotent $(2, 3, 5)$ sub-Riemannian problem
4. Other related problems

Optimal control problem

$$\dot{q} = f(q,u), \qquad q \in M, \quad u \in U,$$
$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed},$$
$$J_{t_1}[q,u] = \int_0^{t_1} g(q(t),u(t))\,dt \to \min.$$

- M, U finite-dimensional analytic manifolds,
- $f(q,u)$, $g(q,u)$ analytic vector field and function depending on u

Focus on the case: M Lie group, f and g left-invariant

Pontryagin Maximum Principle

- Cotangent bundle $T^*M \ni \lambda$, $\pi(\lambda) = q \in M$
- Hamiltonian $h_u^\nu(\lambda) = \langle \lambda, f(q,u) \rangle + \nu g(q,u)$, $\nu \in \mathbb{R}$
- Maximized normal Hamiltonian $H(\lambda) = \max\limits_{u \in U} h_u^{-1}(\lambda)$ analytic

Theorem (PMP)

u_t and q_t optimal $\Rightarrow$ $\exists\ \lambda_t \in T^*_{q_t}M$, $\nu \in \{0, -1\}$, $(\nu, \lambda_t) \neq 0$:

$$\dot{\lambda}_t = \vec{h}^\nu_{u_t}(\lambda_t),$$
$$h^\nu_{u_t}(\lambda_t) = \max_{u \in U} h_u^\nu(\lambda_t).$$

Normal case: $\nu = -1 \quad \Rightarrow \quad \dot{\lambda}_t = \vec{H}(\lambda_t)$

PMP for left-invariant problems

- M Lie group $\quad\Rightarrow\quad$ cotangent bundle is trivialized
- $T^*M \cong L^* \times M$, $L = T_{\mathrm{Id}} M$ Lie algebra of M
- trivialization $F : L^* \times M \to T^*M$, $F(x, q) = \bar{x}_q$:
  $\langle \bar{x}_q, qa \rangle = \langle x, a \rangle$, $x \in L^*$, $a \in L$
- f and g left-invariant $\quad\Rightarrow\quad$ $f = qf(u)$, $g = g(u)$
- maximized normal Hamiltonian $H = H(x)$, $x \in L^*$
- normal Hamiltonian system triangular:

$$\dot{\lambda} = \vec{H}(\lambda) \quad \Leftrightarrow \quad \begin{cases} \dot{x} = \left(\mathrm{ad}\,\frac{dH}{dx}\right)^* x, & x \in L^*, \\ \dot{q} = q\frac{dH}{dx}, & q \in M \end{cases}$$

Optimality of extremal trajectories

- $q_t = \pi(\lambda_t)$ normal extremal trajectory
- q_t regular:

$$\frac{\partial^2}{\partial u^2} h_u^{-1}(\lambda_t) < -\delta, \qquad \delta > 0$$

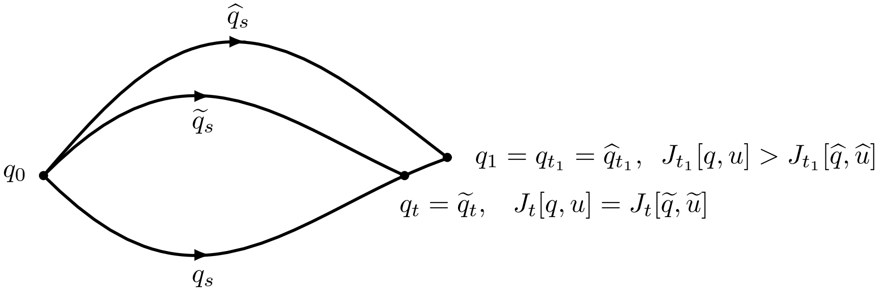
$\Rightarrow$ short arcs of q_t are optimal

Cut time along q_t:

$$t_{\text{cut}}(q_\cdot) = \sup\{t \mid q_s \text{ optimal for } s \in [0, t]\}$$

Maxwell points

Maxwell point q_t: $\exists\ \widetilde{q}_s \not\equiv q_s : \quad q_t = \widetilde{q}_t,\ J_t[q,u] = J_t[\widetilde{q},\widetilde{u}]$



$t = t_{\mathrm{Max}}(q_\cdot)$ — Maxwell time along $q_\cdot$

Maxwell time and cut time

Proposition

If a normal extremal trajectory q_s*,* $s \in [0, t_1]$*, admits a Maxwell time* $t_{\text{Max}}(q_\cdot) \in (0, t_1)$*, then* q_s *is not optimal at* $[0, t_1]$*:*

$$t_{\text{cut}}(q_\cdot) \leq t_{\text{Max}}(q_\cdot).$$

Time t Maxwell set in $N = T^*_{q_0} M$:

$$\text{MAX}_t = \left\{ \lambda \in N \mid \exists \, \widetilde{\lambda} \in N \ : \ \widetilde{q}_s \not\equiv q_s, \ \widetilde{q}_t = q_t, \ J_t[q, u] = J_t[\widetilde{q}, \widetilde{u}] \right\}.$$

