Dynamical properties of entire functions of regular growth

Note Title 03/11/2010

Joint work with Gwyneth Stallard

f is a transcendental entire function

$$M(r) = M(r,f) = \max_{|z|=r} |f(z)|$$

Fatou set $\quad F(f) = \{ z : (f^n) \text{ a normal family near } z \}$

Julia set $\quad J(f) = \mathbb{C} \setminus F(f)$

Baker's conjecture Let f be a t.e.f. If

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} < \frac{1}{2},$$

then $F(f)$ has no unbounded Fatou components.

Partial results: Baker, Stallard, Anderson + Hinkkanen,

Zheng, Singh, R+S, Hinkkanen + Miles

Escaping set $\quad I(f) = \{ z : f^n(z) \to \infty \}$

Eremenko's conjecture Let f be a t.e.f. Then all the components of $I(f)$ are unbounded.

Partial results: Eremenko, Schleicher + Zimmer, R+S,

Barański, Rottenfusser, Rückert, Rempe + Schleicher, Rempe.

The fast escaping set and spiders' webs

Fast escaping set (Bergweiler + Hinkkanen 1999)

$$A(f) = \{z : \exists L \in \mathbb{N} \text{ s.t. } |f^{n+L}(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\},$$

where R so large that $M^n(R) \to \infty$.

> **Theorem 1** (R+S, 2005) Let f be a t.e.f. Then all the components of $A(f)$ are unbounded.

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\}$$

$$A(f) = \bigcup_{L=0}^{\infty} f^{-L}(A_R(f)),$$ increasing union of closed sets.

> E is a *spider's web* if E is connected and
>
> $$E \supset \bigcup_{n=1}^{\infty} \partial G_n, \quad G_n \text{ nested s-conn domains}, \quad \bigcup_{n=1}^{\infty} G_n = \mathbb{C}.$$

If $A_R(f)$ is a spider's web, then

- all components of $F(f)$ are bounded (Baker's conjecture)
- $A(f)$ and $I(f)$ are both connected (Eremenko's conjecture)

$\vdots$

When is $A_R(f)$ a spider's web?

Theorem 2 (R+S, 2005) If f has a multiply connected Fatou component U, then $\overline{U} \subset A(f)$ and $A_R(f)$ is a spider's web.

For which other functions is $A_R(f)$ a spider's web?

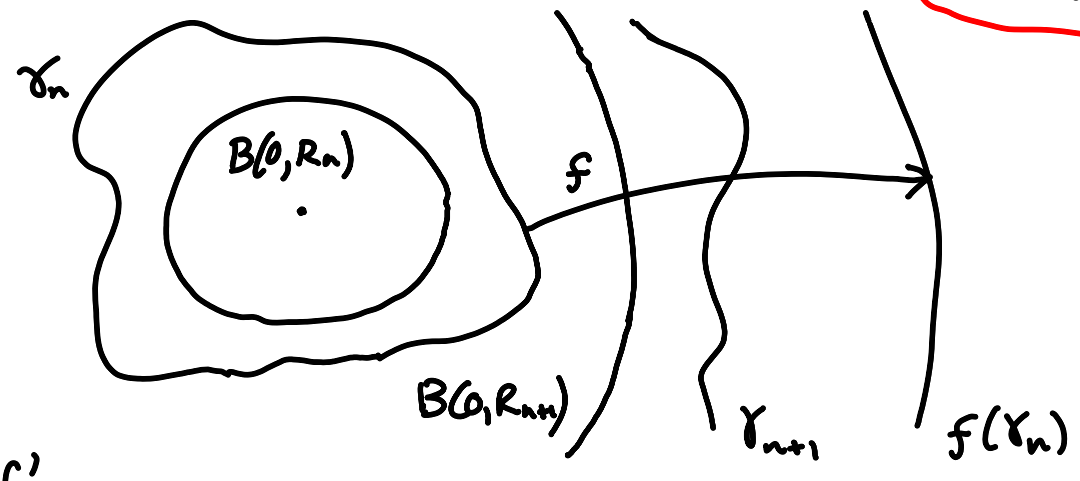
Theorem 3 (R+S, 2010) If there exists a sequence of Jordan curves γ_n such that, for $n \in \mathbb{N}$,

- γ_n surrounds $\{|z| \le R_n\}$
- $f(\gamma_n)$ surrounds γ_{n+1},

then $A_R(f)$ is a spider's web.

$R_n = M^n(R)$

Idea comes from work on Baker's conjecture.



