GEOMETRIC HAMILTON–JACOBI THEORY

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Control, Constraints and Quanta
Bedlewo, 10–16 October 2007

PLAN OF THE TALK

1. Classical Hamilton-Jacobi theory (geometric version)
2. Extensions of the Geometric Hamilton-Jacobi Theory
   - Nonholonomic mechanical systems
   - Singular lagrangian systems
3. Future work

Classical Hamilton-Jacobi theory (geometric version)

The standard formulation of the Hamilton-Jacobi problem is to find a function $S(t, q^A)$ (called the principal function) such that

$$\frac{\partial S}{\partial t} + h(q^A, \frac{\partial S}{\partial q^A}) = 0. \qquad (1)$$

If we put $S(t, q^A) = W(q^A) - tE$, where E is a constant, then W satisfies

$$h(q^A, \frac{\partial W}{\partial q^A}) = E; \qquad (2)$$

W is called the characteristic function.

Equations (1) and (2) are indistinctly referred as the Hamilton-Jacobi equation.

R. Abraham, J.E. Marsden: *Foundations of Mechanics* **(2nd edition). Benjamin-Cumming, Reading, 1978.**

Let M be the configuration manifold, and T^*M its cotangent bundle equipped with the canonical symplectic form

$$\omega_M = dq^A \wedge dp_A$$

where (q^A) are coordinates in M and (q^A, p_A) are the induced ones in T^*M.

Let $h : T^*M \longrightarrow \mathbb{R}$ a hamiltonian function and X_h the corresponding hamiltonian vector field:

$$i_{X_h}\omega_M = dh$$

The integral curves of X_h, $(q^A(t), p_A(t))$, satisfy the Hamilton equations:

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}, \ \frac{dp_A}{dt} = -\frac{\partial h}{\partial q^A}$$

Let λ be a closed 1-form on M, say $d\lambda = 0$; (then, locally $\lambda = dW$)

Hamilton-Jacobi Theorem

The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

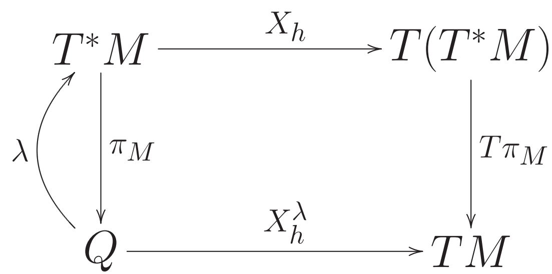
$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(ii) $d(h \circ \lambda) = 0$

Define a vector field on M:

$$X_h^\lambda = T\pi_M \circ X_h \circ \lambda$$



The following conditions are equivalent:

(i) If $\sigma : I \to M$ satisfies the equation

$$\frac{dq^A}{dt} = \frac{\partial h}{\partial p_A}$$

then $\lambda \circ \sigma$ is a solution of the Hamilton equations;

(i)' If $\sigma : I \to M$ is an integral curve of X_h^λ, then $\lambda \circ \sigma$ is an integral curve of X_h;

(i)" X_h and X_h^λ are λ-related, i.e.

$$T\lambda(X_h^\lambda) = X_h \circ \lambda$$

Hamilton-Jacobi Theorem

Let λ be a closed 1-form on M. Then the following conditions are equivalent:

(i) X_h^λ **and** X_h **are** λ**-related;**

(ii) $d(h \circ \lambda) = 0$

If

$$\lambda = \lambda_A(q)\, dq^A$$

then the Hamilton-Jacobi equation becomes

$$h(q^A, \lambda_A(q^B)) = const.$$

and we recover the classical formulation when

$$\lambda_A = \frac{\partial W}{\partial q^A}$$

NONHOLONOMIC MECHANICAL SYSTEMS

Lagrangian mechanics

Let $L = L(q^A, \dot{q}^A)$ **be a lagrangian function, where** (q^A) **are coordinates in a configuration** n**-manifold** Q**.**

The Hamilton 's principle produces the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0,\ 1 \leq A \leq n. \tag{3}$$

A geometric version of Eq. (3) can be obtained as follows.

$L : TQ \longrightarrow \mathbb{R}$**. Consider the (1,1)-tensor field** S **and the Liouville vector field** Δ **defined on the tangent bundle** TQ **of** Q**:**

$$S = \frac{\partial}{\partial \dot{q}^A} \otimes dq^A,\ \Delta = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}.$$

We construct the Poincaré-Cartan 1 and 2-forms

$$\alpha_L = S^*(dL),\ \omega_L = -d\alpha_L,$$

S^* **denotes the adjoint operator of** S**.**

The energy is given by

$$E_L = \Delta(L) - L,$$

so that we recover the classical expressions

$$\omega_L = dq^A \wedge dp_A,\ E_L = \dot{q}^A p_A - L,$$

$p_A = \dfrac{\partial L}{\partial \dot{q}^A}$ **denotes the generalized momenta.**

We say that L is regular if the 2-form ω_L is symplectic, which in coordinates turns to be equivalent to the regularity of the Hessian matrix of L with respect to the velocities, say

$$\left(W_{AB} = \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right)$$

is nonsingular.

