A quasi-boundary-value method for a Cauchy problem for elliptic equations with nonhomogeneous Neumann data

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Outline

The Cauchy problem for elliptic equations

G. Alessandrini, L. Rondi, E. Rosset and S. Vessella, The stability for the Cauchy problem for elliptic equations, Inverse Problems, 25(2009), 123004 (47pp).

F.B. Belgacem, Why is the Cauchy problem severely ill-posed? Inverse Problems, 23(2007), 823–836.

V. Isakov, Inverse Problems for Partial Differential Equations (Applied Mathematical Sciences vol 127) 2nd edn (New York: Springer), 2006.

Our problem

$$
\begin{cases}
u_{xx}(x,y) - \mathcal{L}u(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\
u(x,y) = 0, & x \in [0,1], y \in \partial\Omega, \\
u(0,y) = g_1(y), & y \in \Omega, \\
u_x(0,y) = g_2(y), & y \in \Omega,
\end{cases}
$$

where $\mathcal{L} : D(\mathcal{L}) \subset H \to H$ denotes a linear densely defined self-adjoint and positive definite elliptic operator, and Ω is connected bounded domain.

U. Tautenhahn, Optimal stable solution of Cauchy problems of elliptic equations, J. Analysis and its Applications, 15(1996), 961–984.

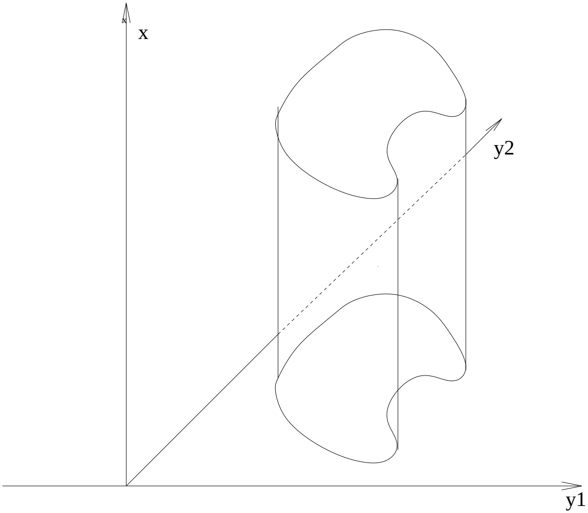

Figure: Three dimensional case

Set $u = u_1 + u_2$,

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where u_1 satisfies

$$\begin{cases} (u_1)_{xx}(x,y) - \mathcal{L}u_1(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\ u_1(x,y) = 0, & x \in [0,1], y \in \partial\Omega, \\ u_1(0,y) = g_1(y), & y \in \Omega, \\ (u_1)_x(0,y) = 0, & y \in \Omega, \end{cases}$$

and u_2 satisfies

$$\begin{cases} (u_2)_{xx}(x,y) - \mathcal{L}u_2(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\ u_2(x,y) = 0, & x \in [0,1], y \in \partial\Omega, \\ u_2(0,y) = 0, & y \in \Omega, \\ (u_2)_x(0,y) = g_2(y), & y \in \Omega, \end{cases}$$

then according to the linearity, $u = u_1 + u_2$ is the solution of the above problem.

Set $u = u_1 + u_2$,
where u_1 satisfies ILL-POSED PROBLEM

$$\begin{cases} (u_1)_{xx}(x,y) - \mathcal{L}u_1(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\ u_1(x,y) = 0, & x \in [0,1], y \in \partial\Omega, \\ u_1(0,y) = g_1(y), & y \in \Omega, \\ (u_1)_x(0,y) = 0, & y \in \Omega, \end{cases}$$

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then according to the linearity, $u = u_1 + u_2$ is the solution of the above problem.

Quasi Boundary Value Method

The quasi-boundary value method was introduced by Abdulkerimov. The idea is replacing the boundary value problem with an approximate one which is well-posed, then the latter has a non-local boundary value condition.

