Siegel disk boundaries and hairs for irrationally indifferent fixed points

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Low-Dimensional Dynamics
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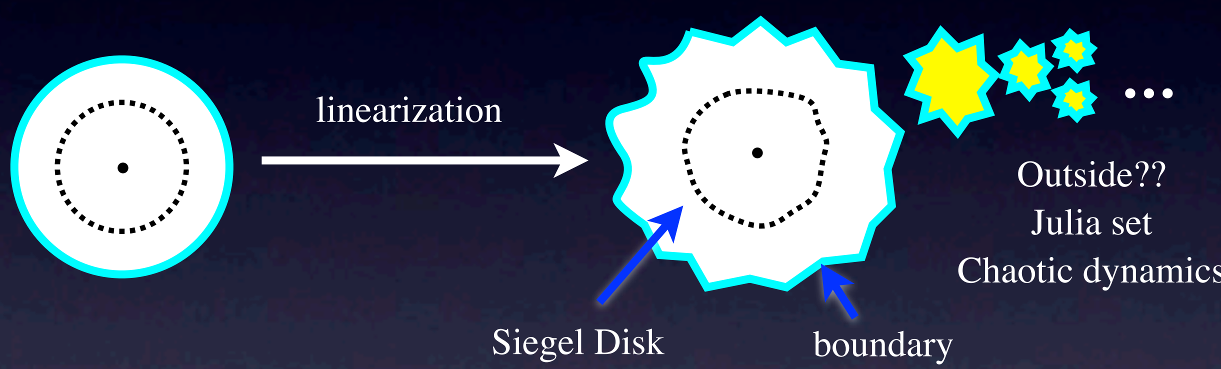
Plan

Theorem: If $f(z) = e^{2\pi i\alpha}h(z)$ such that $\alpha \in Irrat_N$ (high type) and is a Brjuno number and $h \in \mathcal{F}_1$, then the boundary of the Siegel disk of f is a Jordan curve.

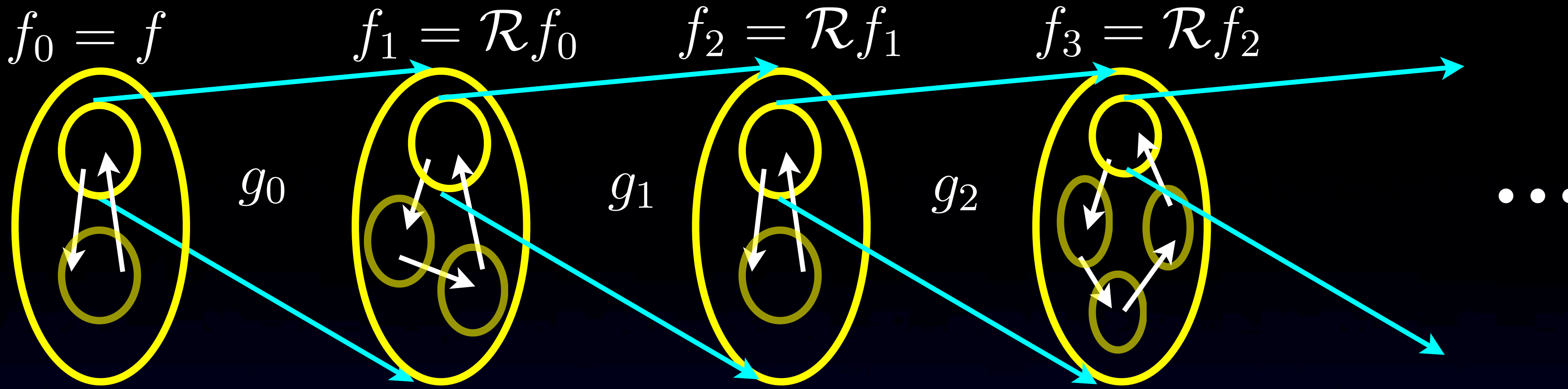
Earlier results by Herman, Petersen-Zackeri (surgery), Buff-Chéritat (function spaces)



In order to study "tame" dynamics, take advantage of "expanding" (chaotic) dynamics as *renormalizers*.

How to recover the original dynamics from its renormalizations in the case of rotation-like dynamics (overlapping domains).
-- solution via *Dynamical Charts*

