Smoothness of hairs for some transcendental entire functions

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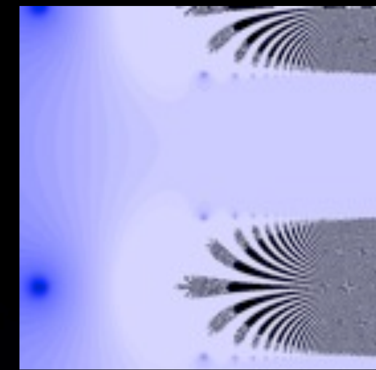
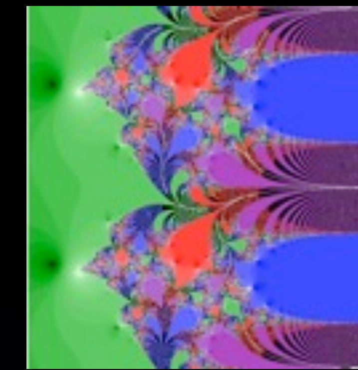
Goal





Hairs for transcendental entire functions

Case of exponential maps $z \mapsto \lambda e^z$

Devaney-Krych (1984) existence of hairs (Cantor bouquet) for symbol sequences satisfying "admissibility condition"

Viana (1988) hairs are C^∞-curves

Our case: $f(z) = P(z)e^{Q(z)}$, where P and Q are polynomials with $deg\, Q \geq 1$ ("structurally finite")

Construction and C^∞-smoothness of hairs

Compare with A Classical Theorem (Dichotomy) by Fatou?:

If the Julia set J_f of a rational function f is a Jordan curve and f is expanding on J_f, then either J_f is a round circle or it has no tangent at any point. (Round or fractal)

$f : \mathbb{C} \to \mathbb{C}$: a transcendental entire function.

$F(f) := \{z \in \mathbb{C} \mid \exists U : \text{nbd of } z,\ \{f^n|_U\}_{n=1}^{\infty} \text{ is normal}\}$ Fatou set

$J(f) := \mathbb{C} \setminus F(f)$ Julia set

$I(f) := \{z \mid f^n(z) \to \infty\}$ escaping set

Eremenko $J(f) = \partial I(f)$

Let $f(z) = P(z)e^{Q(z)}$, where P, Q are polyn. and $deg\, Q \geq 1$.

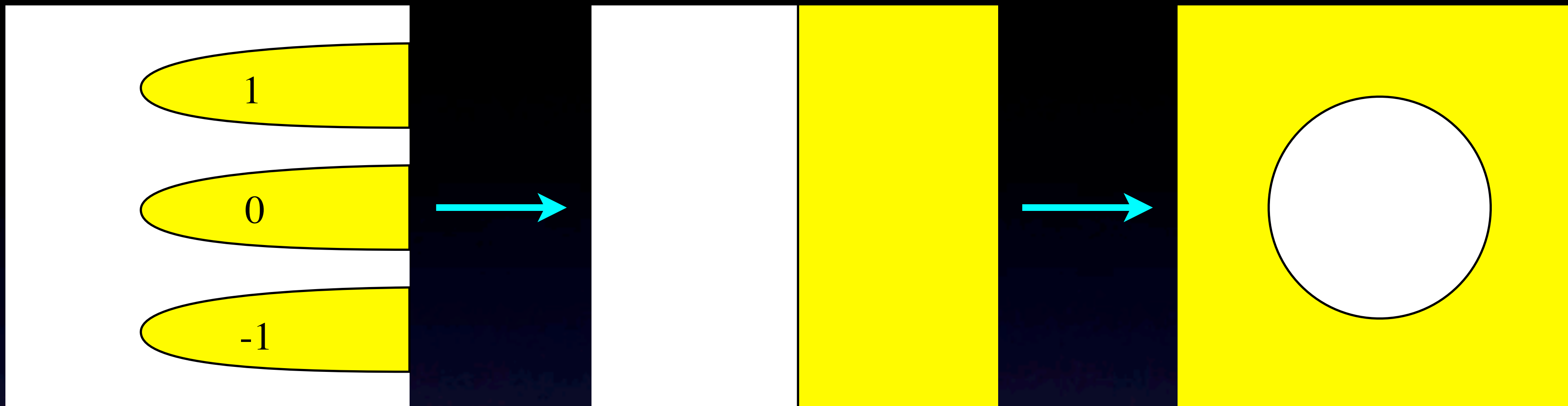
Define the reference function $g(t) = t^m e^{t^d}$, where $m = deg\, P$, $d = deg\, Q$. Take t_* large so that $g : [t_*, \infty) \to [t_*, \infty)$ and $g(t) > t$.