In this case, the equation

$$\boxed{i_X \omega_L = dE_L \qquad (4)}$$

has a unique solution, $X = \xi_L$, called the Euler-Lagrange vector field; ξ_L is a second order differential equation (SODE) that means that its integral curves are tangent lifts of their projections on Q (these projections are called the solutions of ξ_L). A direct computation shows that the solutions of ξ_L are just the ones of Eqs (3).

If $\flat_L : TTQ \longrightarrow T^TQ$ is the musical isomorphism, $\flat_L(v) = i_v\omega_L$, then we have $\flat_L(\xi_L) = dE_L$.

Legendre transformation

Finally, let us recall that the Legendre transformation $FL : TQ \longrightarrow T^*Q$ is a fibred mapping (that is, $\pi_Q \circ FL = \tau_Q$, where $\tau_Q : TQ \longrightarrow Q$ and $\pi_Q : T^*Q \longrightarrow Q$ denote the canonical projections of the tangent and cotangent bundle of Q, respectively).

In local coordinates, we have

$$FL(q^A, \dot{q}^A) = (q^A, p_A),$$

and then we have that L is regular if and only if FL is a local diffeomorphism.

Along this paper we will assume that FL is in fact a global diffeomorphism (in other words, L is hyperregular) which is the case when L is a lagrangian of mechanical type, say

$$L = T - V$$

where

- T is the kinetic energy defined by a Riemannian metric on Q,
- $V : Q \longrightarrow \mathbb{R}$ is a potential energy.

Hamiltonian description

The hamiltonian counterpart is developed in the cotangent bundle T^*Q of Q. Denote by $\omega_Q = dq^A \wedge dp_A$ the canonical symplectic form, where (q^A, p_A) are the canonical coordinates on T^*Q.

The Hamiltonian energy is just $H = E_L \circ FL$ and the Hamiltonian vector field is the solution of the symplectic equation

$$i_{X_H}\omega_Q = dH.$$

As we know, the integral curves $(q^A(t), p_A(t))$ of X_H satisfies the Hamilton equations

$$\left.\begin{array}{rcl} \dot{q}^A & = & \frac{\partial H}{\partial p_A} \\ \dot{p}_A & = & -\frac{\partial H}{\partial q^A} \end{array}\right\} \qquad (5)$$

Since $FL^*\omega_Q = \omega_L$ we deduce that ξ_L and X_H are FL-related, and consequently FL transforms the Euler-Lagrange equations (3) into the Hamilton equations (5).

Example: The rolling disk

Consider a disk rolling without sliding on a horizontal plane.

Let (x, y) be the coordinates of the point of contact with the floor, ψ the angle measured from a chosen point of the rim to the point of contact (rotation angle), ϕ is the angle between the tangent to the disk at the point of contact and the x axis (heading angle), and θ is the angle of inclination of the disk.

The configuration manifold is then $Q = \mathbb{R}^2 \times S^1 \times S^1 \times S^1$.

The lagrangian is $L = T - V$ where

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + R^2\dot{\theta}^2 + R^2\dot{\phi}^2 \sin^2\theta) - mR(\dot{\theta}\cos\phi(\dot{x}\sin\phi - \dot{y}\cos\phi) \\ &+ \dot{\phi}\sin\theta(\dot{x}\cos\phi + \dot{y}\sin\phi)) + \frac{1}{2}I_1(\dot{\theta}^2\dot{\phi}^2\cos^2\theta) + \frac{1}{2}I_2(\dot{\psi} + \dot{\phi}\sin\theta)^2 \end{aligned}$$

and

$$V = mgR\cos\theta$$

Here m is the mass of the disk, R is the radius, and I_1 and I_2 are the principal momenta of inertia.

The rolling without sliding condition means that the point of contact has zero velocity and consequently the following constraints have to be fullfilled along the motion

$$\Phi^1 = \dot{x} - (R\cos\phi)\dot{\psi} = 0,\ \Phi^2 = \dot{y} - (R\sin\phi)\dot{\psi} = 0.$$

All the configurations are available, but not all the velocities.