- $\lambda \in \text{MAX}_t \quad \Rightarrow \quad t_{\text{cut}}(\lambda_\cdot) \leq t$

Symmetries of the exponential mapping

Exponential mapping for time t:

$$\mathrm{Exp}_t \;:\; N = T^*_{q_0}M \to M, \qquad \mathrm{Exp}_t(\lambda) = \pi \circ e^{t\vec{H}}(\lambda) = q_t.$$

Invertible mapping $\Phi \;:\; N \to N$ is a symmetry of Exp_t if $\exists \varphi \;:\; M \to M$ such that the diagram is commutative:

$$\begin{array}{ccc} N & \xrightarrow{\Phi} & N \\ \downarrow{\scriptstyle \mathrm{Exp}_t} & & \downarrow{\scriptstyle \mathrm{Exp}_t} \\ M & \xrightarrow{\varphi} & M \end{array}$$

and $J_t[q_\cdot] = J_t[\varphi(q_\cdot)]$.

Maxwell set corresponding to a group of symmetries

$G = \{\Phi : N \to N\}$ some group of symmetries of Exp_t

Maxwell set corresponding to G:

$$\mathrm{MAX}_t^G = \Big\{\lambda \in N \mid \exists\ \Phi \in G\ :\ \widetilde{\lambda} = \Phi(\lambda) \neq \lambda, \\ \widetilde{q}_s \not\equiv q_s,\ \widetilde{q}_t = q_t,\ J_t[\widetilde{q}] = J_t[q]\Big\}.$$

- $\mathrm{MAX}_t^G \subset \mathrm{MAX}_t$

Symmetries of $\vec{H}$ and of Exp_t

A diffeomorphism $\Phi : T^*M \to T^*M$ is a symmetry of $\vec{H} \in \mathrm{Vec}(T^*M)$ if the diagram is commutative for any $t \in \mathbb{R}$:

$$\begin{array}{ccc} T^*M & \xrightarrow{\Phi} & T^*M \\ \downarrow{\scriptstyle e^{t\vec{H}}} & & \downarrow{\scriptstyle e^{t\vec{H}}} \\ T^*M & \xrightarrow{\Phi} & T^*M \end{array}$$

Proposition

Let $\Phi : T^*M \to T^*M$ be a symmetry of $\vec{H}$ such that:

- $\exists \varphi : M \to M$ *such that* $\pi \circ \Phi = \varphi \circ \pi$,
- $\Phi(N) = N$,
- $J_t[\widetilde{q}.] = J_t[q.]$ *for all* $q_s = \mathrm{Exp}_s(\lambda)$, $\widetilde{q}_s = \varphi(q_s)$, $\lambda \in N$.

Then Φ is a symmetry of Exp_t.

Hamiltonian system of PMP for left-invariant problem:

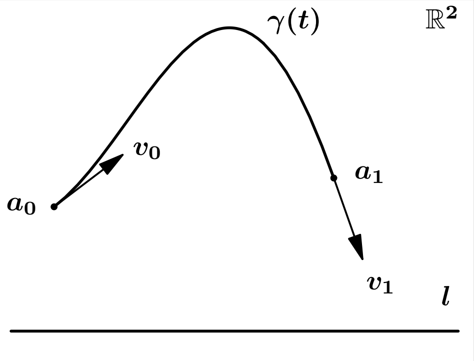
$$\dot{x} = \left(\mathrm{ad}\,\frac{dH}{dx}\right)^* x, \qquad x \in L^* = N, \tag{1}$$

$$\dot{q} = q\frac{dH}{dx}, \qquad q \in M. \tag{2}$$

Construction of symmetries of Exp_t:

- Take a diffeomorphism $\Phi : N \to N$.
- $\forall x_0 \in N$ and $\widetilde{x}_0 = \Phi(x_0) \in N$, find:
  - $x_s, \widetilde{x}_s \in N$ the corresponding solutions to (1),
  - $q_s, \widetilde{q}_s \in M$ the corresponding solutions to (2).
- Suppose that $J_t[q.] = J_t[\widetilde{q}.]$.
- Suppose that $\exists \varphi : M \to M$ such that $\widetilde{q}_t = \varphi(q_t)$.

Then Φ is a symmetry of Exp_t.



Given: $l > 0$, $a_0, a_1 \in \mathbb{R}^2$, $v_0 \in T_{a_0}\mathbb{R}^2$, $v_1 \in T_{a_1}\mathbb{R}^2$, $|v_0| = |v_1| = 1$.