L.Š. Abdulkerimov, Regularization of an ill-posed Cauchy problem for evolution equations in a Banach space, Azerbaidzan. Gos. Univ. Ucen. Zap. Fiz. Mat., 1(1977), 32–36 (MR0492645) (in Russian).

R. Beals, Non-Local Boundary Value Problems for Elliptic Operators, Amer. J. Math., 87(1965), 315–362.

$$\begin{cases} v_{xx}^{\alpha,\delta}(x,y) - \mathcal{L}v^{\alpha,\delta}(x,y) = 0, & x \in (0,1), y \in \Omega \subset \mathbb{R}^n, n \geq 1, \\ v^{\alpha,\delta}(x,y) = 0, & x \in [0,1] \times \partial\Omega, \\ v^{\alpha,\delta}(0,y) = 0, & y \in \Omega, \\ v_x^{\alpha,\delta}(0,y) + \alpha v^{\alpha,\delta}(1,y) = g_2^\delta(y), & y \in \Omega, \end{cases}$$

where the noisy data $g_2^\delta(y)$ satisfy

$$\|g_2^\delta - g_2\| \leq \delta.$$

- After using the method of separation of variables, we can get

$$u_2(x, y) = \sum_{n=1}^{\infty} \kappa_n(x)(g_2, w_n)w_n(y),$$

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- $$v^{\alpha,\delta}(x,y)=\sum_{n=1}^{\infty}\frac{\kappa_n(x)}{1+\alpha\kappa_n(1)}(g_2^{\delta},w_n)w_n(y).$$

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- $\kappa_n(x)=\sinh(\sqrt{\lambda_n}x)/\sqrt{\lambda_n}$, where $\lambda_n, w_n(y)$ are the corresponding eigenvalues and eigenfunctions of elliptic operator $\mathcal{L}$.

Preliminaries

1.

For $s > 0$, $0 < \alpha_2 < 1$ *and* $0 < x < 1$, *the following inequalities hold.*

(a). $\frac{\sinh(xs)}{s} \leq e^{xs}$;

(b). $\frac{\sinh(xs)}{\sinh(s)} \leq e^{(x-1)s}$;

(c). *For* $f_1(s) := \frac{\frac{\sinh(xs)}{s}}{1+\alpha\frac{\sinh(s)}{s}}$, *there holds* $f_1(s) \leq (\frac{1}{\alpha})^x$;

2.

For $s > 0$ *and* $0 < x < 1$, $f_2(x) := (\frac{\sinh(xs)}{s})^{\frac{1}{x}}$ *is strict monotonically increasing.*

An a-priori parameter choice

Theorem 1

Let $v^{\alpha,\delta}(x,y)$ *be the solution of quasi-boundary-value problem and* $u_2(x,y)$ *be the exact solution of original problem. If conditions*

$$\|g_2^\delta - g_2\| \leq \delta.$$

and

$$\|u_2(1,\cdot)\| \leq E.$$

hold and we choose

$$\alpha = \frac{\delta}{E},$$

then there holds error estimate

$$\|v^{\alpha,\delta}(x,\cdot) - u_2(x,\cdot)\| \leq 2\delta^{1-x}E^x, \quad 0 < x < 1.$$

An a-posteriori parameter choice

Theorem 2

Suppose there exist the a-priori bound

$$\|u_2(1,\cdot)\| \leq E.$$

and conditions

$$\|g_2^\delta - g_2\| \leq \delta \qquad \|v_x^{\alpha,\delta}(0,\cdot) - {g_2}^\delta(\cdot)\| = \tau\delta.$$

Here $\tau > 1$ *such that* $0 < \tau\delta < \|{g_2}^\delta\|$. *Then for* $0 < x < 1$, *there holds*

$$\|u_2(x,\cdot) - v^{\alpha,\delta}(x,\cdot)\| \leq CE^x\delta^{1-x},$$

where $C = \left(1 + \sqrt{2\frac{1+(\tau-1)^2}{(\tau-1)^2}}\,\right)^x (1+\tau)^{1-x}$.

