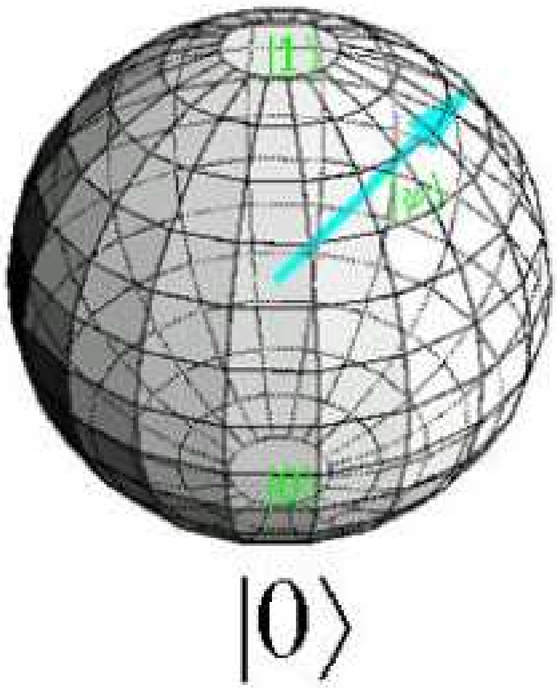
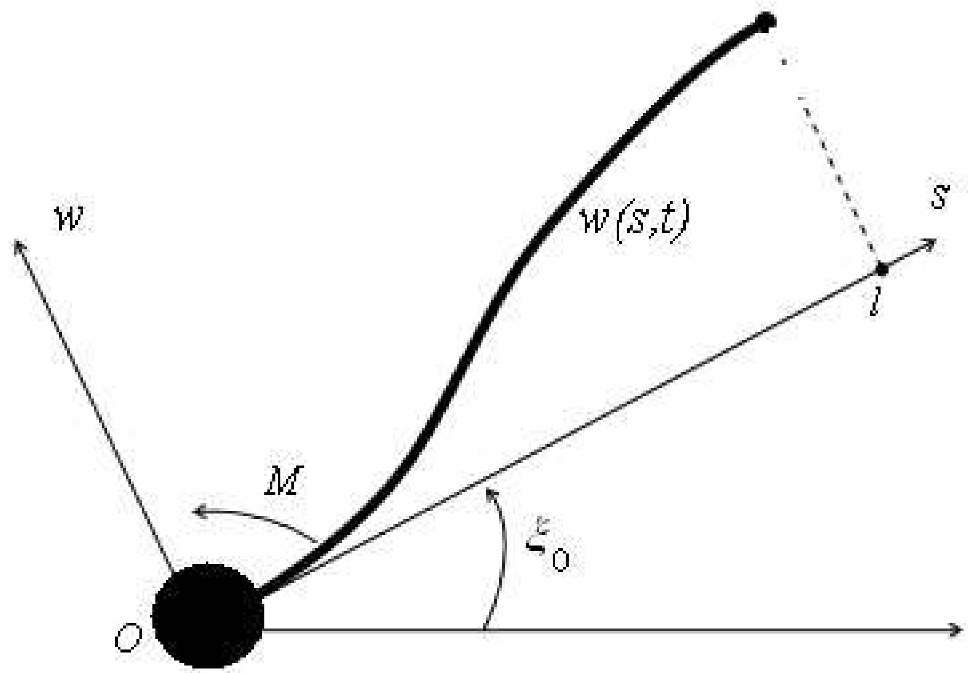


OPTIMAL CONTROL OF AN INFINITE DIMENSIONAL SYSTEM WITH MODAL COORDINATES

Alexander Zuyev

Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine,
R. Luxembourg 74, 83114 Donetsk, Ukraine