$$I^0_{fast}(f) := \{z \mid |f^n(z)| \geq |g^n(t_*)|\ (n \geq 0)\}$$

$$I_{fast}(f) := \{z \mid \exists n \text{ such that } f^n(z) \in I^0_{fast}(f)\}$$

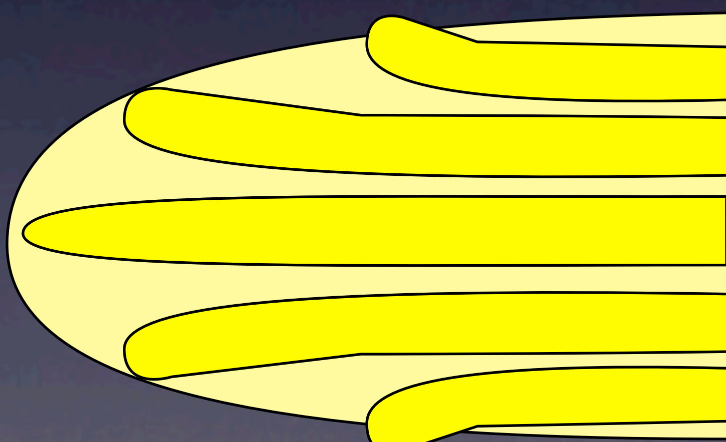
Theorem. *If t_* is very large, then $I^0_{fast}(f)$ consists of C^∞-curves. ($I_{fast}(f)$ consists of their inverse images, which may have branching via critical points.)*

Case of exponential maps $E_\lambda : z \mapsto \lambda e^z$

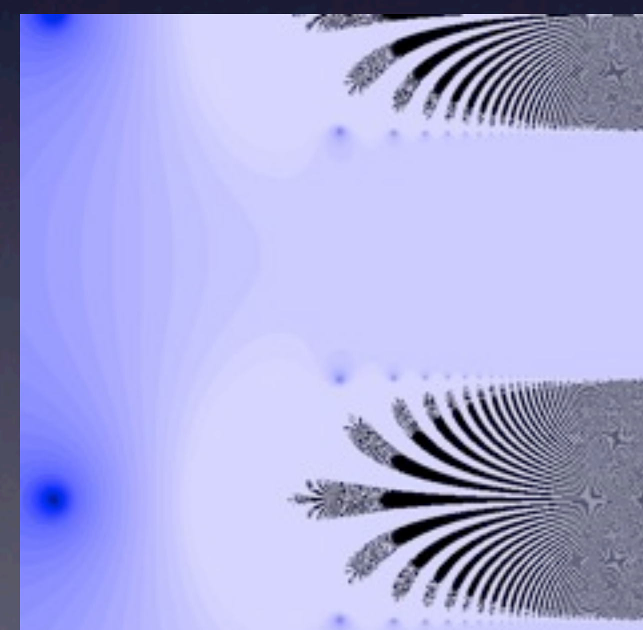


Sub-tracts
indexed by $\mathbb{Z}$

Tract



indexed by $\mathbb{Z} \times \mathbb{Z}$



Cantor bouquet: hairs indexed by $\mathbb{Z}^{\mathbb{N}}$

Theorem (Devaney-Krych, 1984) If $\boldsymbol{s} \in \mathbb{Z}^{\mathbb{N}}$ satisfies a growth condition ("admissibility")

$\Longrightarrow\ \exists h_{\boldsymbol{s}}(t) \subset J(E_\lambda)$ s.t.