Remark

Under the a-priori bound

$$\|u_2(1,\cdot)\| \leq E,$$

the optimal error bounds for problem talked here is

$$\omega(\delta, E) = E^x(\frac{\delta}{2})^{1-x}(\ln\frac{E}{\delta})^{x-1}(1+o(1)), \qquad \textit{for } \delta \to 0, 0 < x < 1.$$

U. Tautenhahn, Optimal stable solution of Cauchy problems of elliptic equations, J. Analysis and its Applications, 15(1996), 961–984.

A simple case

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in (0,1) \times (0,1), \\ u(x,0) = u(x,1) = 0, & x \in [0,1], \\ u(0,y) = 0, & y \in [0,1], \\ u_x(0,y) = g_2(y), & y \in [0,1], \end{cases}$$

Quasi-boundary-value problem

$$\begin{cases} \Delta v^{\alpha,\delta}(x,y) = 0, & (x,y) \in (0,1) \times (0,1), \\ v^{\alpha,\delta}(x,0) = v^{\alpha,\delta}(x,1) = 0, & x \in [0,1], \\ v^{\alpha,\delta}(0,y) = 0, & y \in [0,1], \\ v_x^{\alpha,\delta}(0,y) + \alpha v^{\alpha,\delta}(1,y) = g_2^\delta(y), & y \in [0,1], \end{cases} \tag{1}$$

whose solution is

$$v^{\alpha,\delta}(x,y) = \sum_{n=1}^{\infty} \frac{\kappa_n(x)}{1 + \alpha\kappa_n(1)} (g_2^\delta, \sqrt{2}\sin(n\pi y))\sqrt{2}\sin(n\pi y), \tag{2}$$

with $\kappa_n(x) = \frac{\sinh(n\pi x)}{n\pi}$.

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Quasi-boundary-value problem The Finite Difference Method (FDM)

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whose solution is The Discrete Sine Transform (DST)

$$v^{\alpha,\delta}(x,y)=\sum_{n=1}^{\infty}\frac{\kappa_n(x)}{1+\alpha\kappa_n(1)}(g_2^\delta,\sqrt{2}\sin(n\pi y))\sqrt{2}\sin(n\pi y), \tag{2}$$

with $\kappa_n(x)=\frac{\sinh(n\pi x)}{n\pi}$.

Numerical Examples

- **Example 1.** We consider a simple example with an analytic solution $u(x,y) = \sin(\pi y)\sinh(\pi x)$, i.e., $g_2(y) = \pi\sin(\pi y)$.
-

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- **Example 2.**

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in (0,1)\times(0,1), \\ u(x,0) = u(x,1) = 0, & x \in [0,1], \\ u(0,y) = 0, & y \in [0,1], \\ u(1,y) = h(y), & y \in [0,1], \end{cases}$$

where $h(y)$ is a test function. We give a random signal which starts and ends with the value zero and create $h(y)$ to have these values used together with a cubic interpolating spline.

Numerical results

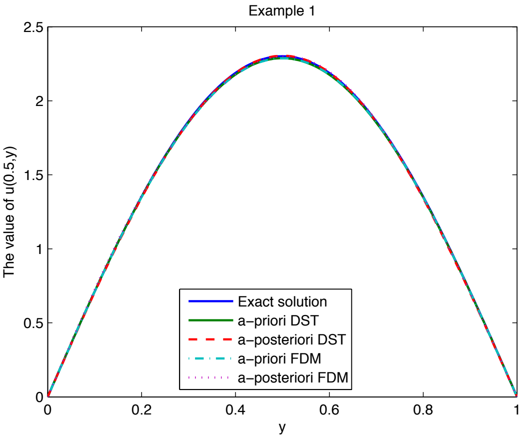

Figure: Example 1: The results at $x_0 = 0.5$ with $\epsilon = 10^{-2}$ and $\alpha_1 = 0.0012, \alpha_2 = 4.2943 \times 10^{-4}, \alpha_3 = 1.0928 \times 10^{-4}$.