1 Introduction: Euler-Bernoulli beam controlled by a torque M



- $\xi_0(t)$ – rotation angle of the rigid body;
- $\eta_0(t)$ – angular velocity of the rigid body;
- $w(s,t)$ – relative displacement of the centerline of the beam,

$$w(s,t) = \sum_{j=1}^{\infty} \xi_j(t) u_j(s).$$

2 Motion equations in operator form

Consider the linearized control system

$$\dot{x} = Ax + Bv, \tag{1}$$

where $x = (\xi_0, \eta_0, \xi_1, \eta_1, \xi_2, \eta_2, \dots)^T \in \ell^2$ is the state vector, $v \in \mathbb{R}^1$ is the control, $A : D(A) \to \ell^2$,

$$A = \mathrm{diag}(A_0, A_1, A_2, \dots),$$

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; A_n = \begin{pmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{pmatrix},$$

and

$$B = (0, 1, \; 0, b_1, \; 0, b_2, \; \dots)^T \in \ell^2.$$

The Hilbert space ℓ^2 is equipped with the standard norm

$$\|x\| = \left(\sum_{n=0}^{\infty} (\xi_n^2 + \eta_n^2) \right)^{1/2}.$$

The coefficients of (1) are given by the formulae

$$\omega_j = c\sqrt{\lambda_j}, \; b_j = -\int_0^l (s + \tilde{d}) u_j(s)\, ds,$$

where $\lambda_1 < \lambda_2 < \ldots$ and $u_1(s)$, $u_2(s)$, ... are solutions of the following Sturm-Liouville problem

$$\frac{d^4}{ds^4}u_j(s) = \lambda_j u_j(s), \quad s \in (0,l),$$

$$u_j(0) = u_j'(0) = u_j''(l) = u_j'''(l) = 0, \; \|u_j\|_{L_2(0,l)} = 1, \quad (j = 1,2,\ldots).$$

Mechanical parameters of the Euler-Bernoulli beam model:

- l – length of the beam;
- $\tilde{d}$ – distance between the rotation axis and the clamped beam's end;
- $c = \sqrt{EI/\rho} > 0$;
- E – Young's modulus;
- I – moment of inertia for the cross-section area;
- ρ – mass per unit length.

Derivation of the motion equations can be found in:

- Zuyev A.L. Partial asymptotic stabilization of nonlinear distributed parameter systems. Automatica, 2005, Vol. 41, No. 1, P. 1-10.
- Zuyev A.L. Control of a system with elastic components in the non-resonance case. Ukrainian Mathematical Bulletin, 2006, Vol. 3, No. 1, P. 131-144.

3 Control of a finite dimensional approximation

Let us fix an arbitrary $N \geq 1$ and consider the finite dimensional subsystem of (1) corresponding to the modes with indices $j \leq N$:

$$\begin{aligned} &\dot{\xi}_0 = \eta_0, \\ &\dot{\eta}_0 = v, \\ &\dot{\xi}_j = \omega_j \eta_j, \\ &\dot{\eta}_j = -\omega_j \xi_j + b_j v, \quad j = \overline{1,N}. \end{aligned} \tag{2}$$

Denote

$$x_N = \begin{pmatrix} \xi_0 \\ \eta_0 \\ \xi_1 \\ \eta_1 \\ \vdots \\ \xi_N \\ \eta_N \end{pmatrix}, \tilde{x}_N = \begin{pmatrix} \tilde{\xi}_0 \\ \tilde{\eta}_0 \\ \tilde{\xi}_1 \\ \tilde{\eta}_1 \\ \vdots \\ \tilde{\xi}_N \\ \tilde{\eta}_N \end{pmatrix}, \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{pmatrix}, \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix}, \tilde{\xi} = \begin{pmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \vdots \\ \tilde{\xi}_N \end{pmatrix}, \tilde{\eta} = \begin{pmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \\ \vdots \\ \tilde{\eta}_N \end{pmatrix}.$$

Theorem 1. *If $b_j \neq 0$ and all $\omega_j^2 \neq 0$ are different then there exists a transformation*

$$\tilde{x}_N = \Phi x_N,\ \tilde{v} = \alpha v + \beta \xi, \quad (|\Phi| \neq 0,\ \alpha \neq 0), \tag{3}$$

that brings the system (2) into the Brunovsky canonical form:

$$\begin{aligned} \dot{\tilde{\xi}}_{j-1} &= \tilde{\eta}_{j-1}, \\ \dot{\tilde{\eta}}_{j-1} &= \tilde{\xi}_j, \qquad (j = \overline{1,N}), \\ \dot{\tilde{\xi}}_N &= \tilde{\eta}_N, \\ \dot{\tilde{\eta}}_N &= \tilde{v}. \end{aligned} \tag{4}$$

The components of (3) are given explicitly [Zuyev A.L., Control of elastic oscillations by using a canonical form. Dynamical Systems, 2006, Vol. 20, P. 27-34].