Nonholonomic mechanical systems

A nonholonomic mechanical system is given by a lagrangian function $L = L(q^A, \dot{q}^A)$ subject to a family of constraint functions

$$\Phi^i(q^A, \dot{q}^A) = 0,\ 1 \leq i \leq m \leq n = \dim Q.$$

If $\Phi^i(q^A, \dot{q}^A) = \Phi^i_A(q)\dot{q}^A$ (respectively, $\Phi^i(q^A, \dot{q}^A) = \Phi^i_A(q)\dot{q}^A + b^i(q)$) is linear (respectively, affine) in the velocities the constraints are called linear (respectively, affine). Otherwise, they are called nonlinear.

Invoking the D' Alembert principle for linear and affine constraints (or the Chetaev principle, for nonlinear constraints) we derive the nonholonomic equations of motion

$$\left.\begin{array}{rcl} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} & = & \lambda_i \frac{\partial \Phi^i}{\partial \dot{q}^A}\, , 1 \leq A \leq n \\ \Phi^i(q^A, \dot{q}^A) & = & 0. \end{array}\right\} \tag{6}$$

where $\lambda_i = \lambda_i(q^A, \dot{q}^A)$ are Lagrange multipliers to be determined.

In a geometrical setting, L is a function on TQ and the constraints are given by a submanifold M of TQ locally defined by $\Phi^i = 0$.

If the constraints are linear (respectively, affine) then M is the total space of a vector (respectively, affine) subbundle of TQ. For general nonlinear constraints, M is a submanifold satisfying $\tau_Q(M) = Q$ to avoid holonomic constraints. From now on, we will assume nonlinear constraints, since the treatment is the same.

Equations (6) can be equivalently reformulated as follows

$$\left.\begin{array}{lcr} i_X\,\omega_L - dE_L & = & \lambda_i S^*(d\Phi^i) \\ X(\Phi^i) & = & 0. \end{array}\right\} \qquad (7)$$

If we realize that the bundle of 1-forms $S^*((TM)^0)$ **is locally generated by the local 1-forms** $\{S^*(d\Phi^i)\}$**, we can rewrite Eqs (7) as follows**

$$\left.\begin{array}{lcr} i_X\,\omega_L - dE_L & \in & S^*((TM)^0) \\ X & \in & TM. \end{array}\right\} \tag{8}$$

We assume the *admissibility condition***:**

$$\dim(TM)^0 = \dim S^*((TM)^0)$$

which is equivalent to say that the matrix

$$\left(\frac{\partial \Phi^i}{\partial \dot{q}^A}\right)$$

has maximal rank m**.**

(For linear constraints the above conditions means that the set of 1-forms $\{\mu^i = \Phi^i_A(q)dq^A\}$ **is linearly independent and, indeed, a local cobasis of the distribution** M**).**

We also assume the *compatibility condition*:

$$F^{\perp} \cap TM = \{0\}$$

where F is the distribution on TQ (along M) such that

$$F^0 = S^*((TM)^0)$$

and $F^{\perp}$ denotes the ω_L-complement of F.

Notice that $F^{\perp} = \langle Z^i \rangle$ where $\flat_L(Z^i) = S^*(d\Phi^i)$, therefore $\flat_L(F^{\perp}) = F^0$.

Consider a possible solution of the equation

$$i_X\,\omega_L - dE_L = \lambda_i S^*(d\Phi^i);$$

then $X = \xi_L + \lambda_i Z^i$. If we impose the condition to the dynamics be tangent to the constraint submanifold we obtain

$$0 = X(\Phi^j) = \xi_L(\Phi^j) + \lambda_i Z^i(\Phi^j) \tag{9}$$

Denote $C^{ij} = Z^i(\Phi^j)$. Notice that if the matrix (C^{ij}) is regular, then we can compute the Lagrange multipliers solving the linear equation (9) at each point of M. In this case we can obtain the nonholonomic dynamics X_{nh} which is the unique solution of Eqs. (8).

A simple calculation gives

$$C^{ij} = \frac{\partial \Phi^i}{\partial \dot{q}^A} W^{AB} \frac{\partial \Phi^j}{\partial \dot{q}^B}$$

where (W^{AB}) is the inverse matrix of (W_{AB}), and shows that if (W_{AB}) is definite (positive or negative) then (C^{ij}) is inversible.

As a consequence, if the lagrangian function L is of mechanical type then the nonholonomic system is admissible and compatible.

Projections

Assume that the nonholonomic system is compatible and admissible, then we have a direct sum decomposition

$$T_x(TQ) = T_xM \oplus F_x^{\perp}$$

for all $x \in M$. In terms of vector bundles we have a Whitney sum decomposition

$$TTQ_{|M} = TM \oplus F^{\perp}$$

with two complementary projections $\mathcal{P} : TTQ_{|M} = TM$ and $\mathcal{Q} : TTQ_{|M} = F^{\perp}$ such that $X_{nh} = \mathcal{P}(\xi_L)$.

