Controllability of Electron Nuclear Spin Dynamics

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Control, Constraints and Quanta

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Outline

A Reminder

Recall,

Definition

A Lie algebra is simple if it has no proper ideals.

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Defintion

A Lie group is simple if it is connected and its algebra is simple.

Systems of Interest

Left (or right) invariant systems on $\mathfrak{G} \times \mathfrak{G}$, where $\mathfrak{G}$ is a simple Lie group :

$$\dot{X}(t) = \sum_{i=1}^{n} u_i(t) \left(\begin{array}{c} \Omega_i^a \\ \Omega_i^b \end{array} \right)_{X(t)}$$

where, $\left(\begin{array}{c} \Omega_i^a \\ \Omega_i^b \end{array} \right) \in \mathfrak{g} \oplus \mathfrak{g}$, $\mathfrak{g} = \mathrm{Lie}(G)$.

Controllability

$$\dot{X}(t) = u_1(t) \begin{pmatrix} \Omega_1^a \\ \Omega_1^b \end{pmatrix}_{X(t)} + u_2(t) \begin{pmatrix} \Omega_2^a \\ \Omega_2^b \end{pmatrix}_{X(t)} \tag{1}$$

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Controllability Subalgebra

$$\mathfrak{A} = \left\langle \begin{pmatrix} \Omega_1^a \\ \Omega_1^b \end{pmatrix}, \begin{pmatrix} \Omega_2^a \\ \Omega_2^b \end{pmatrix} \right\rangle$$

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The system (1) is controllable if and only if $\mathfrak{A} = \mathfrak{g} \oplus \mathfrak{g}$.

Consider the subalgebras of $\mathfrak{g}$ given by $\langle \Omega_1^a, \Omega_2^a \rangle$ and $\langle \Omega_1^b, \Omega_2^b \rangle$.

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Componentwise Controllability

The system (1) is controllable only if $\langle \Omega_1^a, \Omega_2^a \rangle = \langle \Omega_1^b, \Omega_2^b \rangle = \mathfrak{g}$.

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Is this sufficient? No!

Counterexamples

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Then,

$$\begin{aligned} \mathfrak{A} &= \left\langle \begin{pmatrix} \Omega_1^a \\ \Omega_1^a \end{pmatrix}, \begin{pmatrix} \Omega_1^b \\ \Omega_1^b \end{pmatrix} \right\rangle \\ &= \Delta \\ &= \left\{ \begin{pmatrix} v \\ v \end{pmatrix} : v \in \mathfrak{g} \right\} \end{aligned}$$

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Take $\theta \in \mathrm{Aut}(\mathfrak{g})$.

Let $\Omega_1^b = \theta(\Omega_1^a)$ and $\Omega_2^b = \theta(\Omega_2^a)$.

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$$\begin{aligned} \mathfrak{A} &= \left\langle \begin{pmatrix} \Omega_1^a \\ \theta(\Omega_1^a) \end{pmatrix}, \begin{pmatrix} \Omega_2^a \\ \theta(\Omega_2^a) \end{pmatrix} \right\rangle \\ &= \operatorname{graph} \theta \\ &= \left\{ \begin{pmatrix} v \\ \theta(v) \end{pmatrix} : v \in \mathfrak{g} \right\} \end{aligned}$$

The Controllability Condition

Controllability Condition

Let G is a simple Lie group. Then, given componentwise controllability, the system (1) is uncontrollable if and only if $\exists\ \theta \in \mathrm{Aut}(\mathfrak{g})$ mapping $\Omega_i^a \mapsto \Omega_i^b$ for $i = 1, 2$.

Remarks

In general, for systems with more than two controls,

$$\dot{X}(t) = \sum_{i=1}^{n} u_i(t) \begin{pmatrix} \Omega_i^a \\ \Omega_i^b \end{pmatrix}_{X(t)} \qquad (2)$$

the condition is,

Controllability Condition

Let G is a simple Lie group. Then, given componentwise controllability, the system (2) on $G \times G$ is controllable if and only if $\exists\ \theta \in \mathrm{Aut}(\mathfrak{g})$ mapping $\Omega_i^a \mapsto \Omega_i^b$ for $i = 1, \ldots, n$.

Remarks

The Controllabilty Condition is not true for all semisimple groups.

