Finding the optimal strategy in a tally game
CODY2010, November 2010

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Generalized Tally Game

1. I choose probability measure μ on $[0, 1]$ with $\mathbb{E} = \frac{1}{2}$.
2. My opponent **knowing** my μ chooses $x \in \mathbb{R}_0^+$.
3. We draw s according to μ. If $s \geq x$ I pay x.
4. Otherwise $x \to x - s$ and we go back to step 3.

Obviously, I want to chose μ in such a way that given my opponents best strategy I want to minimize the expected payoff.

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Joint Replenishment Problem with Time Windows

This is one of the fundamental problems in Inventory and Supply Chain Management.

- We are given a warehouse and the set of retailers $\{1, \ldots, n\}$.
- We are given the discrete time horizon $\{1, \ldots, T\}$.
- We are given a set of demands $\mathcal{D}$ consisting of triples (i, r, t) for retailer of type i that arrives at time t and must be satisfied by an order placed in the time interval $[r, t]$.
- To satisfy arbitrary many demands in some time period τ a retailer i places an order at warehouse at time τ and incurs the retailer ordering cost K_i, at the same time the warehouse places an order and incurs the warehouse ordering cost of K_0. The goal is define the set set of warehouse and retailer orders to satisfy all the demand.

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IP Formulation & LP Relaxation

- $$\min \sum_{\tau=1}^{T} K_0 x_{\tau 0} + \sum_{i=1}^{n} \sum_{\tau=1}^{T} K_i x_{\tau i}, \tag{1.1}$$
  $$\sum_{\tau=r}^{t} x_{\tau i} \geq 1, \qquad \forall (i,r,t) \in \mathcal{D}, \tag{1.2}$$
  $$x_{\tau i} \leq x_{\tau 0} \qquad \forall i, \tau, \tag{1.3}$$
  $$x_{\tau 0}, x_{\tau i} \in \{0,1\}, \qquad \forall \tau, i. \tag{1.4}$$
- We relax the integrality condition (1.4) with the condition $x_{\tau 0}, x_{\tau i} \in [0,1]$ for all τ, i and solve the resulting linear programming relaxation using any efficient algorithm (interior points, ellipsoid method). Let x^* be an optimal solution for the linear relaxation.

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Rounding Algorithm

Consider the following rounding algorithm that finds an integral solution for our optimization problem:

1. Define intervals $I_t = (\sum_{\tau=1}^{t-1} x_{\tau 0}^*, \sum_{\tau=1}^{t} x_{\tau 0}^*]$ for time t and intervals $I_{ti} = (\sum_{\tau=1}^{t-1} x_{\tau i}^*, \sum_{\tau=1}^{t} x_{\tau i}^*]$ for time t and retailer i.
2. Consecutively draw random variable d_i from the random distribution $f(x)$. Define $D_0 = 0$ and $D_i = D_{i-1} + d_i$.
3. Let Λ be the set of times t such that there is an index i such that $D_i \in I_t$. Open warehouse orders at all times from Λ.
4. For each retailer independently apply the following process. Initialize $y = 0$. Open retailer i order at the latest time t'' from Λ such that $t'' \leq t'$ where $y + 1 \in I_{t'i}$. Set $y = \sum_{\tau=1}^{t} x_{\tau i}^*$ and repeat the process until $y + 1 > \sum_{\tau=1}^{T} x_{\tau i}^*$.

Sketch of the Analysis

It is not hard to show that this algorithm finds a feasible solution. The expected cost of this solution is upper bounded by

$$\frac{W_1}{1-\rho(f)} + \frac{W_2}{\alpha}$$

where $W_1 = \sum_{i=1}^{n} \sum_{\tau=1}^{T} K_i x^*_{\tau i}$ and $W_2 = \sum_{\tau=1}^{T} K_0 x^*_{\tau i}$, i.e. $W_1 + W_2$ is the optimal cost of linear programming relaxation.