Remark To be more precise, the result $X_{nh} = \mathcal{P}(\xi_L)$ holds if the constraint are homogenous, that is, Δ is tangent to the constraint submanifold, $\Delta_{|M} \in TM$. This is the case for linear and affine constraints.

Assuming the regularity of the Lagrangian, we have that the Lagrangian and Hamiltonian formulations are locally equivalent. If we suppose, in addition, that the Lagrangian L is hyperregular, then the Legendre transformation $FL : TQ \to T^*Q, (q^A, \dot{q}^A) \mapsto (q^A, p_A = \partial L/\partial \dot{q}^A)$, is a global diffeomorphism. The constraint functions on T^*Q become $\Psi^i = \Phi^i \circ FL^{-1}$, i.e.

$$\Psi^i(q^A, p_A) = \Phi^i(q^A, \frac{\partial H}{\partial p_A}) \,,$$

where the Hamiltonian $H : T^*Q \to \mathbb{R}$ is defined by $H = E_L \circ FL^{-1}$. Since locally $FL^{-1}(q^A, p_A) = (q^A, \frac{\partial H}{\partial p_A})$, then

$$H = p_A \dot{q}^A - L(q^A, \dot{q}^A) \,,$$

where $\dot{q}^A$ is expressed in terms of q^A and p_A using FL^{-1}.

The equations of motion for the nonholonomic system on T^*Q can now be written as follows

$$\left.\begin{aligned} \dot{q}^A &= \frac{\partial H}{\partial p_A} \\ \dot{p}_A &= -\frac{\partial H}{\partial q^A} - \bar{\lambda}_i \frac{\partial \Psi^i}{\partial p_B} \mathcal{H}_{BA} \end{aligned}\right\} \tag{10}$$

together with the constraint equations

$$\Psi^i(q,p) = 0$$

where $\mathcal{H}_{AB}$ are the components of the inverse of the matrix $(\mathcal{H}^{AB}) = (\partial^2 H/\partial p_A \partial p_B)$. Note that

$$\left(\frac{\partial \Psi^i}{\partial p_B} \mathcal{H}_{BA}\right)(q,p) = \left(\frac{\partial \Phi^i}{\partial \dot{q}^A} \circ FL^{-1}\right)(q,p).$$

The symplectic 2-form ω_L is related, via the Legendre map, with the canonical symplectic form ω_Q on T^*Q. Let $\bar{M}$ denote the image of the constraint submanifold M under the Legendre transformation, and let $\bar{F}$ be the distribution on T^*Q along $\bar{M}$, whose annihilator is given by

$$\bar{F}^0 = FL_*(S^*((T\bar{M})^0))\,.$$

Observe that $\bar{F}^0$ is locally generated by the m independent 1-forms

$$\bar{\mu}^i = \frac{\partial \Psi^i}{\partial p_A}\mathcal{H}_{AB}dq^B\,,\;\; 1 \leq i \leq m\,.$$

The nonholonomic Hamilton equations for the nonholonomic system can be then rewritten in intrinsic form as

$$\left.\begin{array}{rcl} (i_X\omega_Q - dH)_{|\bar{M}} & \in & \bar{F}^0 \\ X_{|\bar{M}} & \in & T\bar{M} \end{array}\right\} \qquad (11)$$

The compatibility condition is now written as $\bar{F}^{\perp} \cap T\bar{M} = \{0\}$**, where "** $\perp$ **" denotes the symplectic complement with respect to** ω_Q**. Equivalently, the matrix**

$$(\bar{C}^{ij}) = \left(\frac{\partial \Psi^i}{\partial p_A} \mathcal{H}_{AB} \frac{\partial \Psi^j}{\partial p_B} \right) \tag{12}$$

is regular. On the Lagrangian side, the compatibility condition is locally written as

$$\det(\bar{C}^{ij}) = \det \left(\frac{\partial \phi^i}{\partial \dot{q}^A} W^{AB} \frac{\partial \phi^j}{\partial \dot{q}^B} \right) \neq 0 \ , \tag{13}$$

where W^{AB} **are the entries of the Hessian matrix** $\left(\dfrac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)_{1 \leq A,B \leq n}$**.**