Find: $\gamma(t)$, $t \in [0, t_1]$:

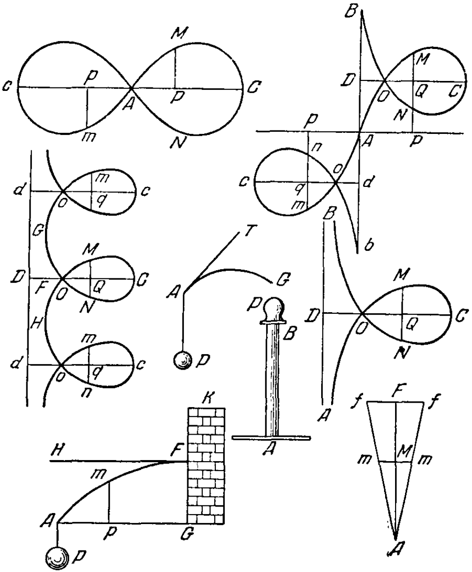
$\gamma(0) = a_0$, $\gamma(t_1) = a_1$, $\dot\gamma(0) = v_0$, $\dot\gamma(t_1) = v_1$. $|\dot\gamma(t)| \equiv 1 \quad \Rightarrow \quad t_1 = l$

Elastic energy $J = \dfrac{1}{2}\displaystyle\int_0^{t_1} k^2 \, dt \to \min$, $k(t)$ — curvature of $\gamma(t)$.

1744: Leonhard Euler

- "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimitrici latissimo sensu accepti", Lausanne, Geneva, 1744
- Problem of calculus of variations
- Euler-Lagrange equation
- Reduction to quadratures
- Qualitative analysis of the integrals
- Types of solutions (elasticae)

1744: Euler's sketches of elasticae



Optimal control problem

$$\dot{x} = \cos\theta,$$
$$\dot{y} = \sin\theta,$$
$$\dot{\theta} = u,$$
$$q = (x, y, \theta) \in \mathbb{R}^2_{x,y} \times S^1_\theta, \qquad u \in \mathbb{R},$$
$$q(0) = q_0 = (x_0, y_0, \theta_0), \ q(t_1) = q_1 = (x_1, y_1, \theta_1), \ \ t_1 \text{ fixed}.$$

$$k^2 = \dot{\theta}^2 = u^2 \quad \Rightarrow \quad J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min .$$

Admissible controls $u(t) \in L_2[0, t_1]$,
trajectories $q(t) \in AC[0, t_1]$

Left-invariant problem on the group of motions of a plane

$$\mathrm{E}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2, \ \theta \in S^1 \right\}$$

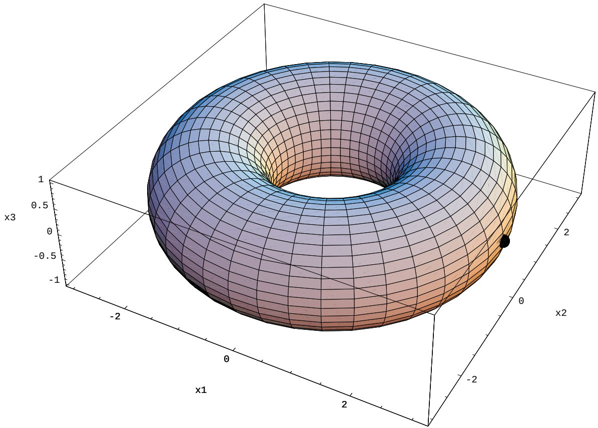
$$\dot{q} = X_1(q) + uX_2(q), \qquad q \in \mathrm{E}(2), \quad u \in \mathbb{R}.$$
$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad t_1 \text{ fixed},$$
$$J = \frac{1}{2}\int_0^{t_1} u^2 dt \to \min,$$

Left-invariant frame on $\mathrm{E}(2)$:

$$X_1(q) = qE_{13}, \quad X_2(q) = q(E_{21} - E_{12}), \quad X_3(q) = -qE_{23}$$

Attainable set and existence of optimal solutions

$\mathcal{A}_{\mathrm{Id}}(1) = \{(x, y, \theta) \mid x^2 + y^2 < 1 \ \ \forall\, \theta \in S^1$ or $(x, y, \theta) = (1, 0, 0)\}.$



$q_1 \in \mathcal{A}_{q_0}(t_1) \quad \Rightarrow \quad \exists$ optimal $u(t) \in L_\infty$

Pontryagin Maximum Principle

$$\dot{q} = X_1(q) + uX_2(q),\ q \in M = \mathbb{R}^2 \times S^1,\ u \in \mathbb{R}, \quad J = \frac{1}{2}\int_0^{t_1} u^2\, dt \to \min$$