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- We have

$$E(x,\mu) = \int_{[0,x)} E(x-s,\mu)d\mu(s) + x\mu([x,1])$$

- and our objective is:

$$\arg\min_{\mu} \max_{x} E(x,\mu)$$

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Best known results so far

Approximation Algorithm for Joint Replenishment with Deadlines, Tim Nonner, Alexander Souza
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- Let μ_0 have a density

$$\begin{cases} x \mapsto 4x & \text{for} \quad x \in [0, 1/2] \\ x \mapsto 2 - 4x & \text{for} \quad x \in [1/2, 1]. \end{cases}$$

Then $E(\bar{x}(\mu_0), \mu_0) \approx 0.327$. (exact)

- Let μ_1 have a density

$$\begin{cases} x \mapsto 12x^2 & \text{for} \quad x \in [0, 1/2] \\ x \mapsto 12(1 - x^2) & \text{for} \quad x \in [1/2, 1]. \end{cases}$$

Then $E(\bar{x}(\mu_1), \mu_1) \approx 0.327 - \epsilon$ (numerically).

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Growth of $x \mapsto E(x, \mu)$ bounded by 1

The analytic approach:

Lemma

Let $\mu \in \mathcal{M}$ and $h(x)$ be a positive, measurable function, so $h(x) \geq 0$. Define $H : [0, \infty) \to \mathbb{R}$ by $H(x) := h(x) + H * \mu(x)$; Then H is a positive function.

- write $G(x) = x - E(x) = x\mu([0, x)) - E * \mu(x)$;
- then $G(x) = \int_{[0,x)} y d\mu(y) + G * \mu(x)$, so $G \geq 0$.
- Now put $H_y(x) = G(x + y) - G(x) = y + E(x) - E(x + y)$.
- Then $H_y(x) = \int_{[x,x+y)} (G(x + y - s) + s) d\mu(s) + H_y * \mu(x)$;
- Conclusion: $E(x + y) \leq E(x) + y$.

Existence and left-continuity of $x \mapsto E(x, \mu)$

The series of convolutions approach:

given g, write $g * \mu(x) = \int_{[0,x)} g(x - z) d\mu(z)$;

$$\begin{aligned} E(x,\mu) &= f(x) + E * \mu(x) \\ &= f(x) + f * \mu(x) + (E * \mu) * \mu(x) \\ &= f(x) + f * \mu(x) + (f * \mu) * \mu(x) + \cdots \\ &= f(x) + \sum_{j=1}^{\infty} f(*\mu)^j(x) \end{aligned}$$

Lemma

If g is left-continuous then $g * \mu$ is left-continuous.

- f is left-cs, so $f * \mu, f(*\mu)^2, f(*\mu)^3 \ldots$ are all left-cs;
- just need to show that the higher terms are small:
- $f(*\mu)^k(x) = \int_{0 \le \sum z_i < x} f(x - (z_1 + \cdots + z_k)) d\mu^k(z_1, \ldots, z_k)$;
- $\mu^k(\sum z_i < x)$ is exponentially small, by large deviations.

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We know now that $E(x+y) \leq E(x) + y$ and E is left-continuous.

Lemma

The maximum of $x \mapsto E(x, \mu)$ is realised.

Suppose $\mu_n \to \mu$. How does $f_n(x) := x\mu_n([x, 1])$ behave?.

- $\limsup_{\varepsilon \to 0} \lim_{\mu_n \to \mu} f_n(x - \varepsilon) = f(x, \mu)$.
- OR, for all $\varepsilon > 0$, there exists $\delta_0 < y$, and if $0 < \delta < \delta_0$, there is N, breathe deeply, if $n \geq N$, for all $\alpha \in [\delta, \delta_0]$

$$|f(y) - f_n(y - \alpha)| \leq \varepsilon.$$

- "left-convergence"

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If g_n left-converges to g and $\mu_n \to \mu$ then $g_n * \mu_n$ left-converges to $g * \mu$.

Theorem

If μ_n converges to μ then $E(\cdot, \mu_n)$ left-converges to $E(\cdot, \mu)$.

By Lemma, left-convergence of $f_n(*\mu_n)^j \to f(*\mu)^j$ holds for all j.

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Theorem

$\mu \mapsto \max_x E(x, \mu)$ is continuous, for probability measures μ supported on $[0, 1]$.