The compatibility condition is not too restrictive, since it is trivially verified by the usual systems of mechanical type (lagrangian = kinetic energy - potential energy), where the $\mathcal{H}_{AB}$ represent the components of a positive definite Riemannian metric. The compatibility condition guarantees the existence of a unique solution of the constrained equations of motion (11) which, henceforth, will be denoted by $\bar{X}_{nh}$ on the Hamiltonian side and X_{nh} on the Lagrangian side. Moreover, if X_H is the Hamiltonian vector field of H $(i_{X_H}\omega_Q = dH)$ then

$$\bar{\lambda}_i = \bar{\mathcal{C}}_{ij} X_H(\Psi^j) \ . \tag{14}$$

Hamilton-Jacobi theory for nonholonomic mechanical systems

Let $L : TQ \longrightarrow \mathbb{R}$ be a lagrangian function subject to nonholonomic constraints given by a submanifold M of TQ. We assume the admissibility and compatibility conditions, and consider the hamiltonian counterpart given by a Hamiltonian function $H : T^*Q \longrightarrow \mathbb{R}$ and a constraint submanifold $\bar{M} = FL(M)$ as in the precedent sections. X_{nh} and $\bar{X}_{nh}$ will denote the corresponding nonholonomic dynamics.

Let γ be a closed 1-form on Q such that $\gamma(Q) \subset \bar{M}$. Then the following conditions are equivalent:

(i) for every curve $\sigma : \mathbb{R} \longrightarrow Q$ such that

$$\dot{\sigma}(t) = T\pi_Q(X_H(\gamma(\sigma(t)))) \tag{15}$$

for all t, then $\gamma \circ \sigma$ is an integral curve of $\bar{X}_{nh}$.

(ii) $\pi_Q^*(d(H \circ \gamma)) \in \bar{F}^0$.

Let $L : TQ \longrightarrow \mathbb{R}$ a lagrangian subject to linear constraints given by a distribution M on Q. Denote by $\bar{M} \subset T^*Q$ the image of $M \subset TQ$ by the Legendre transformation, and by h the corresponding hamiltonian function on T^*Q. In that case, we have proved the following result:

Hamilton-Jacobi Theorem

Let λ be a 1-form on Q taking values into $\bar{M}$ and satisfying $d\lambda \in \mathcal{I}(M^o)$. Then the following conditions are equivalent:

(i) $\bar{X}^{\lambda}_{nh}$ and $\bar{X}_{nh}$ are λ-related;

(ii) $d(h \circ \lambda) \in M^o$

Here, X_{nh} is the nonholonomic dynamics.

D. Iglesias, M. de León, D. Martín de Diego: Towards a Hamilton-Jacobi theory for nonholonomic mechanical systems, *Preprint* (2007).

The mobile robot with fixed orientation

The robot has three wheels with radius R, which turn simultaneously about independent axes, and perform a rolling without sliding over a horizontal floor.

Let (x, y) denotes the position of the centre of mass, θ the steering angle of the wheel, ψ the rotation angle of the wheels in their rolling motion over the floor. So, the configuration manifold is

$$Q = S^1 \times S^1 \times \mathbb{R}^2$$

The lagrangian L is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}J\dot{\theta}^2 + \frac{3}{2}J_\omega\dot{\psi}^2$$

where m is the mass, J is the moment of inertia and J_ω is the axial moment of inertia of the robot.

The constraints are induced by the conditions that the wheels roll without sliding, in the direction in which they point, and that the instantaneous contact point of the wheels with the floor have no velocity component orthogonal to that direction:

$$\begin{aligned}\dot{x}\sin\theta - \dot{y}\cos\theta &= 0,\\ \dot{x}\cos\theta + \dot{y}\sin\theta - R\dot{\psi} &= 0.\end{aligned}$$

Consider

$$\begin{aligned}\gamma_1 &= J\,d\theta\\ \gamma_2 &= 3J_\omega\,d\psi + mR\cos\theta\,dx + mR\sin\theta\,dy\end{aligned}$$

$$\begin{aligned}d\gamma_1 &= 0 \in \mathcal{I}(D^o)\\ d\gamma_2 &= -mRd\theta \wedge (\sin\theta\,dx - \cos\theta\,dy) \in \mathcal{I}(D^o)\end{aligned}$$

Observe that in both cases $d(H\circ\gamma_i) = 0$**, for** $i = 1, 2$**.**

In such a case,

$$\begin{aligned}t &\longmapsto (x_0, y_0, t+\theta_0, \psi_0)\\ t &\longmapsto (tR\cos\theta_0 + x_0, tR\sin\theta_0 + y_0, \theta_0, t+\psi_0)\end{aligned}$$

are the solutions of the nonholonomic system (L, D)**.**

$$\gamma_3 = J\, d\theta + 3J_\omega\, d\psi + mR\cos\theta\, dx + mR\sin\theta\, dy$$

and $d\gamma_3 \in \mathcal{I}(D^o)$**. In such a case, the solution of the nonholonomic problem that we obtain is**

$$t \longmapsto (R\sin(t-\theta_0) + x_0 + R\sin\theta_0, -R\cos(t-\theta_0) + y_0 + R\cos\theta_0, t+\theta_0, t+\psi_0)$$

which is a solution of the nonholonomic problem but not of the free system.