- $T_q M = \mathrm{span}(X_1(q), X_2(q), X_3(q)), \quad X_3 = [X_1, X_2]$
- $T_q^* M = \{(h_1, h_2, h_3)\}, \quad h_i(\lambda) = \langle \lambda, X_i \rangle,\ \lambda \in T^*M$
- Hamiltonian vector fields $\vec{h}_i \in \mathrm{Vec}(T^*M)$
- $h_u^\nu = \langle \lambda, X_1 + uX_2 \rangle + \frac{\nu}{2}u^2 = h_1(\lambda) + uh_2(\lambda) + \frac{\nu}{2}u^2$

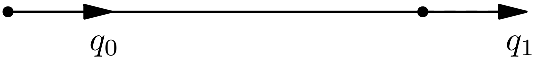
Theorem (PMP)

$u(t)$ and q_t optimal $\quad \Rightarrow \quad \exists\ \lambda_t \in T^*_{q_t}M,\ \nu \in \{0, -1\},\ (\nu, \lambda_t) \neq 0$:

$$\dot{\lambda}_t = \vec{h}^\nu_{u(t)}(\lambda_t) = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t),$$
$$h^\nu_{u(t)}(\lambda_t) = \max_{u \in \mathbb{R}} h^\nu_u(\lambda_t).$$

Abnormal extremal trajectories

$\nu = 0 \quad \Rightarrow \quad u(t) \equiv 0 \quad \Rightarrow \quad \theta \equiv 0, \qquad x = t, \qquad y \equiv 0$



$J = 0 = \min \quad \Rightarrow$

$\Rightarrow \quad$ abnormal extremal trajectories optimal for $t \in [0, t_1]$

Unique trajectory from $q_0 = (0, 0, 0)$ to $(t_1, 0, 0) \in \partial \mathcal{A}_{q_0}(t_1)$.

Normal Hamiltonian system

$\nu = -1 \quad \Rightarrow \quad$ nonuniqueness of extremal trajectories

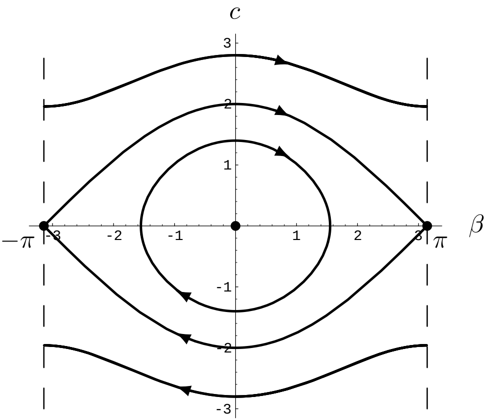
Hamiltonian system:

$$\dot{h}_1 = -h_2 h_3, \qquad \dot{x} = \cos\theta$$
$$\dot{h}_2 = h_3, \qquad \dot{y} = \sin\theta$$
$$\dot{h}_3 = h_1 h_2, \qquad \dot{\theta} = h_2$$

$$r^2 = h_1^2 + h_3^2 \equiv \text{const} \quad \Rightarrow \quad h_1 = -r\cos\beta, \;\; h_3 = -r\sin\beta$$

Equation of pendulum

$$\ddot{\beta} = -r \sin \beta \quad \Leftrightarrow \quad \begin{cases} \dot{\beta} = c, \\ \dot{c} = -r \sin \beta \end{cases}$$



Normal extremal trajectories

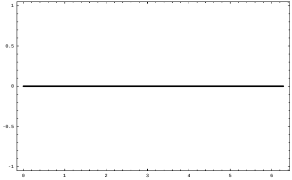
$$\ddot{\theta} = -r \sin(\theta - \gamma), \qquad r, \ \gamma = \text{const},$$
$$\dot{x} = \cos\theta,$$
$$\dot{y} = \sin\theta.$$

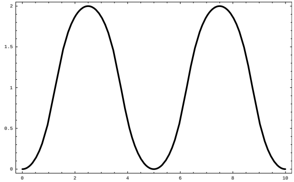
Integrable in Jacobi's functions.

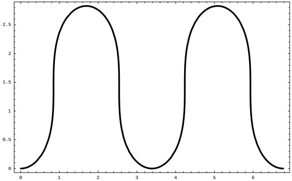
$\theta(t), x(t), y(t)$ parametrized by Jacobi's functions

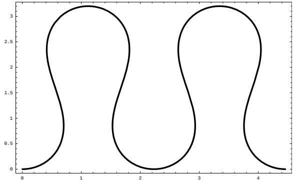
$$\operatorname{cn}(u,k), \quad \operatorname{sn}(u,k), \quad \operatorname{dn}(u,k), \quad \mathrm{E}(u,k).$$