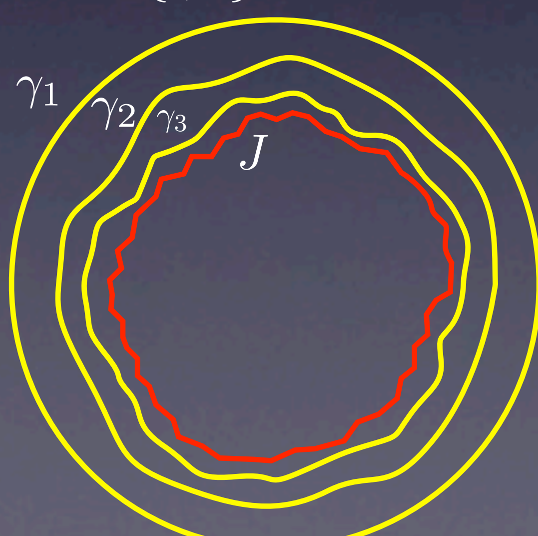
Infinitely renormalizable map



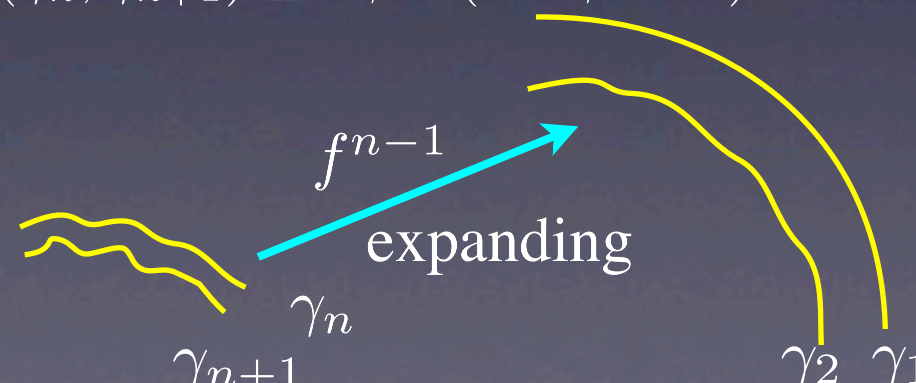
Tame maps f_n are maps *to be renormalized*, and expanding maps g_n are those which *renormalize* f_n's. Take advantage of expansion of g_n (i.e. contraction by local inverse g_n^{-1}), to study the dynamics of f_n's.

How to use Exapansion: Example. $J(z^2 + \varepsilon)$ is a Jordan curve.

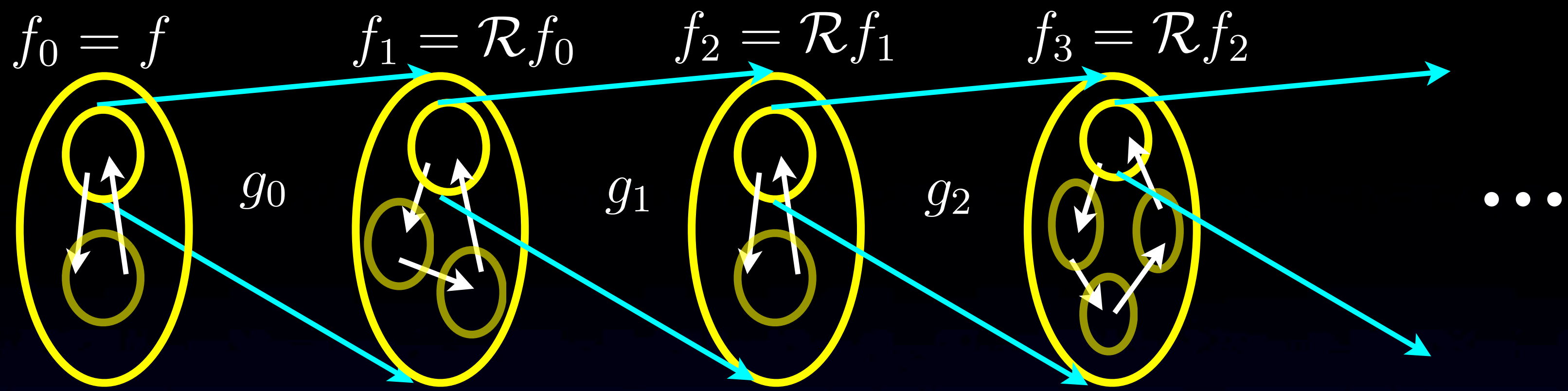
- Construct approximating curves γ_n
- Show that $\{\gamma_n\}$ is a Cauchy sequence and the limit is the object we want



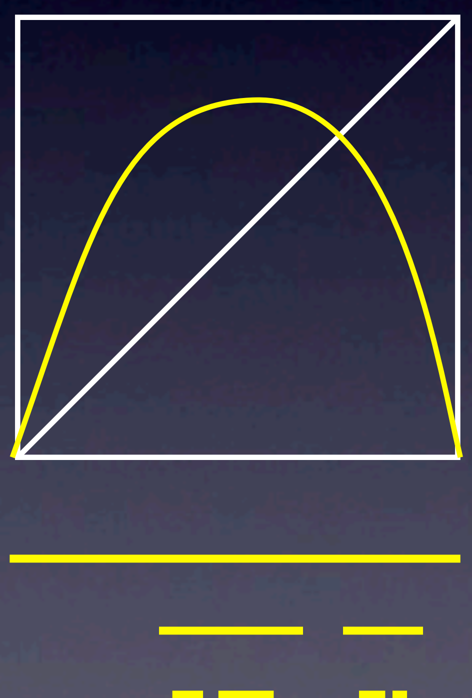
$$d(\gamma_n, \gamma_{n+1}) \le C\beta^n \ (0 < \beta < 1)$$



