Time-optimal synthesis of unitary transformations in coupled fast and slow qubit system

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Control, Constraints and Quanta

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Outline

1. Quantum computing as a control problem
2. Fast and slow qubit system
3. Further comments

Quantum systems and their transformations

(pure) quantum states (= vectors in $\mathbb{C}^m$, $m < \infty$)

- example: qubit (=quantum bit) is an element of $\mathbb{C}^2$ ($\rightarrow$ Bloch sphere)
- combined quantum system: tensor product $\mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_n}$
  - space of all $\mathbb{Z}$-linear comb. of $v_1 \otimes \cdots \otimes v_n$ ($v_j \in \mathbb{C}^{m_j}$, $\otimes$ bilinear)
  - example: two qubits as given by $\mathbb{C}^2 \otimes \mathbb{C}^2$

quantum operations = unitary transformations $U \in \mathrm{SU}(d)$

- Lie group $\mathrm{SU}(d) = \{ G \in \mathrm{GL}(d, \mathbb{C}) | \, G^{-1} = (G^*)^T, \det(G) = 1\}$ (= closed linear matrix group)
- Lie algebra $\mathfrak{su}(d) = \{ g \in \mathfrak{gl}(d, \mathbb{C}) | -g = (g^*)^T, \mathrm{Tr}(g) = 0\}$
  - tangent space to $\mathrm{SU}(d)$ at the identity
  - with bilinear and skew-symmetric multiplication $[g_1, g_2] := g_1 g_2 - g_2 g_1$ where $[g_1, g_2] \in \mathfrak{su}(d)$ and $[[g_1, g_2], g_3] + [[g_3, g_1], g_2] + [[g_2, g_3], g_1] = 0$

Quantum computing as a control problem

physical resources to synthesize unitary transformations $U \in \mathrm{SU}(d)$

- a given set of unitary transformations
- unitary time evolution of the quantum system (Schrödinger equation): $\frac{d}{dt}U(t) = -iH(t)U(t)$ with Hamiltonian $H(t) = H_0 + \sum_{j=1}^{m} v_j(t)H_j$ where $iH(t) \in \mathfrak{su}(d)$ and $v_j(t)$ are control functions

controllability (= universality), i.e., all $U \in \mathrm{SU}(d)$ can be obtained

necessary and sufficient condition: $iH_0, iH_1, \ldots, iH_m$ generate $\mathfrak{su}(d)$ (Brockett (1972,1973), Jurdjevic and Sussmann (1972))

find efficient (control) algorithms for a quantum computer

w.r.t. execution time (or number of applied unitary transformations)

Simulation of unitary transformations

resources (realistic for nuclear spins in nuclear magnetic resonance)

- instantaneous operations $U_j \in \mathrm{SU}(2)^{\otimes n} = \mathrm{SU}(2) \otimes \cdots \otimes \mathrm{SU}(2)$
- time-evolution w.r.t. a coupling Hamilton operator H $(-iH \in \mathfrak{su}(2^n))$

efficient (control) algorithms for $U \in \mathrm{SU}(2^n)$ with execution time t

- $U = \left[\prod_{k=1}^{m} \left(U_k{}^{-1} \exp(-iHt_k) U_k\right)\right] U_0$ and $t = \sum_{k=1}^{m} t_k \qquad (t_k \geq 0)$
- idea: conjugate the orbit $\exp(-iHt_k)$ with instantaneous operations $U_k \in \mathrm{SU}(2^n) \Rightarrow$ piecewise change of the time evolution

remarks

- widely used assumptions in the literature ($\rightarrow$ entanglement)
- we compare our methods to time-optimal control algorithms for two $(n = 2)$ nuclear spins (Khaneja/Brockett/Glaser (2001))

Outline

Our model: coupled fast and slow qubit system (1/2)

the physical system (high field case, in a double rotating frame)

- free evolution w.r.t. the Hamiltonian $H_0 = JI_z + J(2S_zI_z)$
- control Hamiltonian on the first qubit (= electron spin):
  $H_S = w_r^S(t)[S_x \cos\phi_S(t) + S_y \sin\phi_S(t)]$
- control Hamiltonian on the second qubit (= nuclear spin):
  $H_I = w_r^I(t)[I_x \cos\phi_I(t) + I_y \sin\phi_I(t)]$
- time scales $\omega_r^I \ll J \ll \omega_r^S$ (H_0 faster than some local operations!)
- first qubit = fast qubit and second qubit = slow qubit

notation: $S_\mu = (\sigma_\mu \otimes \mathrm{id}_2)/2$ and $I_\nu = (\mathrm{id}_2 \otimes \sigma_\nu)/2$ $\quad (\mu, \nu \in \{x, y, z\})$
where $\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathrm{id}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Our model: coupled fast and slow qubit system (2/2)

how to synthesize slow transformations (first order approximation)