Euler elasticae

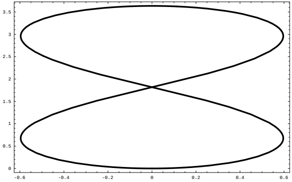


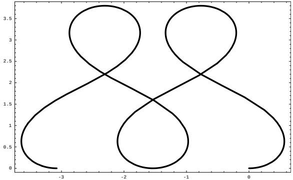


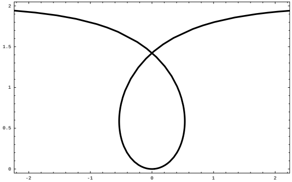


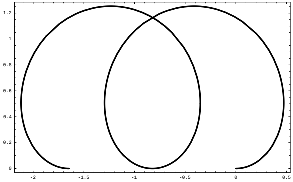


Euler elasticae

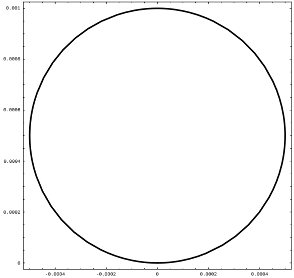




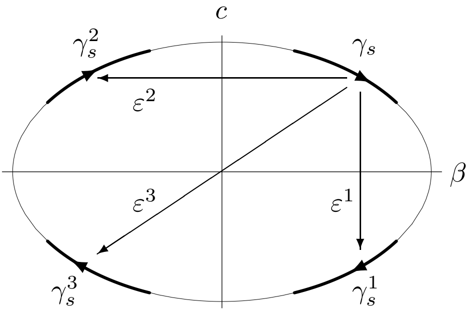




Euler elasticae



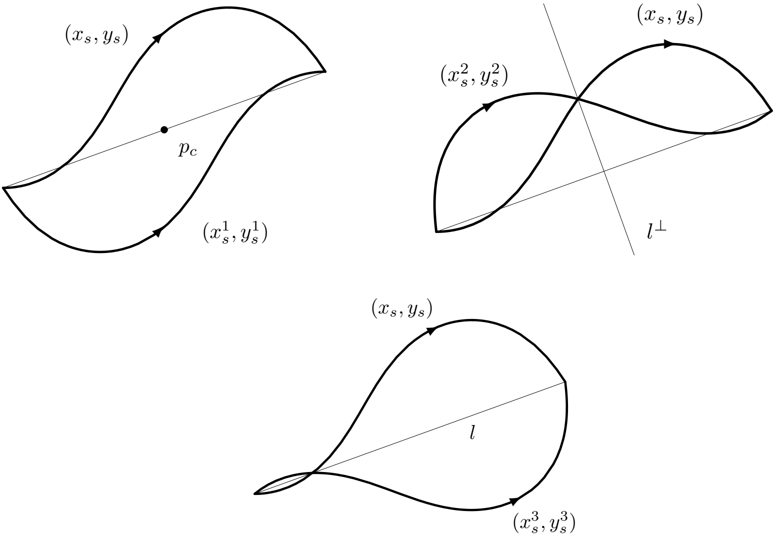
Reflections in the phase cylinder of pendulum $\ddot{\beta} = -r \sin \beta$



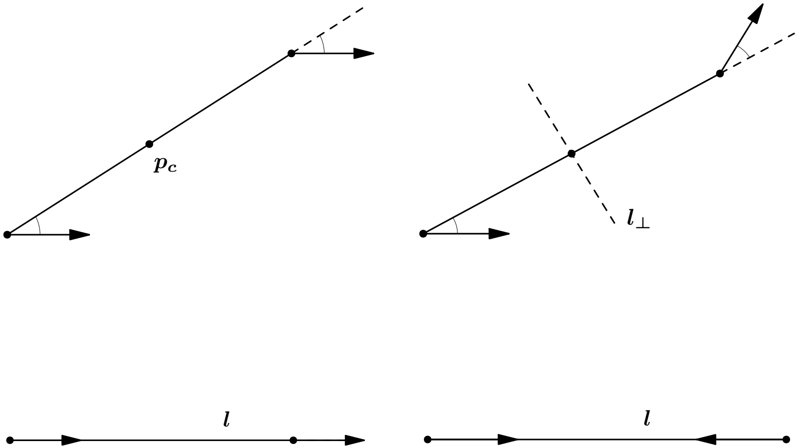
Dihedral group $D_2 = \{\text{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$

	ε^1	ε^2	ε^3
ε^1	Id	ε^3	ε^2
ε^2	ε^3	Id	ε^1
ε^3	ε^2	ε^1	Id

Action of reflections ε^1, ε^2, ε^3 on elasticae



Fixed points of reflections ε^1, ε^2, ε^3



Maxwell points corresponding to reflections

Fixed points of reflections ε^i $\quad \Rightarrow \quad$ Maxwell times:

$$t = t^n_{\varepsilon^i}, \qquad i = 1, 2, \quad n = 1, 2, \dots$$

T = period of pendulum $\quad \Rightarrow$

$$t^n_{\varepsilon^1} = nT, \qquad \left(n - \frac{1}{2}\right) T < t^n_{\varepsilon^2} < \left(n + \frac{1}{2}\right) T.$$