Theorem 1 allows to parametrize the state $x_N(t)$ and control $v(t)$ in terms of $\tilde{\xi}_0(t)$ and its time-derivatives. This is useful for solving the following

Optimal control problem. *Given $\tau > 0$ and*

$$x_N^0 = (\xi_0^0, \eta_0^0, \ldots, \xi_N^0, \eta_N^0)^T \in \mathbb{R}^{2N+2}, \quad x_N^1 = (\xi_0^1, \eta_0^1, \ldots, \xi_N^1, \eta_N^1)^T \in \mathbb{R}^{2N+2},$$

fi nd the control $\bar{v}_N \in L_2[0,\tau]$ that minimizes the functional

$$J(v) = \int_{t_0}^{t_1} |v(t)|^2 dt \tag{5}$$

among all controls $v \in L_2[0,\tau]$ such that $x_N(0) = x_N^0$ and $x_N(\tau) = x_N^1$.

This problem is reduced to the Lagrange problem on $\tilde{\xi}_0(t)$ by applying Theorem 1.

Theorem 2. *Let $b_j \neq 0$, $\omega_j \neq 0$, and all ω_j^2 be different for $j = \overline{1,N}$. Then, for any $x_N^0, x_N^1 \in \mathbb{R}^{2N+2}$ and $\tau > 0$, there exists unique optimal control $\bar{v}_N(t)$, $t \in [0,\tau]$. The optimal control is smooth and given by the formula*

$$\bar{v}_N(t) = k_0 + k_1 t + \sum_{j=1}^{N} \left(U_j \cos(\omega_j t) + V_j \sin(\omega_j t)\right), \; t \in [0,\tau],, \tag{6}$$

where k_0, k_1, U_j, V_j satisfy the linear algebraic system

$$(M+F)\begin{pmatrix} k_0 \\ \tau k_1 \\ U_1 \\ V_1 \\ \vdots \\ U_N \\ V_N \end{pmatrix} = \begin{pmatrix} (\xi_0^1 - \xi_0^0)/\tau - \eta_0^0 \\ \eta_0^1 - \eta_0^0 \\ (\xi_1^1 \sin\omega_1\tau + \eta_1^1 \cos\omega_1\tau - \eta_1^0)/b_1 \\ (\xi_1^1 \cos\omega_1\tau - \eta_1^1 \sin\omega_1\tau - \xi_1^0)/b_1 \\ \vdots \\ (\xi_N^1 \sin\omega_N\tau + \eta_N^1 \cos\omega_N\tau - \eta_N^0)/b_N \\ (\xi_N^1 \cos\omega_N\tau - \eta_N^1 \sin\omega_N\tau - \xi_N^0)/b_N \end{pmatrix}, \tag{7}$$

$$M = \operatorname{diag}\left(\begin{pmatrix} \tau/2 & \tau/6 \\ \tau & \tau/2 \end{pmatrix}, \begin{pmatrix} \tau/2 & 0 \\ 0 & -\tau/2 \end{pmatrix}, \ldots, \begin{pmatrix} \tau/2 & 0 \\ 0 & -\tau/2 \end{pmatrix}\right), \tag{8}$$

$$f_{1,2i+1} = \frac{1-\cos(\omega_i\tau)}{\omega_i^2\tau},\; f_{1,2i+2} = \frac{\omega_i\tau - \sin(\omega_i\tau)}{\omega_i^2\tau},\; f_{2,2i+1} = \frac{\sin(\omega_i\tau)}{\omega_i},\; f_{2,2i+2} = \frac{1-\cos(\omega_i\tau)}{\omega_i},$$

$$f_{2j+1,1} = \frac{\sin(\omega_j\tau)}{\omega_j},\; f_{2j+1,2} = \frac{\omega_j\tau\sin(\omega_j\tau) + \cos(\omega_j\tau) - 1}{\omega_j^2\tau},\; f_{2j+1,2j+1} = \frac{\sin(2\omega_j\tau)}{4\omega_j},$$