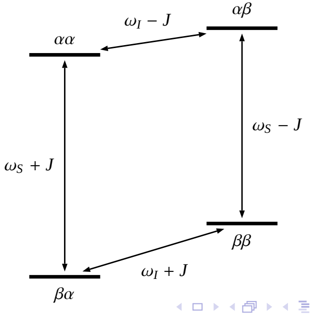
$H_0 + H_I = 2JS^\beta I_z + w_r^I(t)(S^\alpha + S^\beta)(I_x \cos\phi_I + I_y \sin\phi_I)$ truncates to

$$H^\alpha(\phi_I) = 2JS^\beta I_z + w_r^I(t)S^\alpha(I_x \cos\phi_I + I_y \sin\phi_I)$$

where $S^\beta = (\mathrm{id}_4/2 + S_z) = \begin{pmatrix} \mathrm{id}_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}$, $S^\alpha = (\mathrm{id}_4/2 - S_z) = \begin{pmatrix} 0_2 & 0_2 \\ 0_2 & \mathrm{id}_2 \end{pmatrix}$

energy diagram (w.r.t. lab frame)

ω_S, ω_I = natural precession frequency of first and second qubit



Mathematical structure of our model (1/3)

Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ $(\mathfrak{g} = \mathfrak{su}(4),\ \mathfrak{G} = \mathrm{SU}(4))$

condition: $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ ($\mathfrak{k}$ Lie algebra, $\mathfrak{K}$ its Lie group)

fast operations: $-iS_\mu$ ($\mu \in \{x, y, z\}$) and $-iH_0 \Rightarrow$
$\mathfrak{K} = \exp(\mathfrak{k})$ where $\mathfrak{k} = \{-iS_\mu, -i2S_\nu I_z, -iI_z :\ \mu, \nu \in \{x, y, z\}\}$

slow operations: e.g., $-iH^\alpha(\phi_I) \Rightarrow$
$\mathfrak{P} = \exp(\mathfrak{p})$ where $\mathfrak{p} = \{-iI_\gamma, -i2S_\mu I_\gamma :\ \gamma \in \{x, y\}, \mu \in \{x, y, z\}\}$

compare to two nuclear spins: $\mathfrak{k} \oplus \mathfrak{p} = \text{local} \oplus \text{nonlocal}$

Mathematical structure of our model (2/3)

Weyl orbit $\mathcal{W}(p) = \{KpK^{-1}\colon\ K \in \mathfrak{K}\} \cap \mathfrak{a}$ of $p \in \mathfrak{p}$

- max. Abelian subalgebra $\mathfrak{a} = \{a_1(-iS^\beta I_x) + a_2(-iS^\alpha I_x)\colon a_j \in \mathbb{R}\} \subset \mathfrak{p}$
- Kostant's convexity theorem (1973): the slow operations and the corresponding Weyl orbit determine the time-optimal control
- $\mathcal{W}[(b_1, b_2)] = \mathcal{W}[b_1(-iS^\beta I_x) + b_2(-iS^\alpha I_x)] = \{(b_1, b_2), (b_1, -b_2), (-b_1, b_2), (-b_1, -b_2), (b_2, b_1), (b_2, -b_1), (-b_2, -b_1), (-b_2, b_1)\}$
- majorization condition: (a_1, a_2) is in the convex closure of $\mathcal{W}[(b_1, b_2)]$ iff $\max\{|a_1|, |a_2|\} \leq \max\{|b_1|, |b_2|\}$ and $|a_1| + |a_2| \leq |b_1| + |b_2|$

compare to two nuclear spins

- Bennett et al. (2002) introduced a similar majorization condition
- Zeier/Grassl/Beth (2004) (see also Yuan/Khaneja (2005 and 2006)) proved the connection to the convex closure of the Weyl orbit

Mathematical structure of our model (3/3)

$\mathfrak{KAK}$ decomposition $(\mathfrak{A} = \exp(\mathfrak{a}))$

- max. Abelian subalgebra $\mathfrak{a} = \{a_1(-iS^\beta I_x) + a_2(-iS^\alpha I_x) \colon a_j \in \mathbb{R}\} \subset \mathfrak{p}$
- $G = K_1 \exp[a_1(-iS^\beta I_x) + a_2(-iS^\alpha I_x)]K_2 \in \mathfrak{G}$ $(K_j \in \mathfrak{K})$

remark: $\mathfrak{KAK}$ decomposition is not unique

$\Rightarrow$ consider all $(a_1, a_2) + \pi(z_1, z_2)$ where $z_1, z_2 \in \mathbb{Z}$

- Zeier/Yuan/Khaneja (arXiv:0709.4484v1): majorization condition simplifies for $a_1, a_2 \in [-\pi, \pi]$
  $\Rightarrow$ sufficient to consider only $z_1 = z_2 = 0$
- similar to two nuclear spins:
  - Vidal/Hammerer/Cirac (2002) used a similar majorization condition
  - Zeier/Grassl/Beth (2004) (see also Dirr et al. (2006)) analyzed the nonuniqueness of the $\mathfrak{KAK}$ decomposition

Time-optimal control of fast and slow qubit system

Zeier/Yuan/Khaneja (arXiv:0709.4484v1)