Upper bound of cut time:

$$t_{\text{cut}} \leq \min(t^1_{\varepsilon^1}, t^1_{\varepsilon^2}) \leq T.$$

Global optimality of elasticae

$$q_1 \in \mathcal{A}_{q_0}(t_1), \qquad \text{optimal } q_t = ?$$

$$q_t = \mathrm{Exp}_t(\lambda) \text{ optimal for } t \in [0, t_1] \quad \Rightarrow \quad t_1 \leq \min(t^1_{\varepsilon_1}(\lambda), t^1_{\varepsilon_2}(\lambda))$$

$$N' = \{\lambda \in T^*_{q_0}M \mid t_1 \leq \min(t^1_{\varepsilon_1}(\lambda), t^1_{\varepsilon_2}(\lambda))\}$$

$\mathrm{Exp}_{t_1} \; : \; N' \to \mathcal{A}_{q_0}(t_1)$ surjective, without singularities, with multiple points

$\exists$ open dense $\widetilde{N} \subset N'$, $\widetilde{M} \subset \mathcal{A}_{q_0}(t_1)$ such that

$$\mathrm{Exp}_{t_1} \; : \; \widetilde{N} \to \widetilde{M} \text{ double covering}$$

Global structure of exponential mapping

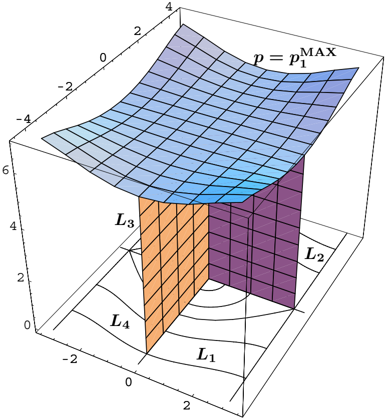


Figure: $\widetilde{N} = \cup_{i=1}^{4} L_i$

$\xrightarrow{\mathrm{Exp}_{t_1}}$

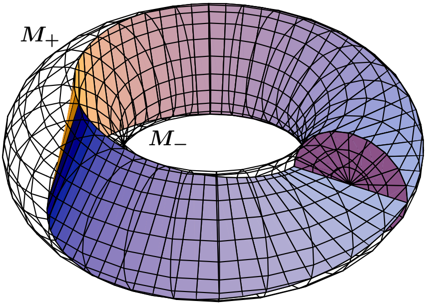
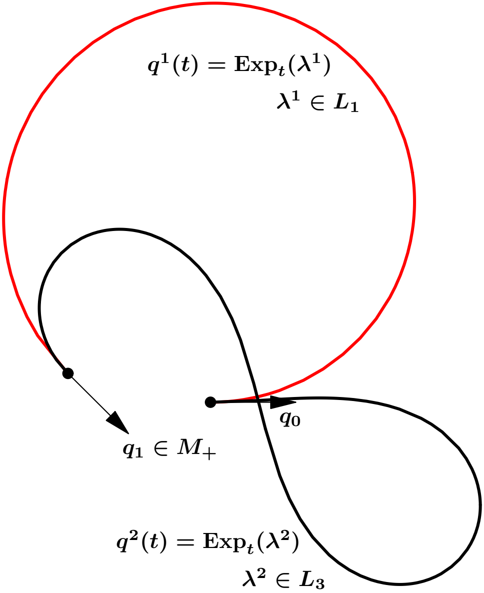


Figure: $\widetilde{M} = M_+ \cup M_-$

$\mathrm{Exp}_{t_1} \;:\; L_1,\; L_3 \to M_+$ diffeo, $\qquad \mathrm{Exp}_{t_1} \;:\; L_2,\; L_4 \to M_-$ diffeo

Competing elasticae



$$? \ : \ J[q^1_\cdot] \lesseqgtr J[q^2_\cdot]$$

Movies . . .

Statement of nilpotent $(2, 3, 5)$ sub-Riemannian problem

- Lie algebra $L = \mathrm{span}(X_1, \dots, X_5)$ with

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4, \qquad [X_2, X_3] = X_5,$$

- M — the corresponding connected simply connected Lie group,
- SR structure on M: $\Delta = \mathrm{span}(X_1, X_2)$, $\langle X_i, X_j \rangle = \delta_{ij}$, $i, \ j = 1, \ 2$,
- SR problem:

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \qquad q \in M, \quad u = (u_1, u_2) \in U = \mathbb{R}^2,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

Geometric model: Generalized Dido's problem

Given: $a_0, a_1 \in \mathbb{R}^2$ connected by a curve $\gamma_0 \subset \mathbb{R}^2$,
$S \in \mathbb{R}$,
$c \in \mathbb{R}^2$.