How to recover f from $\mathcal{R}^n f$'s

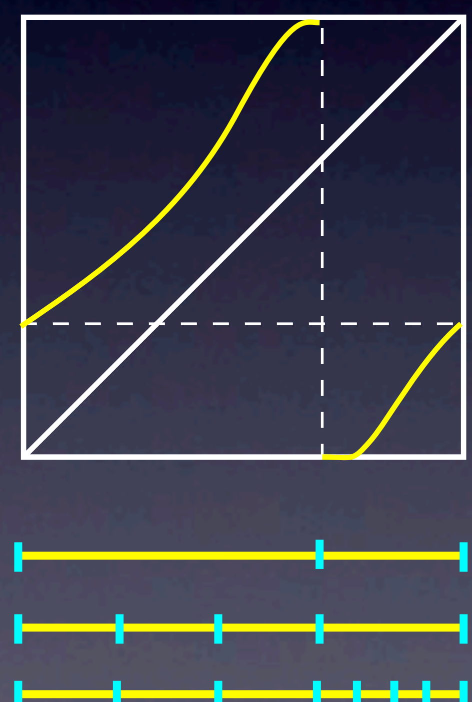


Feigenbaum



proper subintervals

Circle map



partition of interval

commuting pairs

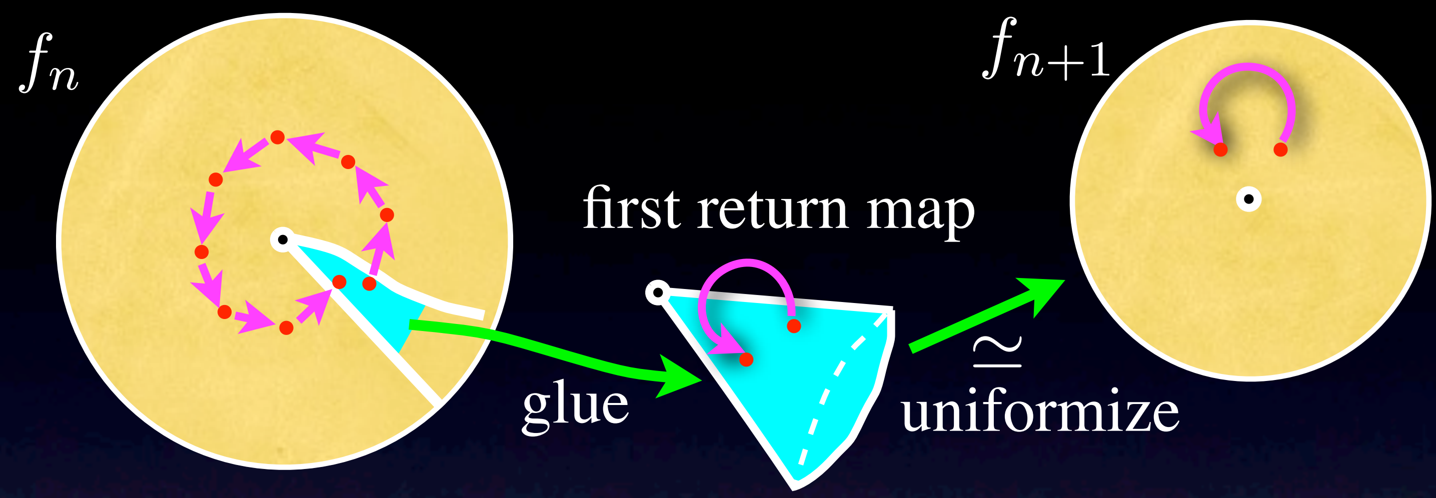
Near-parabolic

covering by sector or croissant-like domains

gluing/identification needed to define the renormalization

Yoccoz renormalization for Siegel-Bruno Theorem

$$f_n(z) = e^{2\pi i \alpha_n} z + \dots \rightsquigarrow f_{n+1}(z) = e^{2\pi i \alpha_{n+1}} z + \dots$$