(i) $E_\lambda(h_{\boldsymbol{s}}(t)) = h_{\sigma(\boldsymbol{s})}(g(t)), \quad g(t) := |\lambda| e^t, \quad \sigma$: shift map

(ii) $E_\lambda^n(h_{\boldsymbol{s}}(t)) \to \infty\ (n \to \infty)$

$h_{\boldsymbol{s}}(t)$ is called a hair.

Theorem (Viana, 1988) $h_{\boldsymbol{s}}(t)$ is a C^∞ curve.

In the proof, many estimates were exponential specific.

We need to reformulate the proof and we will see how different the proofs are. In fact, the estimates for exponential maps turned out to be simpler than general cases.

Strategy

Set up an appropriate symbolic dynamics ($(\mathbb{Z}/d\mathbb{Z}\times\mathbb{Z})^{\mathbb{N}}$). Fix a symbol sequence $\boldsymbol{s}=(s_n)$. Symbol $\Leftrightarrow$ Subtract.

Define initial curves $h_{n,n}(t)$ corresponding to symbol s_n and parametrized by $t\in[g^n(t_*),\infty)$. Need to satisfy the conditions below.

Pull-back by the dynamics to define $h_{n,n-1}, h_{n,n-2},\ldots,h_{n,1},h_{n,0}$. The inverse branches are specified by symbols $s_{n-1},s_{n-2},\ldots,s_1,s_0$.

$$h_{n,j}(t) \text{ is defined on } [g^j(t_*),\infty),$$
$$f\circ h_{n,j}(t)=h_{n,j+1}\circ g(t)\ (0\le j<n)$$

Show that $h_{n,n}(t)$ and $h_{n+1,n}$ have bounded C^r distance. (initial distance)

By pull-back, show that $h_{n,j}(t)$ and $h_{n+1,j}$ have exponentially small C^r distance with respect to $n-j$.

C^r-distance estimate

pull-back

initial curve

initial distance

$$
\begin{array}{cccccc}
h_{0,0} & & & & & \\
h_{1,0} & h_{1,1} & & & & \\
h_{2,0} & h_{2,1} & h_{2,2} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
h_{n,0} & h_{n,1} & h_{n,2} & \cdots & h_{n,n} & \\
h_{n+1,0} & h_{n+1,1} & h_{n+1,2} & \cdots & h_{n+1,n} & h_{n+1,n+1} \\
\vdots & \vdots & \vdots & & \vdots & \vdots \quad \ddots \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\
h_0 & h_1 & h_2 & \cdots & h_n & h_{n+1} \quad \cdots
\end{array}
$$

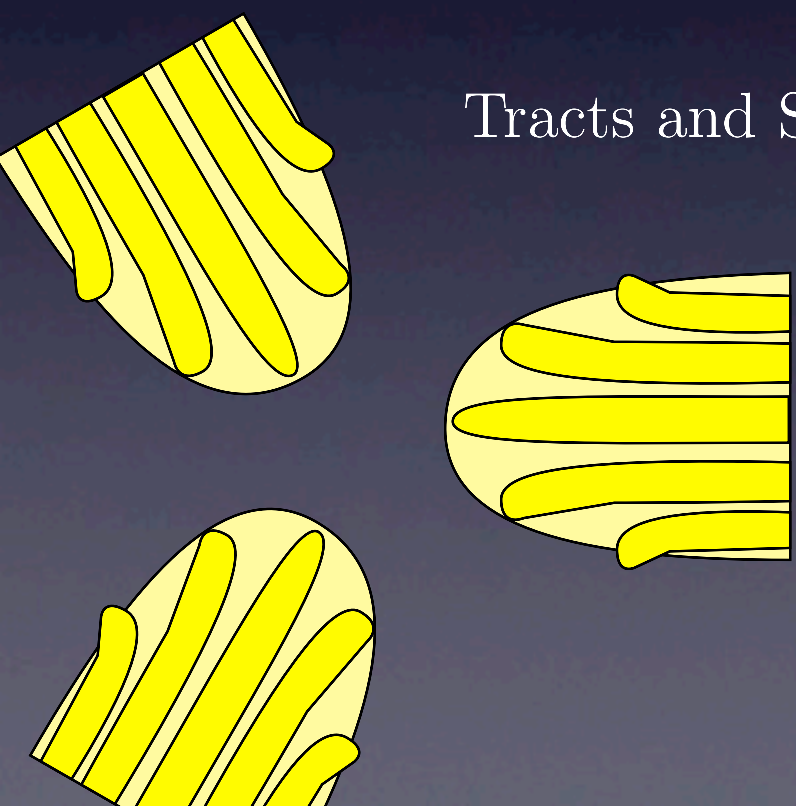
If $||h_{n,j} - h_{n+1,j}||_{C^r} \leq const\, \kappa_r^{n-j}$ with $\kappa_r < 1$, then for each j, $\{h_{n,j}\}_{n=j}^{\infty}$ converges in C^r-topology and the limit h_j satisfies $f \circ h_j(t) = h_{j+1} \circ g(t)$.