Find: the shortest curve $\gamma \subset \mathbb{R}^2$ connecting a_0 and a_1 such that
$area(D) = S$,
$center\ of\ mass(D) = c$,
$\partial D = \gamma_0 \cup \gamma_1$.

Motivation and history

Nilpotent approximation to a generic rank 2 sub-Riemannian problem on a 5-dimensional manifold at a generic point, in particular to:

- pair of rigid bodies in $\mathbb{R}^3$ rolling one on another without slipping or twisting,
- car with 2 off-hooked trailers.

R. Brockett and L. Dai (1993):

- the problem first considered,
- integrability in elliptic functions.

Optimal control problem

$$\dot{q} = u_1 X_1 + u_2 X_2, \qquad q = (x, y, z, v, w) \in M = \mathbb{R}^5, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0 = 0, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min,$$
$$X_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} - \frac{x^2 + y^2}{2}\frac{\partial}{\partial w},$$
$$X_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2 + y^2}{2}\frac{\partial}{\partial v}.$$

Normal extremals

Normal Hamiltonian system of PMP:

$$\dot{\lambda} = \vec{H}(\lambda) \quad \Leftrightarrow \quad \begin{cases} \dot{\theta} = c, \\ \dot{c} = -\alpha \sin(\theta - \beta), \\ \dot{\alpha} = 0, \\ \dot{\beta} = 0, \\ \dot{q} = (\cos\theta) X_1(q) + (\sin\theta) X_2(q), \end{cases}$$

$$\lambda = (\theta, c, \alpha, \beta; q) \in T^*M \cap \{H(\lambda) = 1/2\}.$$

(x_t, y_t) Euler elasticae

Symmetries of the exponential mapping:

- rotations $e^{r\vec{h}_0}$: : $\beta \mapsto \beta + r$,
- reflections of the pendulum ε^i, $i = 1, 2, 3$.

Maxwell time and cut time

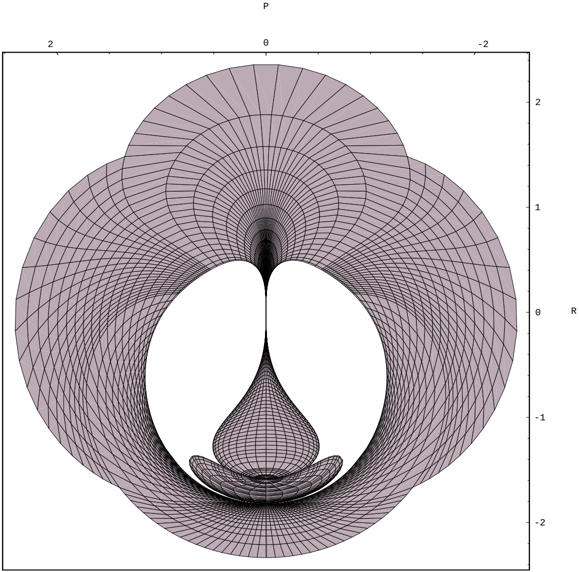
- $G = \langle e^{r\vec{h}_0}, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$ group of symmetries of Exp,
- MAX^G computed,
- for any extremal trajectory $q_\cdot$, the first Maxwell time is evaluated :

$$t^1_{\mathrm{Max}}(q_\cdot) \in (0, +\infty].$$

- Conjecture: $t_{\mathrm{cut}}(q_\cdot) = t^1_{\mathrm{Max}}(q_\cdot)$.

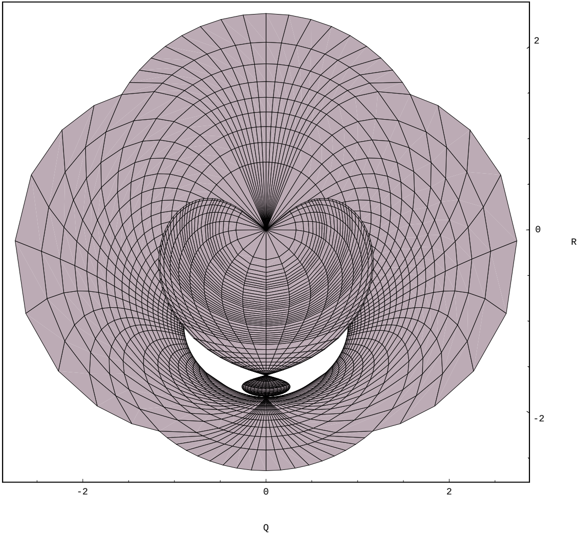
Maxwell strata corresponding to inflectional elasticae