- By left-convergence, given $\varepsilon > 0$, for large n there is a point x_n such that $|E(x_n, \mu_n) - E(x, \mu)| < \varepsilon$;
- so $\lim \max_y E(y, \mu_n) \geq \max_y E(y, \mu)$;
- Let x_n maximise $E(\cdot, \mu_n)$ and $x_n \to x$ (subsequence). $E(x_n - \varepsilon, \mu_n) \geq E(x_n, \mu_n) - \varepsilon$. Take $lim_{\varepsilon \to 0} \lim_{n \to \infty}$:
- Get $E(x, \mu) \geq \lim E(x_n, \mu_n)$, as required.

Corollary (Optimal measure exists)

There exists a measure μ_0 minimising $\max_x E(x, \mu)$ over all probability measures with support on $[0, 1]$ and expected value $1/2$.

Local search

We consider discrete measures supported on points $\frac{k}{n}$, $k = 1, \ldots, n$.
We start with a measure μ with $\mu([0,1]) = 1$ and $\mathbb{E}(\mu) = \frac{1}{2}$.

1. We chose $0 < k_1 < k_2 < k_3 \leq n$ at random.
2. We construct a unique measure ν supported on $\{\frac{k_1}{n}, \frac{k_2}{n}, \frac{k_3}{n}\}$ with $\nu([0,1]) = 0$ and $\mathbb{E}(\nu) = 0$.
3. We try to find t such that $\mu + t \cdot \nu$ is a measure (no negative values allowed!) and that $\sup_x E(x, \mu + t \cdot \nu) < \sup_x E(x, \mu)$.
4. If we succeed: we replace μ with $\mu + t \cdot \nu$.

Rinse & repeat

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1. We chose $0 < k_1 < k_2 < k_3 \leq n$ at random.
2. We construct a unique measure ν supported on $\{\frac{k_1}{n}, \frac{k_2}{n}, \frac{k_3}{n}\}$ with $\nu([0,1]) = 0$ and $\mathbb{E}(\nu) = 0$.
3. We try to find t such that $\mu + t \cdot \nu$ is a measure (no negative values allowed!) and that $\sup_x E(x, \mu + t \cdot \nu) < \sup_x E(x, \mu)$.
4. If we succeed: we replace μ with $\mu + t \cdot \nu$.

Rinse & repeat

Local search

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Local Search in action

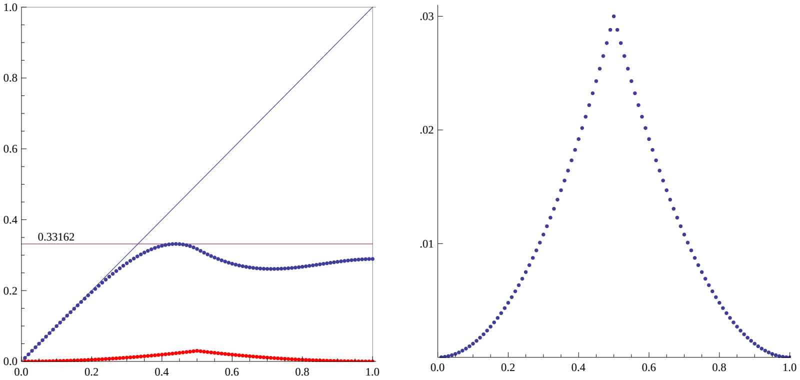

Figure: Starting measure - "two parabolas"