The minimal time to synthesize $G \in \mathrm{SU}(4)$ is $\min\{(|t_1| + |t_2|)/\omega_r^I\}$ such that $G = K_1 \exp[t_1(-iS^\beta I_x) + t_2(-iS^\alpha I_x)]K_2$

remarks

- slow operations: $-iH^\alpha(0)$, we use the Weyl orbit of $-iS^\alpha I_x$: $b_1 = 0$ and $b_2 = 1 \Rightarrow \mathcal{W}[(b_1, b_2)] = \{(-1,0),(1,0),(0,-1),(0,1)\}$
- relies on Kostant's convexity theorem (1973)
- the control problem is reduced to convex optimization
- similar to two nuclear spins: see Khaneja/Brockett/Glaser (2001)

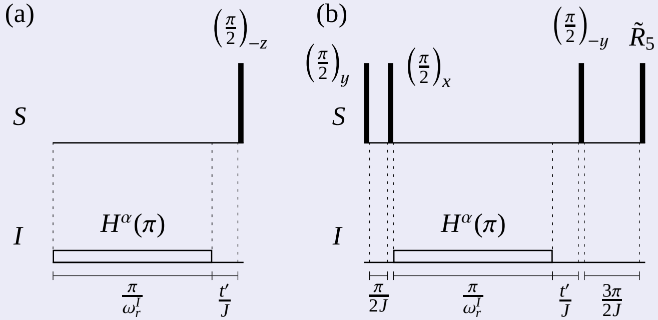
Examples of time-optimal controls (1/2)

minimum time $t_{\min}$ for $\mathrm{CNOT}[2,1]$, $\mathrm{CNOT}[1,2]$, and SWAP

1. $e^{i\pi/4}\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix} = \exp[\pi(-i2S_xI_z + iS_x + iI_z)/2] \Rightarrow t_{\min} = 0$
   (as it is contained in $\mathfrak{K}$ = fast operations)
2. $e^{i\pi/4}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} = \exp[\pi(-i2S_zI_x + iS_z + iI_x)/2] =$
   $\exp(i\pi S_z/2)\exp(-it'H_0/J)\exp\left[-i\pi H^{\alpha}(\pi)/w_r^I\right]$
   (where $t' = -\pi J/w_r^I \mod 2\pi \geq 0$)
   $\Rightarrow t_{\min} = \pi/\omega_r^I$
3. $e^{i\pi/4}\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix} = \exp[\pi(i2S_xI_x + i2S_yI_y + i2S_zI_z)/2] =$
   $e^{i\pi S_z/2}e^{-i\pi S_x/2}e^{-i3\pi H_0/(2J)}e^{i\pi S_y/2}e^{-it'H_0/J}\exp\left[-i\pi H^{\alpha}(\pi)/w_r^I\right]$
   $\times e^{-i\pi S_x/2}e^{-i\pi H_0/(2J)}e^{-i\pi S_y/2}$
   $\Rightarrow t_{\min} = \pi/\omega_r^I$

Examples of time-optimal controls (2/2)

corresponding pulse sequences: (a) $e^{i\pi/4}\mathrm{CNOT}[1,2]$, (b) $e^{i\pi/4}\mathrm{SWAP}$, see Zeier/Yuan/Khaneja (arXiv:0709.4484v1)



Outline

Beyond two nuclear spins

approach for choosing a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

- for two nuclear spins: $\mathfrak{k}$ = local part, $\mathfrak{p}$ = non-local part
- n nuclear spins ($n > 2$): local operations $\subsetneq \mathfrak{K}$ (e.g., $\mathrm{SU}(2)^{\otimes n} \subsetneq \mathfrak{K}$)

lower bounds on the execution time

- assume that all elements of $\mathfrak{K}$ can be applied instantaneously, and not only the elements of $\mathrm{SU}(2)^{\otimes n}$
  $\Rightarrow$ we get the execution time (under this assumption)
- $\mathrm{SU}(2)^{\otimes n} \subseteq \mathfrak{K} \Rightarrow$ the execution time can only be greater

determine suitable $\mathfrak{K}$ (Childs et al. (2003), Zeier/Grassl/Beth (2004))

- n even: $\mathfrak{K}$ is conjugated to the orthogonal group $\mathrm{O}(2^n)$
- n odd: $\mathfrak{K}$ is conjugated to the (unitary) symplectic group $\mathrm{Sp}(2^{n-1})$

Algebraic structure analysis for multi-qubit systems

general case of $\mathfrak{G} = \mathrm{SU}(2^n)$, $\mathfrak{L} = \mathrm{SU}(2)^{\otimes n}$, and $\mathfrak{l} = \text{Lie algebra}(\mathfrak{L})$

$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$, where $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$ and $[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}$ (but not $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{l}$ for $n > 2$)
$\Rightarrow$ no Cartan decomposition

de Rham cohomology of $\mathfrak{G}/\mathfrak{L} = \mathrm{SU}(2^n)/\mathrm{SU}(2)^{\otimes n}$

- computed for $n = 2, 3$ (Zeier (2006))
- potential structure insight to the simulation of unitary transformations
- connections to the structure of entanglement

for more information see http://www.eecs.harvard.edu/~zeier/

Thank you for your attention!