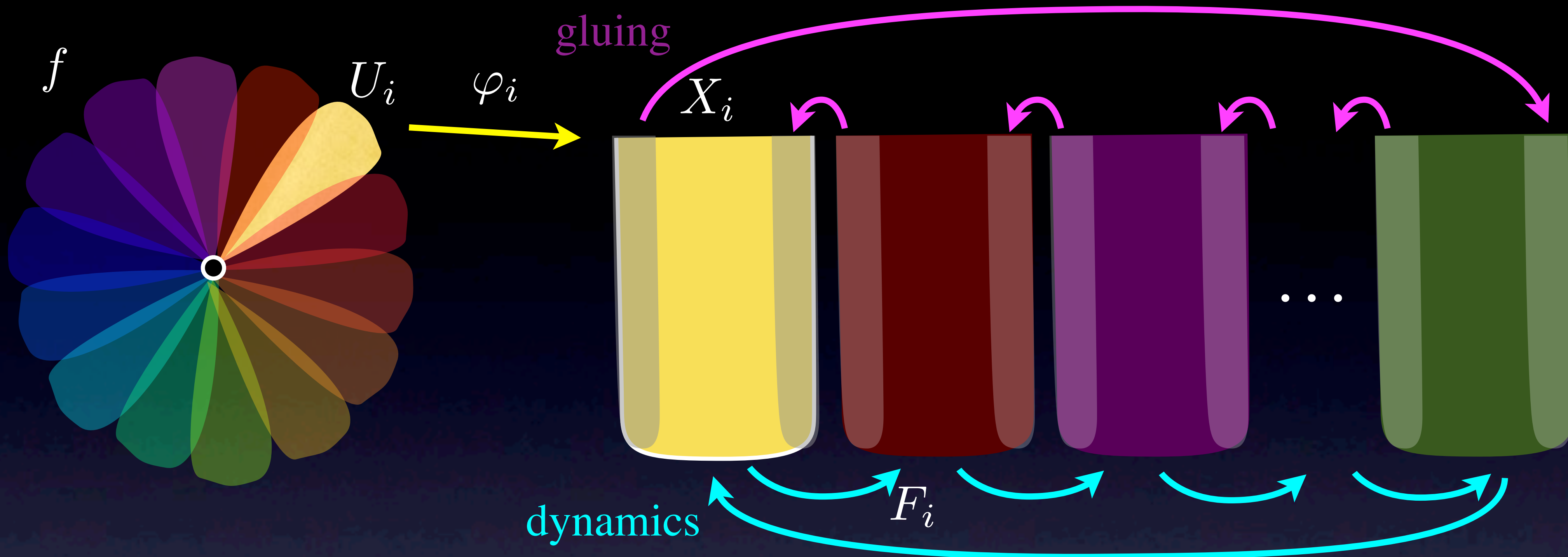
The first return time and the first return map are discontinuous on the sector. The first return map becomes continuous after gluing.

It is better to think that we are taking the quotient of an open set containing the sector. The punctured neighborhood is covered by the iterates of the open set.

Due to the overlapping, there are problems when we want to recover f from the renormalization.

For example, if the renormalization has a fixed point, what is the period for the corresponding periodic point for f?

Dynamical charts for irrational rotation-like dynamics



Open sets U_i $(i \in I)$, which cover a punctured nbd of fixed pt

Maps $\varphi_i : U_i \to X$, a small number of model spaces

A small number of model dynamics $F = \varphi_{r(i)} \circ f \circ \varphi_i^{-1}$
(for example, F_{can} and id)

Index set I with induced dynamics $r : I \to I$
(represents the combinatorics of the dynamics)

Gluing $\varphi_{i,j} = \varphi_i \circ \varphi_j^{-1}$ on overlaps, compatible with model dynamics
(absorbs the difference for particular maps)

Special for irrational dynamics: Refinements
one system of charts $\longrightarrow$ a new system of refined charts

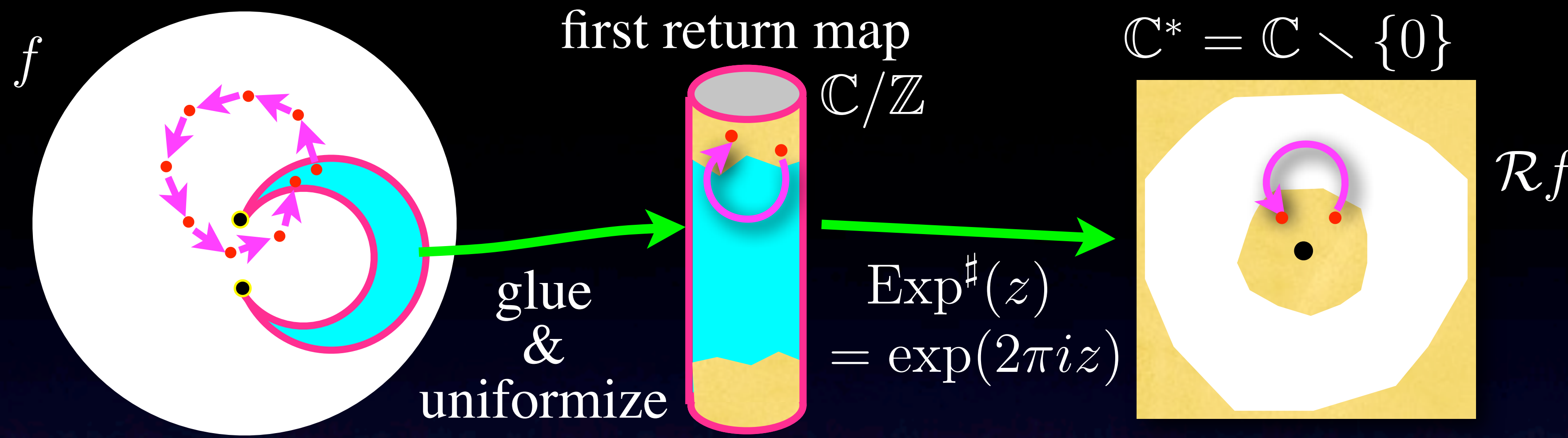
Convention

$z \to 0$ in U_i

$\Updownarrow$

$Im\, w \to +\infty$ in X_i

Cylinder/Near-parabolic renormalization



Theorem (Inou & S.): For some V and N, the near-parabolic renormalization $\mathcal{R}$ from

$$\{e^{2\pi i\alpha} f : \alpha \in Irrat_N, f \in \mathcal{F}_1\} = Irrat_N \times \mathcal{F}_1$$