However, usual C^r-norm does not work. So we consider weighted C^r-norms: for an weight function $\rho : [\tau, \infty) \to \mathbb{R}_{>0}$, define

$$||\psi||_{\rho,\tau} = \sup_{t \in [\tau,\infty)} |\psi(t)|\rho(t).$$

For suitable weight functions ρ_r, C^0-distance will be measured by $|| \cdot ||_{\rho_*,\tau}$ and C^{r+1}-distance $(r \geq 0)$ between $h(t)$ and $\tilde{h}(t)$ will be measured by $||(\log h'(t))^{(r)} - (\log \tilde{h}'(t))^{(r)}||_{\rho_r,\tau}$.

We will see later how this estimate goes.



Tracts and Sub-tracts for $d = 3$.

Initial curves are given by

$$f(\zeta t(1 + w(t))) = \zeta' t^m e^{t^d}$$

where ζ, ζ' are roots of unity determined by the symbol sequence and $w(t)$ is small.

Estimates of differentiability

For simplicity, let us consider the case $s_0 = s_1 = s_2 = \ldots$
$h_{m,n}$'s are renamed so that $h_{m,m-n}$ becomes new h_n.

$$\begin{array}{c|cccccccc}
h_0 & h_{0,0} & & & & & & \\
h_1 & h_{1,0} & h_{1,1} & & & & & \\
h_2 & h_{2,0} & h_{2,1} & h_{2,2} & & & & \\
 & \vdots & \vdots & \vdots & \ddots & & & \\
 & h_{n,0} & h_{n,1} & h_{n,2} & \cdots & h_{n,n} & & \\
 & h_{n+1,0} & h_{n+1,1} & h_{n+1,2} & \cdots & h_{n+1,n} & h_{n+1,n+1} & \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots \\
 & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\
 & h_0 & h_1 & h_2 & \ldots & h_n & h_{n+1} & \ldots
\end{array}$$

They satisfy $f \circ h_{n+1}(t) = h_n \circ g(t)$.

Initial curve: $h_0(t)$;
Initial distance: $h_0(t) - h_1(t)$.

Suppose $h_n(t)$ are already defined and C^0-distance $h_{n+1} - h_n$ is exponentially small with respect to some weight function.

From $f \circ h_{n+1}(t) = h_n \circ g(t)$, we have

$$\log h'_{n+1} = \log h'_n \circ g + \log g' - \log f' \circ h_{n+1}.$$

Define $\psi_n(t) := \log h'_n(t)$. Then

$$\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n).$$

If $\psi_n - \psi_{n-1} \to 0$ as $t \to \infty$, by composing g, $(\psi_n - \psi_{n-1}) \circ g$ may go to 0 faster.

$$\begin{aligned} ||\psi \circ g||_{\rho_0,\tau} &= \sup_{t\geq\tau} |\psi(g(t))|\rho_0(t) = \sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))} \cdot |\psi(g(t))|\rho_0(g(t)) \\ &\leq \left(\sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) \cdot \left(\sup_{t'\geq g(\tau)} |\psi(t')|\rho_0(t') \right) \\ &= \left(\sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))} \right) ||\psi||_{\rho_0,g(\tau)} \end{aligned}$$

So if we define ρ_0 so that $\displaystyle\sup_{t\geq\tau} \frac{\rho_0(t)}{\rho_0(g(t))} < 1$ then $||\cdot||_{\rho_0,\tau}$-norm is contracted by composing g.