First component in the plane $V = 0$



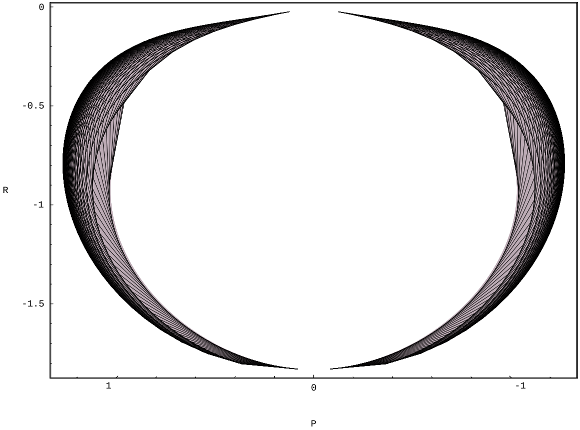
Maxwell strata corresponding to inflectional elasticae

First component in the plane $z = 0$

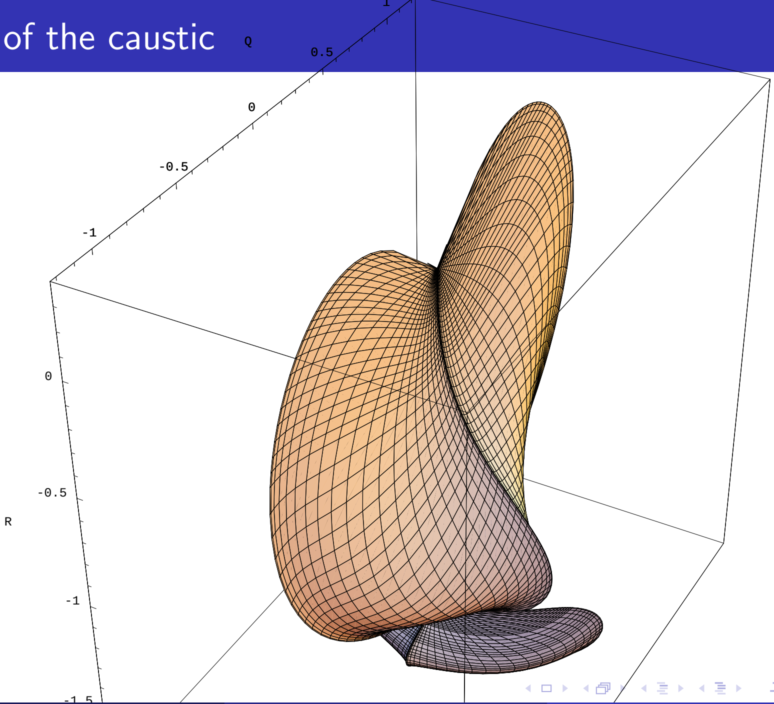


Maxwell strata corresponding to non-inflectional elasticae

First component in the plane $V = 0$



Part of the caustic



Other related problems

- The ball-plate problem:
  - studied in detail by V.Jurdjevic (1993),
  - vertical subsystem of the Hamiltonian system of PMP contains equations of the pendulum,
  - integrable in elliptic functions,
  - projections of extremal trajectories are Euler elasticae.
- Sub-Riemannian problem on $\mathrm{E}(2)$:
  - vertical subsystem of the Hamiltonian system of PMP contains equations of the pendulum,
  - integrable in elliptic functions,
  - $G = \langle \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle$ — symmetries of the exponential mapping.

Optimality of extremal trajectories: open question

Summary

- Symmetry analysis is a very powerful tool for the study of optimal control problems.
- Combined with other techniques of geometric control theory, it can provide solution to classical and new problems.

Publications
http://www.botik.ru/PSI/CPRC/sachkov/

[1] Yu. L. Sachkov, Exponential mapping in generalized Dido's problem, *Sbornik: Mathematics*, **194** (2003).
[2] Yu. L. Sachkov, Discrete symmetries in the generalized Dido problem, *Sbornik: Mathematics*, **197** (2006), 2: 235–257.
[3] Yu. L. Sachkov, The Maxwell set in the generalized Dido problem, *Sbornik: Mathematics*, **197** (2006), 4: 595–621.
[4] Yu. L. Sachkov, Complete description of the Maxwell strata in the generalized Dido problem, *Sbornik: Mathematics*, **197** (2006), 6: 901–950.
[5] Yu. L. Sachkov, Maxwell strata in Euler's elastic problem, *Journal of Dynamical and Control Systems* (accepted), available at `arXiv:0705.0614 [math.OC]`, 3 May 2007.
[6] Yu. L. Sachkov, Conjugate points in Euler's elastic problem, *Journal of Dynamical and Control Systems* (accepted), available at `arXiv:0705.1003 [math.OC]`, 7 May 2007.
[7] Yu. L. Sachkov, Optimality of Euler elasticae, *Doklady Mathematics*, **417** (2007), No. 1, pp. 23–25.