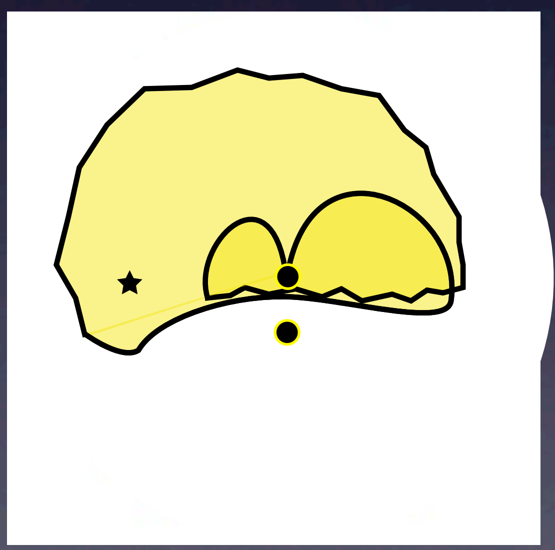
is well defined and expanding along α direction and uniformly contracting along $\mathcal{F}_1$ direction. Moreover $\mathcal{R}(e^{2\pi i\alpha}z + z^2)$ belong to the above set for $\alpha \in Irrat_N$.

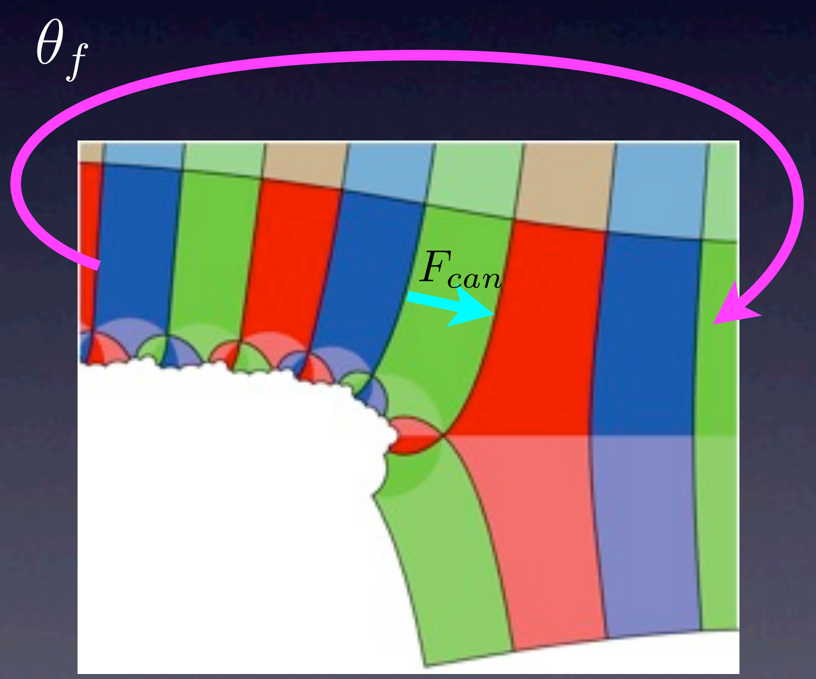
If $f(z) = e^{2\pi i\alpha}z + z^2$ with α irrational of high type, then $\mathcal{R}f, \mathcal{R}^2 f, \mathcal{R}^3 f, \ldots$ are defined and in a nice class.

A Consequence from the proof of Inou-S. theorem

There exist a canonical domain Ω_{can} and a canonical dynamics $F_{can} : \Omega_{can} \to \Omega_{can}$ such that $f, \mathcal{R}f, \mathcal{R}^2 f, \ldots$ admit the following description:

A punctured neighborhood is isomorphic to Ω_{can} cut off at a certain width ($\sim \frac{4}{3|\alpha|}$) with left and right edges glued by a homeomorphism θ_f. The dynamics is induced by F_{can}.



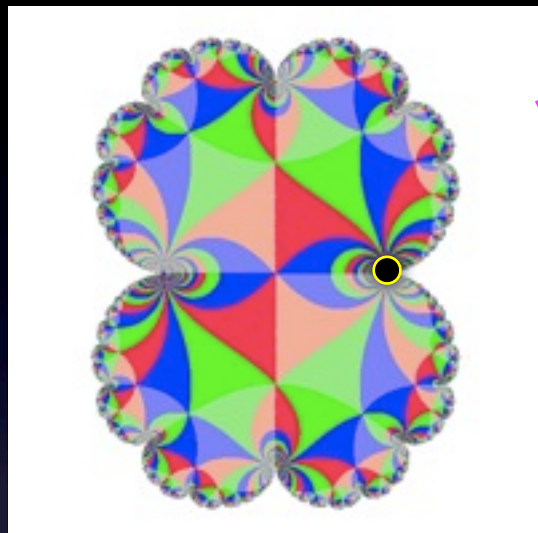


f near 0 ⟷ Model space Ω_{can} and model map F_{can}
modulo gluing

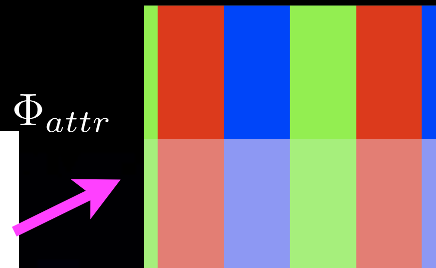
Model space and map: Checkerboard pattern