Now let $\alpha_0(\tau) := \sup_{t \geq \tau} \frac{\rho_0(t)}{\rho_0(g(t))}$ and assume $\alpha_0(t_*) \leq \kappa_0 < 1$. In addition, assume that $|(\log f')'(z)| \times |h_1(t) - h_0(t)| \leq C$ for z near $h_0(t)$. Then

$$
\begin{aligned}
||\psi_{n+1} - \psi_n||_{\rho_0,\tau} &\leq \alpha_0(\tau)||\psi_n - \psi_{n-1}||_{\rho_0,g(\tau)} + ||\log f' \circ h_{n+1} - \log f' \circ h_n||_{\rho_0,\tau} \\
&\leq \kappa_0||\psi_n - \psi_{n-1}||_{\rho_0,g(\tau)} + C \leq \kappa_0||\psi_n - \psi_{n-1}||_{\rho_0,\tau} + C.
\end{aligned}
$$

This shows the convergence of ψ_n which implies that the limit $h = \lim_{n\to\infty} h_n$ is C^1.

Higher order derivatives

Differentiating

$$\psi_{n+1} - \psi_n = (\psi_n - \psi_{n-1}) \circ g - (\log f' \circ h_{n+1} - \log f' \circ h_n).$$

and using $h'_{n+1} = e^{\psi_{n+1}}$, we have

$$\psi'_{n+1} = (\psi'_n \circ g) \cdot g' + (\log g')' - \big((\log f')' \circ h_{n+1}\big)\, e^{\psi_{n+1}},$$

$$\psi''_{n+1} = (\psi''_n \circ g) \cdot (g')^2 + (\psi'_n \circ g) \cdot g'' + (\log g')'' - \big((\log f')'' \circ h_{n+1}\big)\, e^{2\psi_{n+1}} - \big((\log f')' \circ h_{n+1}\big)\, e^{\psi_{n+1}} \psi'_{n+1}.$$

More generally, for $k = 1, 2, \ldots,$ we have

$$\psi^{(k)}_{n+1} = \big(\psi^{(k)}_n \circ g\big)\,(g')^k \; + \sum_{\substack{1 \le \ell < k \\ j_1 \ge \cdots \ge j_\ell \ge 1 \\ j_1 + \cdots + j_\ell = k}} \mathit{const}\,\big(\psi^{(\ell)}_n \circ g\big)\, g^{(j_1)} \ldots g^{(j_\ell)} \; + \; (\log g')^{(k)}$$

$$- \sum_{\substack{1 \le \ell \le k,\, 0 \le \nu \\ j_1 \ge \cdots \ge j_\nu \ge 1 \\ \ell + j_1 + \cdots + j_\nu = k}} \mathit{const}\,\Big((\log f')^{(\ell)} \circ h_{n+1}\Big)\, e^{\ell \psi_{n+1}} \psi^{(j_1)}_{n+1} \ldots \psi^{(j_\nu)}_{n+1},$$

=0 for exponentials

where the coefficients "*const*" are some constants depending the indices $\ell, j_1, j_2, \ldots$.

$$\psi_{n+1}^{(k)} - \psi_n^{(k)} = \left(\left(\psi_n^{(k)} - \psi_{n-1}^{(k)} \right) \circ g \right) (g')^k + \sum_{\substack{1 \le \ell < k \\ j_1 \ge \cdots \ge j_\ell \ge 1 \\ j_1 + \cdots + j_\ell = k}} const \left(\left(\psi_n^{(\ell)} - \psi_{n-1}^{(\ell)} \right) \circ g \right) g^{(j_1)} \cdots g^{(j_\ell)}$$

k-th derivative

lower order derivative

$$- \sum_{\substack{1 \le \ell \le k,\, 0 \le \nu \\ j_1 \ge \cdots \ge j_\nu \ge 1 \\ \ell + j_1 + \cdots + j_\nu = k}} const \left[\left((\log f')^{(\ell)} \circ h_{n+1} - (\log f')^{(\ell)} \circ h_n \right) e^{\ell \psi_{n+1}} \psi_{n+1}^{(j_1)} \cdots \psi_{n+1}^{(j_\nu)} \right.$$