Local Search in action

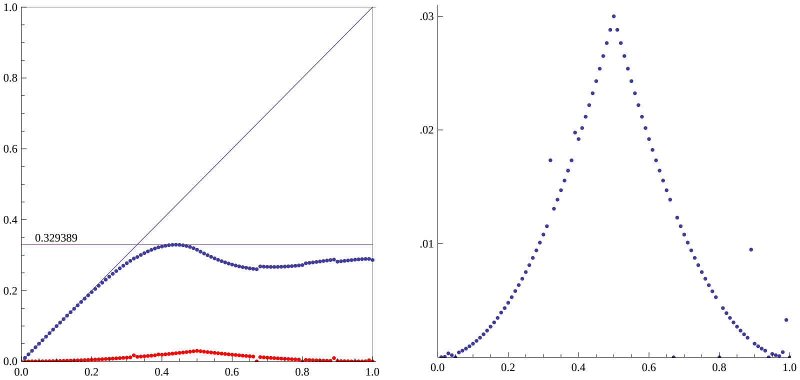

Figure: After 5 steps of optimization

Local Search in action

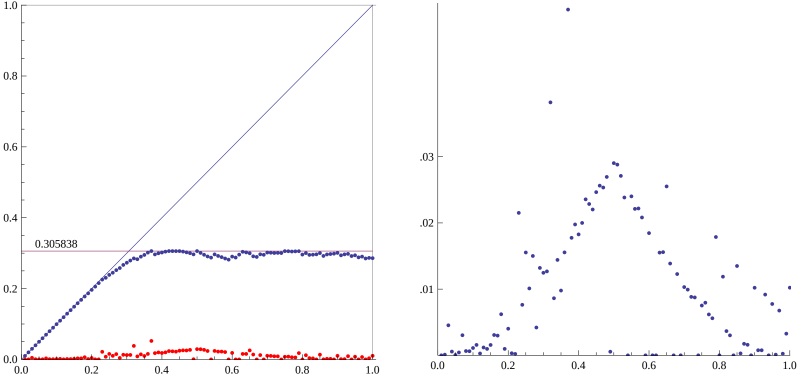

Figure: After 15 steps of optimization

Local Search in action

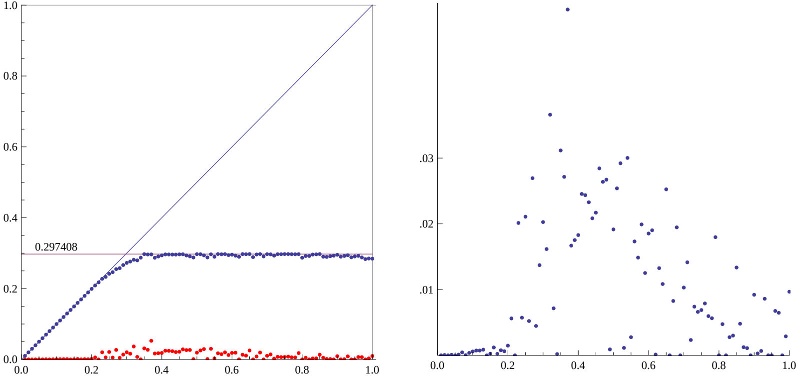

Figure: After $\sim$ 100 steps of optimization

Local Search in action

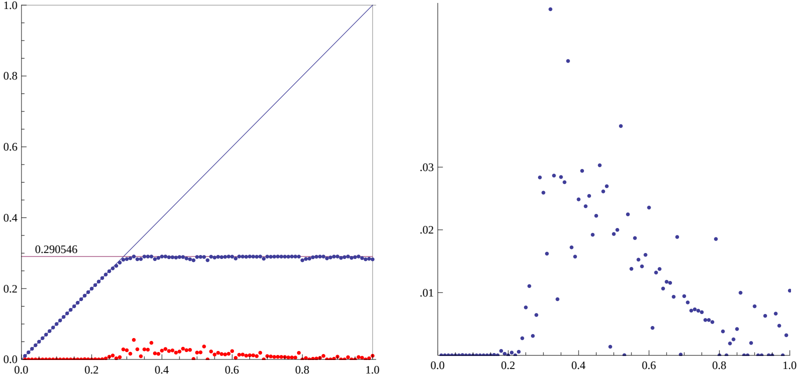

Figure: After $\sim$ 200 steps of optimization

Local Search in action

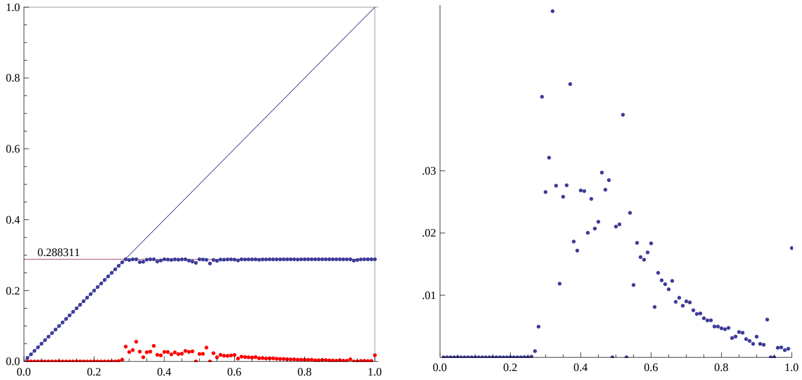

Figure: After $\gg$ 1000 steps of optimization

So...