parabolic fixed point

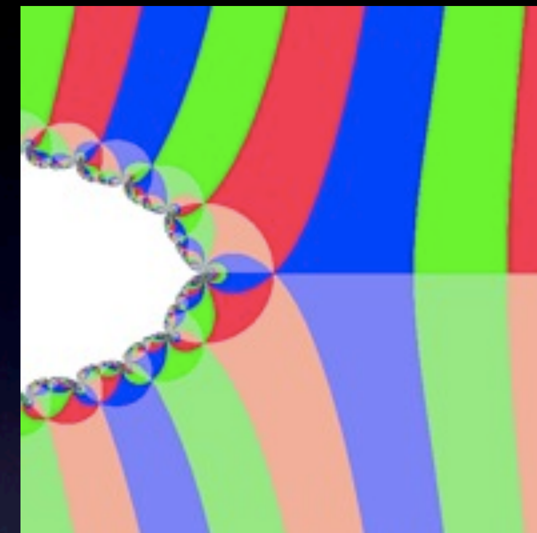
$z + z^2$



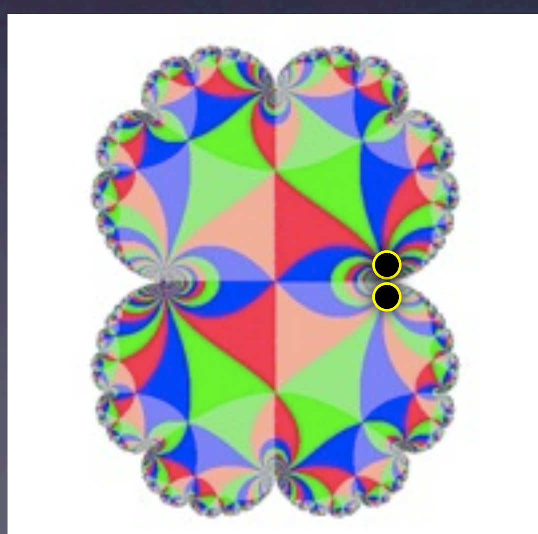
Φ_{attr}



F_{can} $(\infty = \text{fixed pt})$



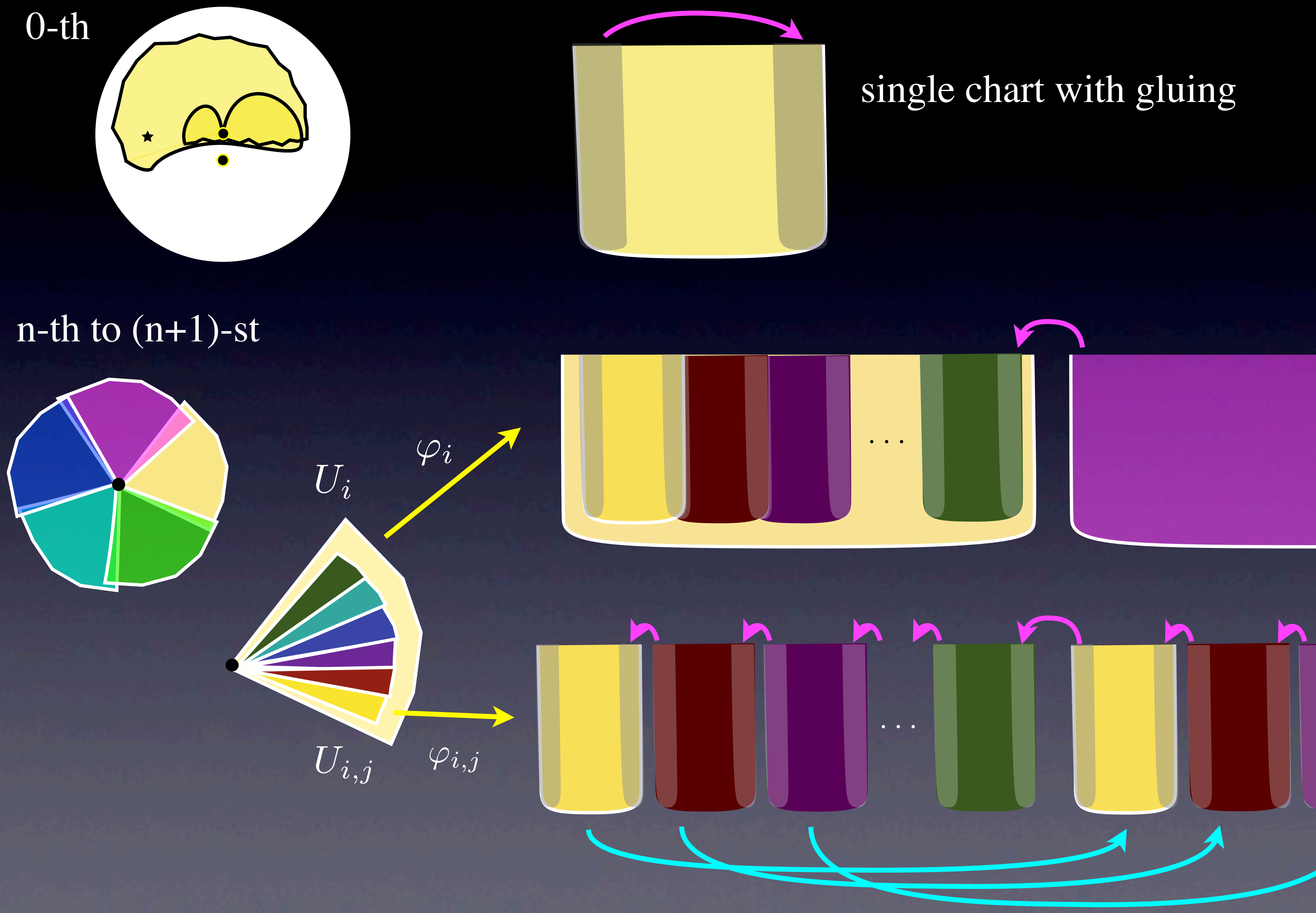
irrat. indiff. (near-parabolic)