$$+ \left((\log f')^{(\ell)} \circ h_n \right) (e^{\ell \psi_{n+1}} - e^{\ell \psi_n}) \psi_{n+1}^{(j_1)} \cdots \psi_{n+1}^{(j_\nu)}$$

$$+ \left((\log f')^{(\ell)} \circ h_n \right) e^{\ell \psi_n} (\psi_{n+1}^{(j_1)} - \psi_n^{(j_1)}) \psi_{n+1}^{(j_2)} \cdots \psi_{n+1}^{(j_\nu)}$$

$$\left. + \cdots + \left((\log f')^{(\ell)} \circ h_n \right) e^{\ell \psi_n} \psi_n^{(j_1)} \cdots \psi_n^{(j_{\nu-1})} (\psi_{n+1}^{(j_\nu)} - \psi_n^{(j_\nu)}) \right].$$

In order to get a contraction from k-th order derivative term, define $\alpha_k(\tau) := \sup_{t \ge \tau} \frac{\rho_k(t)|g'(t)|^k}{\rho_k(g(t))}$ and assume that

$$\lim_{\tau \to \infty} \alpha_k(\tau) < \kappa_k < 1.$$

In order to control other terms, we need more assumptions involving the weight functions and derivatives of g.

$$D_k(t) := \sup_{z \in B_f(t)} \left| \left(\log f'\right)^{(k)}(z) \right|,$$

where $\quad B_f(t) := \{z \in U : |f(z) - h_0(g(t))| \leq R(g(t))\}$
and $R(t) \doteqdot |h_1(t) - h_0(t)|$.

$\mathbf{C}_k$: h_0, h_1 are C^{k+1} and $\psi_0 = \log h_0'$ and $\psi_1 = \log h_1'$ satisfy

$$||\psi_1^{(k)} - \psi_0^{(k)}||_{\rho_k, \tau_*} < \infty \quad \text{and} \quad ||\psi_0^{(k)}||_{\sigma_k, \tau_*} < \infty.$$

$\mathbf{D}_k$: $\lim\limits_{\tau \to \infty} \alpha_k(\tau) < 1$.

$\mathbf{E}_k$: For $1 \leq \ell < k$ and $j_1, \ldots, j_\ell \geq 1$ with $j_1 + \cdots + j_\ell = k$,

$$\sup_{t \geq \tau_*} \frac{\rho_k(t)|g^{(j_1)}(t) \cdots g^{(j_\ell)}(t)|}{\rho_\ell(g(t))} < \infty.$$

$\mathbf{F}_k$: For $1 \le \ell \le k$, $\nu \ge 0$, $j_1, \ldots, j_\nu \ge 1$ with $\ell + j_1 + \cdots + j_\nu = k$,

$$\sup_{t \ge \tau_*} D_{\ell+1}(t) R(t) \frac{\rho_k(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t)} < \infty;$$

$$\sup_{t \ge \tau_*} D_\ell(t) \frac{\rho_k(t)}{\rho_0(t)\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t)} < \infty;$$

$$\text{if } \nu \ge 1, \text{ then for } 1 \le i \le \nu, \quad \sup_{t \ge \tau_*} D_\ell(t) \frac{\rho_k(t)}{\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t)} \frac{\sigma_{j_i}(t)}{\rho_{j_i}(t)} < \infty.$$