Consequences

Let B be the Killing form on $\mathfrak{g}$. Automorphisms of $\mathfrak{g}$ preserve B,

$$B(\theta(X), \theta(Y)) = B(X, Y), \forall X, Y \in \mathfrak{g}, \forall \theta \in \mathrm{Aut}(\mathfrak{g})$$

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So,

A Sufficient Condition

The system (1) is controllable if

- $B(\Omega_i^a, \Omega_i^a) \neq B(\Omega_i^b, \Omega_i^b)$ for some $i \in \{1, 2\}$ or,
- $B(\Omega_1^a, \Omega_2^a) \neq B(\Omega_1^b, \Omega_2^b)$

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Is this necessary? Only for $\mathfrak{so}(3)$.

The Sufficient Condition Is Not Necessary

In general, let $\mathfrak{g}_0$ be a compact simple Lie algebra, but not $\mathfrak{so}(3)$.

The negative of the Killing form on $\mathfrak{g}_0$ is an inner product.

Let $O(\mathfrak{g}_0)$ be the orthogonal group with respect to $-B$. Then, strictly,

$$\mathrm{Aut}(\mathfrak{g}_0) \subset O(\mathfrak{g}_0)$$

The Sufficient Condition Is Not Necessary

Let S denote the unit sphere in $\mathfrak{g}_0$ under $-B$, and suppose that $\mathrm{Aut}(\mathfrak{g}_0)$ does not act transitively on it.

Find Ω_1^a and Ω_2^a that generate $\mathfrak{g}_0$, and find a $\theta \in O(\mathfrak{g}_0)$ such that $\theta(\Omega_1^a)$ is not in the $\mathrm{Aut}(\mathfrak{g}_0)$ orbit of Ω_1^a.

Then, $B(\Omega_i^a, \Omega_i^a) = B(\Omega_i^b, \Omega_i^b)$ for $i = 1, 2$ and $B(\Omega_1^a, \Omega_2^a) = B(\Omega_1^b, \Omega_2^b)$.

But, by the Controllability Condition,

$$\mathfrak{A} = \left\langle \begin{pmatrix} \Omega_1^a \\ \theta(\Omega_1^a) \end{pmatrix}, \begin{pmatrix} \Omega_2^a \\ \theta(\Omega_2^a) \end{pmatrix} \right\rangle = \mathfrak{g}_0 \oplus \mathfrak{g}_0$$

A Question

Does $\mathrm{Aut}(\mathfrak{g}_0)$ ever act transitively on S?

$\mathfrak{so}(3)$

The sufficient condition for controllability is necessary in $\mathfrak{so}(3)$.

Claim

$$\mathrm{Aut}(\mathfrak{so}(3)) = SO(\mathfrak{so}(3))$$

Let (Ω_1^a, Ω_2^a) and (Ω_1^b, Ω_2^b) be two pairs in $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Claim

There exists a $\theta \in SO(\mathfrak{so}(3)) = \mathrm{Aut}(\mathfrak{so}(3))$ mapping $\Omega_i^a \mapsto \Omega_i^b$ for $i = 1, 2$ if and only if $||\Omega_i^a|| = ||\Omega_i^b||$ and $\langle \Omega_1^a, \Omega_2^a \rangle = \langle \Omega_1^b, \Omega_2^b \rangle$.

Conclusion

Given componentwise controllability, the system (1) is uncontrollable if and only if $||\Omega_i^a|| = ||\Omega_i^b||$ and $\langle \Omega_1^a, \Omega_2^a \rangle = \langle \Omega_1^b, \Omega_2^b \rangle$.

Using Eigenvalues

Observation

All automorphisms of $\mathfrak{so}(2n+1)$ are inner. Inner automorphisms of $\mathfrak{so}(2n+1)$ preserve eigenvalues.

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The system (1) on $SO(2n+1) \times SO(2n+1)$ is controllable if the eigenvalues of Ω_i^a are not equal to those of Ω_i^b for some i.

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Other algebras (except for $\mathfrak{sp}(n)$) have outer automorphisms, which complicate things.

Summary

Controllability Condition

Let G is a simple Lie group. Then, given componentwise controllability, the system (1) is controllable if and only if $\exists\ \theta \in \mathrm{Aut}(\mathfrak{g})$ mapping $\Omega_i^a \mapsto \Omega_i^b$ for $i = 1, 2$.

Future Work

Stronger conditions!

Things I Deferred

The proof, and a counterexample for non-simple semisimple algebras.

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