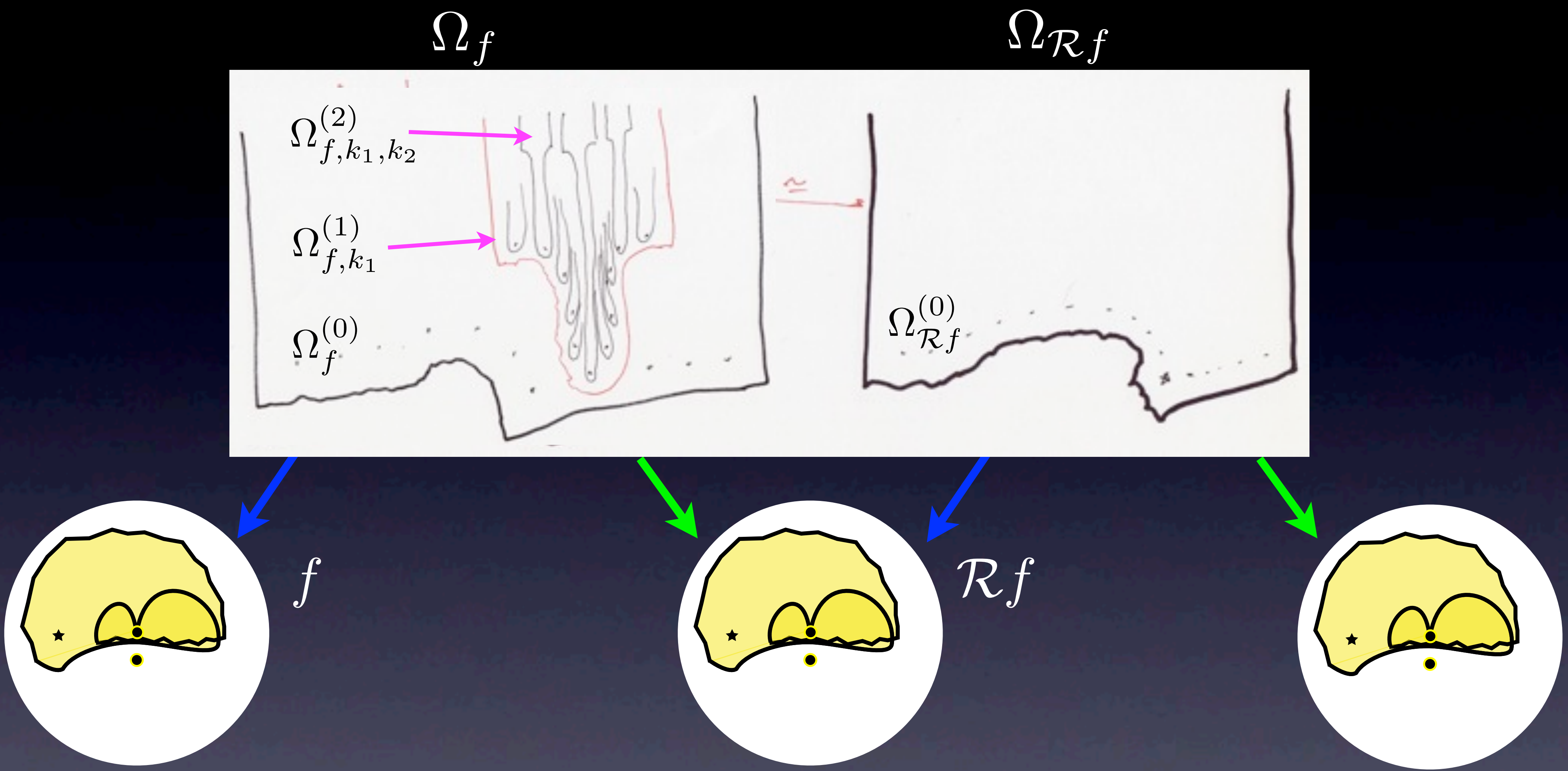
Need to cut off at
a certain width and
glue left and right edges

