Properties of state spaces

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Spaces versus algebras (rings)

Remember Gelfand's relation

compact spaces $\Leftrightarrow$ algebra of continuous functions

Other examples:

algebra of (k-times) differentiable functions

algebra of holomorphic functions on disk, ...

$\mathcal{H}$ Hilbert space $\Leftrightarrow \mathcal{B}(\mathcal{H}) \Leftrightarrow \Omega(\mathcal{H})$

Unital *-algebra $\mathcal{A} \Rightarrow$ state space $\Omega(\mathcal{A})$

Alfsen, Shulz: Which compact convex sets K are state spaces of C*-algebras (Jordan algebras)?

Necessary: The face generated by two extremal points is a ball.

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$\Omega_d = \Omega(\mathcal{H})$, $\dim \mathcal{H} = d$.

K any compact set in a finite dimensional linear space. Assume ex$[K]$ compact.

Let T map K into K' preserving convex combinations. T can be extended to an affine map.

If T maps Ω affine into K, there is a unique linear extension of T to Herm$(\mathcal{H})$.

Aut$[K]$: Affine one-to-one maps from K onto K.

Aut$[\Omega]$: Wigner symmetries $\omega \to V\omega V^*$

Two-point-concavity on a convex set K

$G = G(\xi, \xi')$ defined for $\xi, \xi' \in K$ is concave if

$$G(\xi, \xi') \geq \sum \sqrt{p_j p'_j} G(\xi_j, \xi'_j) \tag{1}$$

for all pairs of convex combinations

$$\xi = \sum p_j \xi_j, \quad \xi = \sum p'_k \xi'_k \ . \tag{2}$$

The infimum of a set of concave functions in two variables is again concave.

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K convex and compact: $\Rightarrow$
Given $g(\eta, \eta')$ for all $\eta, \eta' \in \mathrm{ex}[K]$, there is a minimal concave function, say $g^{\cap}(\xi, \xi')$, coinciding with g at extremal points.

$$g^{\cap}(\xi, \xi') = \inf \sum \sqrt{p_j p'_j} g(\pi_j, \pi'_j) \tag{3}$$

If G is a function of two points we also write

$$G^{\cap} := g^{\cap} \text{ if } g(\eta, \eta') = G(\eta, \eta') \tag{4}$$

on ex$[K]$.
(This is similar to the concave roof extension in one argument.)

Main example: $K := \Omega_d$.

$$F(\rho,\omega) = \mathrm{tr}\,(\,\rho^{1/2}\omega\rho^{1/2}\,)^{1/2}, \quad \Pr(\rho,\omega) = F(\rho,\omega)^2\ . \tag{5}$$

Statement: F is concave and $F = F^{\cap}$

Indeed, one knows for positive operators

$$F(\sum A_j, \sum B_k) \geq \sum F(A_j, B_j) \tag{6}$$

and equality takes place if and only if

$$F(A_j, B_k) = 0 \text{ for all } j \neq k\ . \tag{7}$$

Setting, say, $A_j = p_j \pi_j$, ..., the assertion follows.

For simplices, Pr(.,.) is the Kakutani mean. (Bhattachayya)

Mielnik's construction.

With a good definition of fidelity,

$$\text{fidelity} = \sqrt{\text{transition probability}},$$

for extremal states we could apply (3). The task to define transition probabilities for pairs of extremal points is due to Mielnik:

Let K compact. $[0,1]_K$ denotes the set of all affine functions l satisfying $0 \leq l(\xi) \leq 1$ for all $\xi \in K$.

For $K = \Omega$: $l(\omega) = \mathrm{tr}\, A\omega$, $\mathbf{0} \leq A \leq \mathbf{1}$

$$\mathrm{Pr}_K(\eta_1, \eta_2) := \min l(\eta_2), \quad l(\eta_1) = 1, \quad l \in [0,1]_K \tag{8}$$

is Mielnik's transition probability for pairs $\eta_1, \eta_2 \in \mathrm{ex}[K]$.

Now one applies (3) to extend (8) from ex$[K]$ to K.

Examples

$$\mathrm{Pr}_\Omega(\pi_1, \pi_2) = \mathrm{tr}\,\pi_1\pi_2, \quad \pi_j \text{ pure.} \tag{9}$$

Mielnik's procedure gives the correct expression.
Consider the n-dimensional unit ball $\mathbf{B}_n$ and ex$[\mathbf{B}_n] = \mathbf{S}_n$, the unit sphere.
For two vectors on the sphere:

$$\mathrm{Pr}_B(\vec{y}, \vec{y}') = \frac{1 + \vec{y}\vec{y}'}{2} \ . \tag{10}$$

For two general vectors one applies (3), (5):

$$\mathrm{Pr}_B(\vec{x}, \vec{x}') = \frac{1 + \vec{x}\vec{x}' + \sqrt{1 - \vec{x}\vec{x}}\sqrt{1 - \vec{x}'\vec{x}'}}{2} \ . \tag{11}$$

Maps and Monotonicity

Let T be an affine map from Ω_n into Ω_m.

$$\mathrm{Pr}(\omega, \rho) \leq \mathrm{Pr}(T\omega, T\rho) \tag{12}$$

Now assume $G(.,.)$ monotone for *cpt-maps*,

$$G(\omega, \rho) \leq G(T\omega, T\rho), \quad T \text{ cpt.} \tag{13}$$

Statement: (Categorical definition of $\mathrm{Pr}(.,.)$)

$$G(\pi_1, \pi_2) = \mathrm{Pr}(\pi_1, \pi_2), \quad G(\omega_1, \omega_2) \leq \mathrm{Pr}(\omega_1, \omega_2) \tag{14}$$

Then : $G(\omega_1, \omega_2) = \mathrm{Pr}(\omega_1, \omega_2)$
Because: Given ω_1, ω_2, there is a cpt-map satisfying $T\pi_j = \omega_j$ with equality in (12).