SINGULAR LAGRANGIAN SYSTEMS

Let $L : TQ \longrightarrow \mathbb{R}$ be a singular lagrangian. That is, the Hessian matrix

$$\left(W_{AB} = \frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B} \right)$$

is not regular, or, equivalently, the closed 2-form ω_L is not symplectic.

Therefore, the the equation

$$\boxed{i_X \, \omega_L = dE_L \qquad (16)}$$

has no solution in general, or the solutions are not defined everywhere.

We can apply the Dirac-Bergman theory of constraints.

We assume that L is almost regular:

- $M_1 = FL(TQ)$ **is a submanifold of T^*Q;**
- **The Legendre mapping $FL_1 : TQ \longrightarrow M_1$ is a submersion with connected fibers.**

M_1 **is the submanifold of primary constraints.**

If L is almost regular, then E_L projects onto a function

$$h_1 : M_1 \longrightarrow \mathbb{R}$$

Denote by $j_1 : M_1 \longrightarrow \mathbb{R}$ the natural inclusion and put

$$\omega_1 = j_1^*(\omega_Q)$$

Consider the equation

$$i_X\,\omega_1 = dh_1 \qquad (17)$$

There are two possibilities:

- **There is a solution** X **defined at all the points of** M_1**; such** X **is called a global dynamics and it is a solution modulo** $\ker \omega_1$**. In other words, there are only primary constraints.**
- **Otherwise, we select the submanifold** M_2 **formed by the points of** M_1 **where a solution exists. But such a solution** X **is not necessarily tangent to** M_2**, so we have to impose a tangency condition, and we obtain a new submanifold** M_3 **along it there exists a solution. Continuing this process, we obtain a chain of submanifolds**

$$\cdots M_k \hookrightarrow \cdots M_2 \hookrightarrow M_1 \hookrightarrow T^*Q$$

If the algorithm stabilizes at some k**, say** $M_{k+1} = M_k$**, then we say that** M_k **is the final constraint submanifold and then there exists a well-defined solution** X **of (17) along** M_f**.**

Case I: There is a global dynamics

In this case there exists a vector field X on M_1 such that

$$i_X \omega_1 = dh_1$$

Moreover, $\pi_1(M_1) = Q$.
Assume that γ is a closed 1-form on Q such that $\gamma(Q) \subset M_1$.
Define a vector field X^γ on Q by putting

$$\boxed{X^\gamma = T\pi_1 \circ X \circ \gamma}$$

$$\begin{array}{ccc} TM_1 & \xrightarrow{T\pi_1} & TQ \\ X \uparrow & & \uparrow X^\gamma \\ M_1 & \underset{\gamma_1}{\overset{\pi_f}{\rightleftarrows}} & Q \end{array}$$

Here γ_1 is the restriction to γ.

We have

$$
\begin{aligned}
\gamma^*(i_{X-T\gamma(X^\gamma)}\,\omega_1) &= \gamma^*(i_X\,\omega_1) - \gamma^*(i_{T\gamma(X^\gamma)}\,\omega_1)\\
&= \gamma^* d(h_1) - \gamma^*(i_{T\gamma X^\gamma}\,\omega_1)\\
&= d(h_1\circ\gamma)
\end{aligned}
$$

since $\gamma^*(i_{T\gamma X^\gamma}\,\omega_1) = i_{X^\gamma}\,(-d\gamma) = 0$**.**

Therefore, we deduce the following

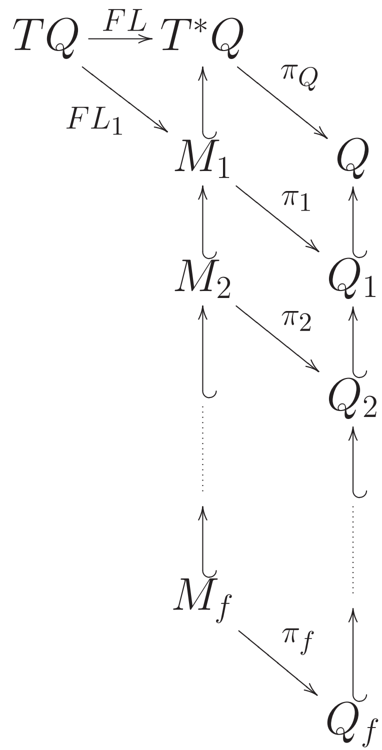
$$
X - T\gamma(X^\gamma) \in \ker\,\omega_1 \Leftrightarrow d(h_1\circ\gamma) = 0
$$