θ_f

Successive construction (refinement of dynamical charts)



Construction of successive charts

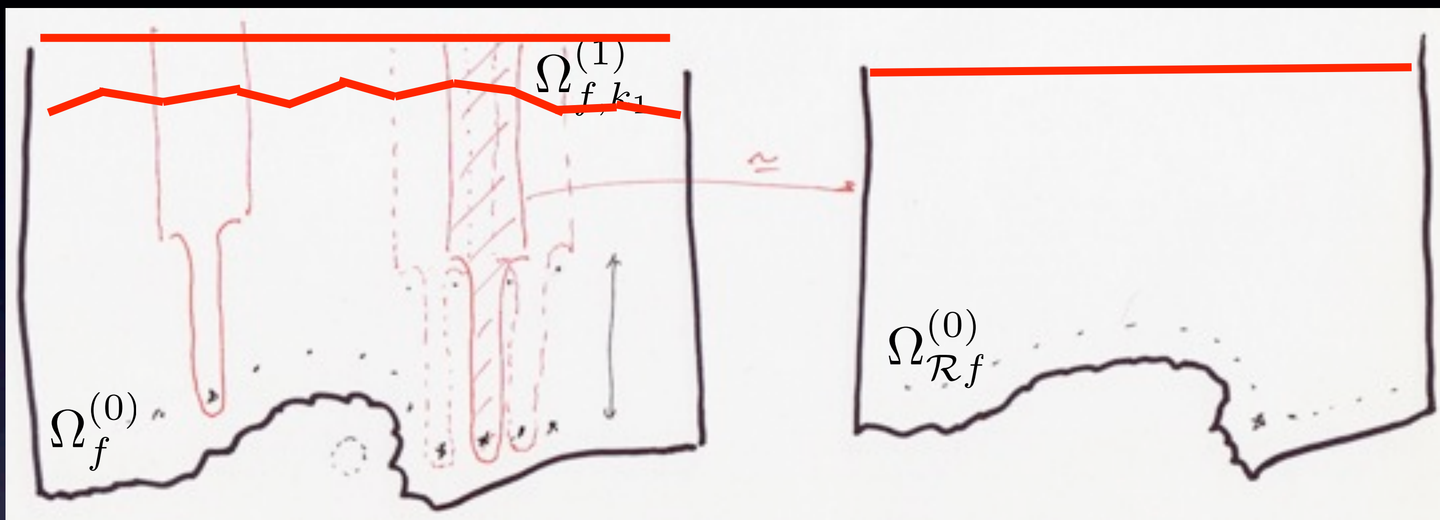


$$\Omega^{(0)} \supset \Omega^{(1)}_{k_1} \supset \cdots \supset \Omega^{(n)}_{k_1,k_2,\ldots,k_n} \supset \Omega^{(n+1)}_{k_1,k_2,\ldots,k_n,k_{n+1}} \supset \cdots$$

each $\Omega^{(n)}_{k_1,k_2,\ldots,k_n}$ is isomorphic to truncated checkerboard pattern $\Omega_{\mathcal{R}^n f}$ they are glued via $\theta_{\mathcal{R}^n f}$

Theorem: If $f(z) = e^{2\pi i\alpha} z + z^2$ with α of sufficiently high type and Brjuno, then the boundary of Siegel disk is a Jordan curve.

Idea of the proof: Construct approximating curves and show that they converge.

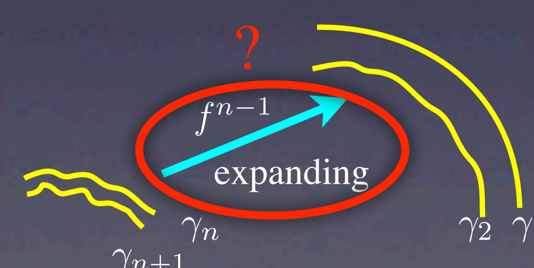


approximation of bdry of Siegel disk

Construct approximate boundary curves by joining segments in $\Omega^{(n)}_{f,k_1,\ldots,k_n}$ in its canonical coordinate. They converge exponentially.

height of the approx. curve: $h_n = B(\alpha_n)$

mapping $\Omega^{(1)}_{f,k_1} \to \Omega^{(0)}_{\mathcal{R}f}$ is uniformly expanding (with respect to Poincaré metrics of $\Omega^{(0)}_{f}$ and $\Omega^{(0)}_{\mathcal{R}f}$)



One only needs to work in model space (i.e. truncated checkerboard pattern)

Thank you!