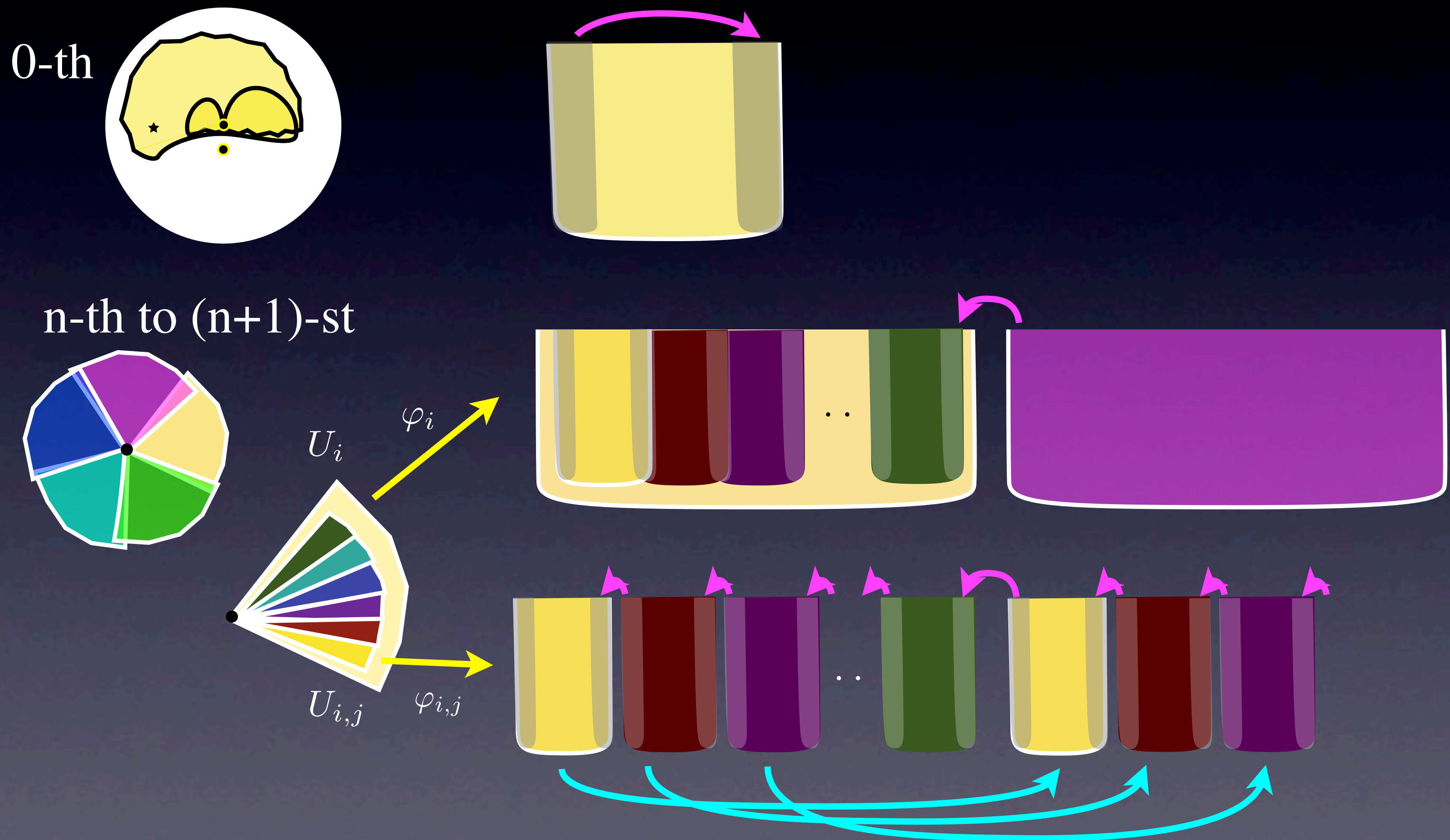
Here if $\nu = 0$, set $\sigma_{j_1}(t) \cdots \sigma_{j_\nu}(t) = 1$. Note that the last condition should be satisfied only when $\nu \ge 1$.

If these conditions are satisfied, $\{\psi_n^{(k)}\}_{n=0}^\infty$ converges with respect to C^k-norm with weight ρ_k, and this implies that the limit $h(t)$ is C^k.

For $f(z) = P(z)e^{Q(z)}$, we can take weight functions of the form $\rho_k(t) = const \dfrac{e^{\varepsilon t}}{t^{\mu_k}}$ with suitable $\mu_k > 0$.

Future plan

Use the same estimates for the renormalization of irrationally indifferent fixed points of high type.



Quadratic polynomial is transcendental if you consider renormalizations

Thank you!