We should remark that ω_1 **(as it happens with** ω_Q**) vanishes acting on two vertical tangent vectors.**

Also, notice that even in the case when X **and** $T\gamma(X^\gamma)$ **are different, both give the solutions of the singular problem. Therefore,** γ **applies the integral curves of** X^γ **into the integral curves of** X **(the solutions of our system) with the Hamilton-Jacobi equation**

$$
d(h_1\circ\gamma) = 0
$$

Case II: There are secondary constraints



We assume that each $\pi_l : M_l \longrightarrow Q_l$ is a fibration (Q_l is assumed to be a manifold for each l).

Assume that γ is a 1-form on Q such that

- $\gamma(Q) \subset M_1$;
- $\gamma(Q_f \subset M_f$;
- $\gamma_f^* \omega_1 = 0$.

Define a vector field X^γ on Q by putting

$$\boxed{X^\gamma = T\pi_f \circ X \circ \gamma_f}$$

$$\begin{array}{ccc} TM_f & \xrightarrow{T\pi_f} & TQ_f \\ X \uparrow & & \uparrow X^\gamma \\ M_f & \underset{\gamma_f}{\overset{\pi_f}{\rightleftarrows}} & Q_f \end{array}$$

Here γ_f is the restriction to γ.

We have a solution X of the equation

$$(i_X\,\omega_1 = dh_1)_{|M_f}$$

where X is a vector field on M_f.

Proceeding as above, we have

$$\begin{aligned} \gamma_f^*(i_{X-T\gamma_f(X^\gamma)}\,\omega_1) &= \gamma_f^*(i_X\,\omega_1) - \gamma_f^*(i_{T\gamma_f(X^\gamma)}\,\omega_1) \\ &= \gamma_f^* d(h_1) - \gamma_f^*(i_{T\gamma X^\gamma}\,\omega_1) \\ &= d(h_1 \circ \gamma_f) \end{aligned}$$

since $\gamma_f^*(i_{T\gamma_f X^\gamma}\,\omega_1) = i_{X^\gamma}\,\gamma_f^*\omega_1 = 0$.

Therefore, we deduce the following

$$X - T\gamma(X^\gamma) \in \ker\,\omega_f \Leftrightarrow d(h_1 \circ \gamma) = 0$$

where γ_f is the restriction of ω_1 to M_f

But any solution of the equation $(i_X\,\omega_1 = dh_1)_{|M_f}$ is a solution of the equation $i_X\,\omega_f = dh_f$.

So,

$$\boxed{d(h_1 \circ \gamma) = 0}$$

could be still considered as the Hamilton-Jacobi equation in this context.

Vakonomic dynamics or variational nonholonomic systems

Let $L : TQ \longrightarrow \mathbb{R}$ be a lagrangian subjected to nonholonomic constraints given by a submanifold M of TQ.

M is locally defined by constraint functions

$$\Phi^i(q^A, \dot{q}^A) = 0, \; 1 = 1, \ldots, k$$

Then the vakonomic problem is equivalent to solve the Euler-Lagrange equations of the extended lagrangian

$$\mathcal{L}(q^A, \lambda_i, \dot{q}^A, \dot{\lambda}_i) = L(q^A, \dot{q}^A) + \lambda_i \, \Phi^i(q^A, \dot{q}^A)$$

Notice that $\mathcal{L}$ is singular, so that we can apply the above machinery.

Optimal Control Theory

A control system of ordinary differential equations is usually given by

$$\dot{x}^i = \Gamma^i(x(t), u(t))$$

where

- x^i, $1 \leq i \leq n$ are called the state variables
- u^a, $1 \leq a \leq m$ are called the control functions

Consider the following optimal control problem:
Given initial and final states x_0 and x_f, the objective is to find a smooth curve $c(t) = (x(t), u(t))$ such that

- $x(t_0) = x_0, x(T_f) = x_f$,
- $c(t)$ satisfies the control equation,
- and minimizes the functional

$$\mathcal{I}(c) = \int_{t_0}^{t_f} L(x(t), u(t))\, dt$$

for some cost function $L = L(x, u)$.