$$f_{2j+1,2j+2} = \frac{\sin^2(\omega_j \tau)}{2\omega_j},\ f_{2j+1,2i+1} = \frac{1}{2}\left(\frac{\sin(\omega_i+\omega_j)\tau}{\omega_i+\omega_j} + \frac{\sin(\omega_i-\omega_j)\tau}{\omega_i-\omega_j}\right),$$

$$f_{2j+1,2i+2} = \frac{1}{2}\left(\frac{\cos(\omega_i-\omega_j)\tau}{\omega_j-\omega_i} - \frac{\cos(\omega_i+\omega_j)\tau}{\omega_i+\omega_j} + \frac{2\omega_i}{\omega_i^2-\omega_j^2}\right),$$

$$f_{2j+2,1} = \frac{\cos(\omega_j\tau)-1}{\omega_j},\ f_{2j+2,2} = \frac{\omega_j\tau\cos(\omega_j\tau)-\sin(\omega_j\tau)}{\omega_j^2\tau},\ f_{2j+2,2j+1} = -\frac{\sin^2(\omega_j\tau)}{2\omega_j},$$

$$f_{2j+2,2j+2} = \frac{\sin(2\omega_j\tau)}{4\omega_j},\ f_{2j+2,2i+1} = \frac{1}{2}\left(\frac{\cos(\omega_i+\omega_j)\tau}{\omega_i+\omega_j} + \frac{\cos(\omega_j-\omega_i)\tau}{\omega_j-\omega_i} + \frac{2\omega_j}{\omega_i^2-\omega_j^2}\right),$$

$$f_{2j+2,2i+2} = \frac{1}{2}\left(\frac{\sin(\omega_i+\omega_j)\tau}{\omega_i+\omega_j} + \frac{\sin(\omega_i-\omega_j)\tau}{\omega_j-\omega_i}\right), \quad i,j=\overline{1,N},\ i\neq j. \tag{9}$$

4 Infinite dimensional system: spillover analysis

The goal is to estimate solutions of the infinite dimensional system (1) by applying the family of controllers $\bar{v}_N(t)$ corresponding to finite dimensional approximations (2). For $v \in L_2(0,\tau)$, denote by $x(t;x^0,v(\cdot))$ the solution of the Cauchy problem (1) with $x(0;x^0,v(\cdot)) = x^0$.

Theorem 3 (approximate controllability). *Let $b_j \neq 0$, $\omega_j > 0$, and $\omega_i \neq \omega_j$ for all $i \neq j$. Suppose that*

$$\sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \frac{1}{(\omega_i-\omega_j)^2} < \infty, \tag{10}$$

and denote

$$S = \{x \in \ell^2 \,|\, \lim_{n\to\infty} \left(\sum_{j=n+1}^{\infty} b_j^2\right)\left(\sum_{j=1}^{n} \frac{\xi_j^2+\eta_j^2}{b_j^2}\right) = 0\}.$$

Then $\exists \tau > 0$:

$$\forall x^0, x^1 \in S \subset \ell^2, \forall \varepsilon > 0 \ \exists N_0 = N_0(x^0,x^1,\varepsilon) \ : \ \|x(\tau;x^0,\bar{v}_N) - x^1\| < \varepsilon, \ \forall N \geq N_0. \tag{11}$$

For the Euler-Bernoulli beam, there are useful estimates of the eigenfrequencies for checking (10), see, e.g.,

- Luo Z.-H., Guo B.-Z., Morgul O. Stability and Stabilization of Infinite Dimensional Systems. - London: Springer-Verlag, 1999. - 403 p.

Corollary of Theorem 3. *Let the coeffi cients of system (1) correspond to the Euler-Bernoulli model. Then, for each* x^0, x^1 *from a dense subset of* ℓ^2 *and arbitrary* $\varepsilon > 0$*, the following estimate holds*

$$\|x(\tau; x^0, \bar{v}_N) - x^1\| < \varepsilon$$

if N *is large enough.*