'Proof'

- by definition $\{|z| < R\} \subset A_R(f)^c$
- show that the component of $A_R(f)^c$ which contains $\{|z| < R\}$ does not cross γ_0.

Two questions

1. When can you apply Theorem 3?

2. How do you exclude m-conn Fatou components?

Theorem 4 (R+S, 2010) Let f be a t.e.f. If $\exists\, m > 1$ s.t.

(a) $\exists\, R_0 > 0$ s.t. $\forall\, r \geq R_0$

$$\exists\, \rho \in (r, r^m) \quad \text{s.t.} \quad \min_{|z|=\rho} |f(z)| \geq M(r)$$

minimum modulus condition

(b) $\exists\, r_n,\ n \geq 0$, s.t.

$$r_n \geq M^n(R) \quad \text{and} \quad M(r_n) \geq r_{n+1}^m,$$

regular growth condition

then $A_R(f)$ is a spider's web.

Proof Wlog $r_n > R_0$, so $\exists\, \rho_n \in (r_n, r_n^m)$ s.t.

$$\min_{|z|=\rho_n} |f(z)| \geq M(r_n) \geq r_{n+1}^m \geq \rho_{n+1}.$$

Thus

$$f \text{ maps } \{|z| = \rho_n\} \text{ outside } \{|z| = \rho_{n+1}\}$$

and

$$\gamma_n = \{|z| = \rho_n\} \text{ surrounds } B(0, M^n(R)). \qquad \blacksquare$$

General minimum modulus condition – used by

Mihaljević-Brandt + Peter, and Sixsmith:

(a) $\exists\, R_0 > 0$ s.t. $\forall\, r \geq R_0$ $\exists$ a s-conn domain $G = G(r)$ s.t.

$$B(0,r) \subset G \subset B(0, r^m) \quad \text{and} \quad \min_{z \in \partial G} |f(z)| \geq M(r)$$

Regularity conditions

ψ-regularity A t.e.f. f is ψ-regular if, for all $m>1$, $\exists$ an increasing function ψ_m and $R_0>0$ s.t for $r \geq R_0$

$$\psi_m(r) \geq r \quad \text{and} \quad M(\psi_m(r)) \geq \psi_m(M(r))^m .$$

This implies Theorem 4(b) holds, for all $m>1$:

R+S, 2009

Take $r_n = \psi_m(M^n(R))$, so

$$r_n \geq M^n(R) \quad \text{and} \quad M(r_n) = M(\psi_m(M^n(R))) \geq \psi_m(M^{n+1}(R))^m .$$

log-regularity A t.e.f. is log-regular if $\exists\, c>0$ s.t.

$$\frac{\varphi'(t)}{\varphi(t)} \geq \frac{1+c}{t}, \quad \text{for large } t.$$

$\varphi(t) = \log M(e^t)$

log-regularity $\Rightarrow$ ψ-regularity, with $\psi(r) = r^k$.

Equivalent definition Let f be a t.e.f.

(a) If f is log-regular with constant c, $\exists\, R_0>0$ s.t.

$$M(r^k) \geq M(r)^{kd}, \text{ for } r \geq R_0, \quad \text{whenever } k>1,\ d=k^c.$$

(b) If

$$M(r^k) \geq M(r)^{kd}, \text{ for } r \geq R_0,$$

for some $k, d > 1$, then f is log-regular.

If

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} < \infty \quad \text{and} \quad \lambda(f) = \underline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r} > 0,$$

then f is log-regular.

Hinkkanen

Functions with $A_R(f)$ spiders' webs

Many families of t.e.f.s have min. mod. property – for example:

Theorem 5 (R+S, 2010) A t.e.f. f has an $A_R(f)$ spider's web if

$\rho(f) < \frac{1}{2}$ and f has regular growth.

Example: $f(z) = \frac{1}{2}(\cos z^{1/4} + \cosh z^{1/4}) = 1 + \frac{z}{4!} + \frac{z^2}{8!} + \cdots$

Composite functions with $A_R(f)$ spiders' webs: Sixsmith

Theorem 6 Let f be a t.e.f. If

- f is log-regular and satisfies the general min. mod. condition,
- $g(z) = P(f(Q(z)), z)$, $P(w,z)$, $Q(z)$ polynomials,

then $A_R(g)$ is a spider's web.

Example: $f(z) = \cos z + \cosh z$.

Theorem 7 Let $f_1, f_2, \cdots, f_k$ be t.e.f.s, $f = f_1 \circ f_2 \circ \cdots \circ f_k$,

(a) all f_j satisfy the general min. mod. condition,

(b) all f_j, $j = 1, 2, \cdots, k$ are ψ-regular (with same ψ_m, $m > 1$),

or one f_j is log-regular.