In geometric terms we have a control fiber bundle

$$\pi : C \longrightarrow B$$

and a vector field Γ **along** π**:**

$$\Gamma = \Gamma^i(x,u)\,\frac{\partial}{\partial x^i}$$

L **is a function** $L : C \longrightarrow \mathbb{R}$

and the optimal control problem is equivalent to a vakonomic problem given by:

- **a lagrangian L** $: TC \longrightarrow \mathbb{R}$
- **a constraint submanifold** $M = \{v \in TC \,|\, T\pi(v) = \Gamma(\tau_C(v))\}$ **of** TC**.**

Therefore, the optimal control problem is equivalent to study the singular lagrangian system defined on $T(C \times \mathbb{R}^n)$ **with a singular lagrangian function**

$$\mathcal{L}(x^i, u^a, \lambda_i, \dot{x}^i, \dot{u}^a, \dot{\lambda}_i) = L(x^i, u^a) + \lambda_i(\dot{x}^i - \Gamma^i(x,u))$$

We will apply the constraint algorithm to this lagrangian $\mathcal{L}$ where now $Q = C \times \mathbb{R}^n$.

Compute the momenta $(p_{x^i}, p_{u^a}, p_{\lambda_i})$:

$$p_{x^i} = \lambda_i, p_{u^a} = 0, p_{\lambda_i} = 0$$

equations which define M_1.

Therefore we have

$$h_1(x^i, u^a, \lambda_i) = -L(x, u) + \lambda_i \Gamma^i(x, u)$$

and

$$\omega_1 = dx^i \wedge d\lambda_i$$

where (x^i, u^a, λ_i) can be considered as local coordinates for M_1.

Consider the equation

$$i_X\,\omega_1 = dh_1 \qquad (18)$$

A generic solution on M_1 is of the form

$$X = A^i\,\frac{\partial}{\partial x^i} + B^a\,\frac{\partial}{\partial u^a} + C_i\,\frac{\partial}{\partial \lambda_i}$$

which, using (18) provides

$$A^i = \Gamma^i, C^i = \frac{\partial L}{\partial x^i} - \lambda_j\,\frac{\partial \Gamma^j}{\partial x^i}$$

so that

$$X = \Gamma^i\,\frac{\partial}{\partial x^i} + B^a\,\frac{\partial}{\partial u^a} + \left(\frac{\partial L}{\partial x^i} - \lambda_j\,\frac{\partial \Gamma^j}{\partial x^i}\right)\,\frac{\partial}{\partial \lambda_i}$$

In addition, we obtain a new constraint

$$\Psi^a = \frac{\partial L}{\partial u^a} - \lambda_j\,\frac{\partial \Gamma^j}{\partial u^a}$$

which is a secondary constraint defining M_2 inside M_1.

The tangency condition implies

$$X(\Psi^b) = 0$$

that is

$$\Gamma^i \frac{\partial \Psi^b}{\partial x^i} + B^a \frac{\partial \Psi^b}{\partial u^a} + \left(\frac{\partial L}{\partial x^i} - \lambda_j \frac{\partial \Gamma^j}{\partial x^i} \right) \frac{\partial \Psi^b}{\partial \lambda_i} = 0$$

Observe that if the following matrix

$$\left(\frac{\partial \Psi^b}{\partial u^a} \right)$$

is regular, then we can obtain the B**´s explicitly.**

In this case $M_f = M_2$**.**

Since we can obtain the u**´s as explicit functions of the rest of coordinates, say**

$$u^a = \zeta^a(x, \lambda)$$

we have local coordinates (x^i, λ^i) **on** M_2**.**

As in the general case, take γ be a 1-form on $Q = C \times \mathbb{R}^n$ such that

- $\gamma(Q_1) \subset M_1$;
- $\gamma(Q_2) \subset M_2$;
- $\gamma_2^*(\omega_Q) = 0$.

Notice that

$$h_1 \circ \gamma_2 = -L(x^i, \zeta^a(x, \lambda) + \lambda_i \, \Gamma(x^j, \zeta^a(x, \lambda))$$

so that the Hamilton-Jacobi becomes

$$-L(x^i, \zeta^a(x, \lambda) + \lambda_i \, \Gamma(x^j, \zeta^a(x, \lambda)) = cte$$

Since $\gamma_2^*(\omega_Q) = 0$ **we deduce**

$$\lambda^i = \frac{\partial W}{\partial x^i}$$

and then we obtain the Hamilton-Jacobi equation

$$\frac{\partial W}{\partial x^i} \, \Gamma(x^j, \zeta^a(x, \lambda)) - L(x^i, \zeta^a(x, \lambda)) = cte \tag{19}$$

(The Hamilton-Jacobi-Bellman (HJB) equation)

FUTURE WORK

- **Intrinsic formulation of the theory for vakonomic dynamics in a Skinner-Rusk context.**
- **Applications to optimal control theory.**