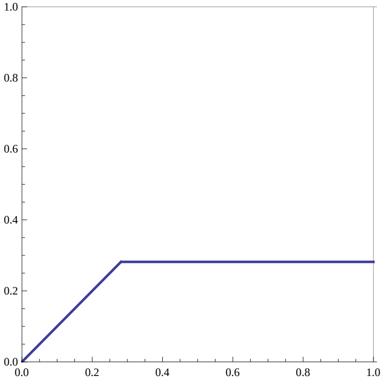

Figure: A payoff function for optimal measure?

Indeed

A measure with density function

$$g(x) = \begin{cases} 0 & \text{for} \quad x \in [0, h) \\ \frac{1}{x} & \text{for} \quad x \in [h, 2h) \\ \frac{1}{x}\left(1 - \ln\left(\frac{x}{h} - 1\right)\right) & \text{for} \quad x \in [2h, 3h) \\ \frac{\mathrm{Li}_2\left(2 - \frac{x}{h}\right) + \ln\left(\frac{x}{h} - 2\right)\ln\left(\frac{x}{h} - 1\right) - \ln\left(\frac{x}{h} - 1\right) + \frac{\pi^2}{12} + 1}{x} & \text{for} \quad x \in [3h, 4h) \\ \dots \end{cases}$$

has expected payoff function

$$\min(x, h).$$

Some calculations

- Recall:

$$\min(x,h) = \int_{[0,x)} \min(x-s,h)d\mu(s) + x\mu((x,+\infty))$$

So $\mu|_{(0,h)} = 0$ and

$$h = \int_{(x-h,x)} (x-s)d\mu(s) + h\mu([0,x-h) + x\mu((x,+\infty)).$$

- Differentiating by x we get

$$xg(x) = \mu([x-h,+\infty))$$

which we recursively solve.

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which we recursively solve.

Conjecture

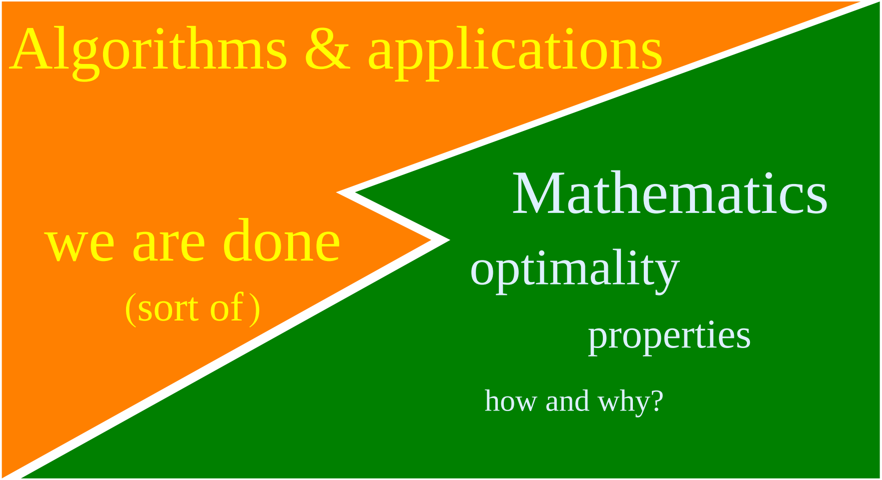
The function

$$\min(x, h_0),$$

where $h_0 = 0.28166214011768503`$ is the optimal payoff function in Generalized Tally Game.

$$0.28166214011768503` \ll 0.327 - \epsilon$$

What next?


Algorithms & applications
we are done
(sort of)
Mathematics
optimality
properties
how and why?