Example:

$$\operatorname{tr} \omega^s \rho^{1-s} \geq \Pr(\omega, \rho), \quad 0 < s < 1 \tag{15}$$

because the left hand side fulfills (13), (14).

Now let G just monotone, i. e. fulfill (13).
Then G must be unitary invariant. The only unitary invariants for pairs of pure states are functions of their transition probability. (13) also requires them increasing.
Hence there is a positive, increasing real function $f = f(x)$ on $0 \leq x \leq 1$ such that

$$G(\pi_1, \pi_2) = f(\operatorname{tr} \pi_1 \pi_2) \tag{16}$$

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Statement: If for all tcp-maps

$$G(\omega, \rho) \leq G(T\omega, T\rho)$$

then there is on $[0, 1]$ a monotone increasing function $f \geq 0$ with

$$G(\omega_1, \omega_2) \geq f(\,\mathrm{Pr}(\omega_1, \omega_2)\,) \quad (17)$$

and equality holds for pairs of pure states.

Example: Take $f(x) = 1 - \sqrt{1-x}$ and convince yourself of

$$1 - (1/2) \parallel \pi_1 - \pi_2 \parallel_1 = f(\mathrm{tr}\, \pi_1 \pi_2)$$

to conclude its correctness for general pairs, i. e.

$$\parallel \omega_1 - \omega_2 \parallel_1 \leq 2\sqrt{1 - \mathrm{Pr}(\omega_1, \omega_2)} \tag{18}$$

In the same manner one obtains with $s_j \geq 0$:

$$\parallel s_1\omega_1 - s_2\omega_2 \parallel_1^2 \leq (s_1 - s_2)^2 + 4 s_1 s_2 (1 - \mathrm{Pr}(\omega_1, \omega_2)) \tag{19}$$

Thus it seems worthwhile to define the 1-distance in general.

Let K convex and compact. Remember $[0,1]_K$.

$$|\xi_2, \xi_1|_K^+ = \max l(\xi_2) - l(\xi_1), \quad l \in [0,1]_K \tag{20}$$

For affine maps on proves the contracting property

$$|\xi_2, \xi_1|_K^+ \geq |T(\xi_2), T(\xi_1)|_{T(K)}^+ \tag{21}$$

and the subadditivity property

$$|\xi_3, \xi_1|_K^+ \leq |\xi_3, \xi_2|_K^+ + |\xi_2, \xi_1|_K^+ \tag{22}$$

Examples:

$$|\omega_2 - \omega_1|_\Omega^+ = \operatorname{tr}(\omega_2 - \omega_1)^+$$

For a ball with radius r :

$$|\vec{x}_2, \vec{x}_1|_B^+ == \frac{1}{2r}\sqrt{(\vec{x_2} - \vec{x_1})^2}$$

The *1-distance* is defined by

$$|\xi_2, \xi_1|_K := |\xi_2, \xi_1|_K^+ + |\xi_1, \xi_2|_K^+ \quad (23)$$

It satisfies the triangle inequality and is contracting with respect to affine maps.
((The notation “1-distance” can be misleading: For balls it is an Euclidian distance.))

Spectra

For $\xi \in K$ consider the set of all coefficient vectors $\{p_j\}$ which can appear in an extremal decomposition

$$\xi = \sum p_j \eta_j , \quad p_1 \geq p_2 \geq \ldots , \quad \eta_j \in \mathrm{ex}[K] \ . \tag{24}$$

A coefficient vector is called *eigenvalue vector,* if it cannot properly majorized by another coefficient vector.

Denote the set of these vectors by spec$_K[\xi]$.

In case $K = \Omega_d$ one gets the decreasingly ordered set of eigenvalues followed by zeros.

In case $\vec{x} \in K = \mathbf{B}_n$: Draw a line through $\vec{x}$ and the center. It intersects the boundary sphere at two points, say $\vec{y}$, $\vec{y}^{\perp}$ satisfying $\vec{x} = (1-p)\vec{y} + p\vec{y}^{\perp}$, $1-p \geq p$. Then spec$_K[\vec{x}] = \{1-p, p\}$

One needs the compactness of ex$[K]$ to prove

a) There are eigenvalue vectors for all $\xi \in K$

b) Denote by rank$[\xi]$ the smallest possible number of non-zero coefficients in a coefficient vector. Then there is an eigenvalue vector with rank$[\xi]$ non-zero components.

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Unsolved: ω and ω' density operators. Consider

$$\inf \sum (p_j)^s (p'_j)^{1-s} \tag{25}$$

running over all coefficient vectors of ω resp. ω'. Is it the trace of $\omega^s (\omega')^{(1-s)}$?