Then $A_R(f)$ is a spider's web.

Generalises a result of Singh, and a result of Cao + Wang.

Second question.

Theorem 8 (R+S) If f is a log-regular t.e.f., then f has no multiply connected Fatou components.

Proof We combine 3 facts.

(1) Eremenko 1989: proof that $I(f) \neq \emptyset$.

Every large enough annulus $A(r, 4r)$ contains z_0 s.t.

$$|f^n(z_0)| > M^n(r), \quad n \in \mathbb{N}.$$

Eremenko point

(2) Zheng 2006: if U is a m-conn. Fatou component, then

$$f^n(U) \supset A(r_n, R_n), \text{ large } n, \quad \text{where } r_n \to \infty, \ \frac{R_n}{r_n} \to \infty$$

(3) Baker 1981: if U is a Fatou component of f, $z, z' \in U$, then

$$\frac{\log |f^n(z')|}{\log |f^n(z)|} = O(1) \text{ as } n \to \infty.$$

Let U be m-connected. By (1) and (2), $\exists\, z_0, z_0' \in U$ s.t.

use $f^n(U)$

$$|z_0'| \doteq 2|z_0| \quad \text{and} \quad |f^n(z_0')| \geq M^n(2|z_0|), \quad n \in \mathbb{N}.$$

Then $2|z_0| = |z_0|^k$, $k > 1$, and by log-regularity $\exists\, d > 1$ s.t.

$$M^n(|z_0|^k) \geq M^n(|z_0|)^{kd^n}, \quad n \in \mathbb{N}.$$

Hence

$$\frac{\log |f^n(z_0')|}{\log |f^n(z_0)|} \geq \frac{\log M^n(|z_0|^k)}{\log M^n(|z_0|)} \geq kd^n \to \infty \quad \text{as } n \to \infty,$$

which contradicts (3). ∎

A sufficient condition for log-regularity

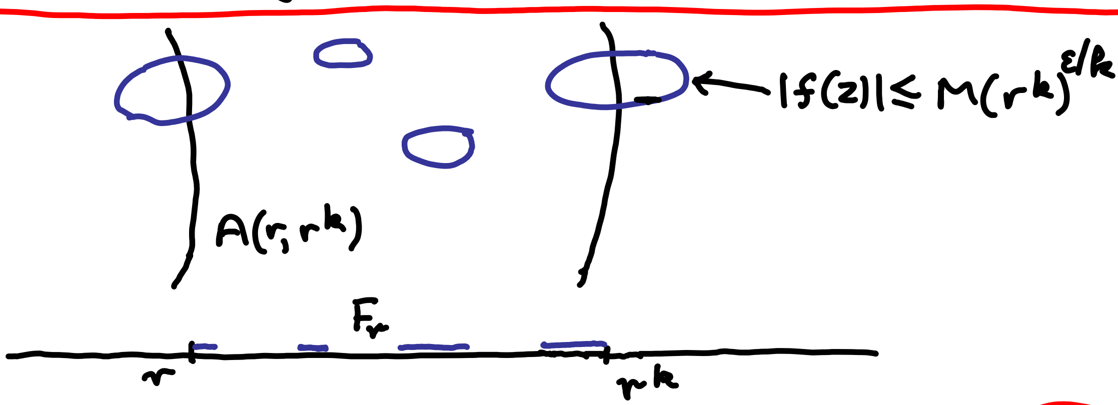
Theorem 9 Let f be a t.e.f. Suppose $\exists\, \varepsilon,\ 0<\varepsilon<1,\ k>1$ s.t.

$$F_r = \left\{ \rho \in (r, r^k) : \min_{|z|=\rho} |f(z)| \leq M(r^k)^{\varepsilon/k} \right\}, \quad r>0,$$

satisfies

$$\int_{F_r} \frac{dt}{t} \geq c(\varepsilon,k) = 2\log\left(\frac{18\sqrt{2}k}{1-\varepsilon}\right), \quad \text{for } r \geq r_0.$$

Then f is log-regular.



Proof is based on a harmonic measure estimate. Tsuji's book

Corollary If $\exists\, \varepsilon,\ 0<\varepsilon<1$, and a path Γ to ∞ s.t.

$$|f(z)| \leq M(|z|)^{\varepsilon}, \qquad \text{for } z \in \Gamma,$$

then f is log-regular and so has no multiply connected Fatou components.

Generalises a result of Bergweiler and a result of Zheng – but a stronger result is possible ... .