Numerical results

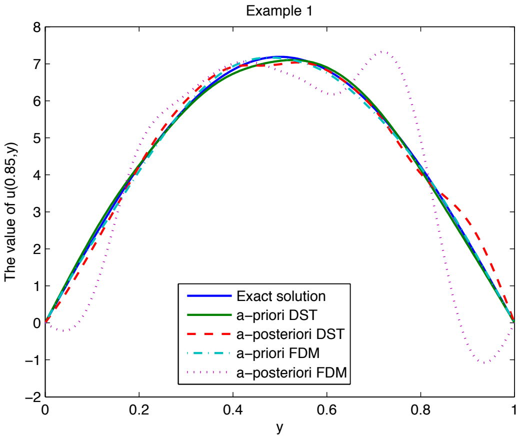

Figure: Example 1: The results at $x_0 = 0.85$ with $\epsilon = 10^{-2}$ and $\alpha_1 = 0.0012, \alpha_2 = 1.3420 \times 10^{-4}, \alpha_3 = 6.1348 \times 10^{-5}$.

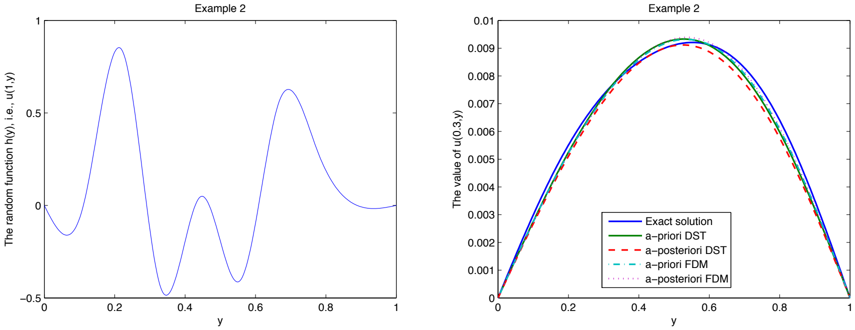

Figure: Example 2: Left: The random function $h(y)$; Right: The results at $x_0 = 0.3$ with $\epsilon = 10^{-3}$, $\alpha_1 = 0.0028$, $\alpha_2 = 0.0106$, $\alpha_3 = 0.0029$.

Numerical results

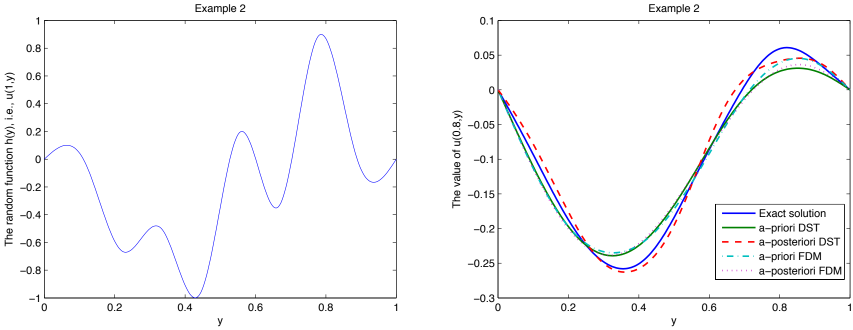

Figure: Example 2: Left: The random function $h(y)$; Right: The results at $x_0 = 0.8$ with $\epsilon = 10^{-3}$, $\alpha_1 = 0.0021$, $\alpha_2 = 4.6174 \times 10^{-4}$, $\alpha_3 = 0.0020$.

Numerical results

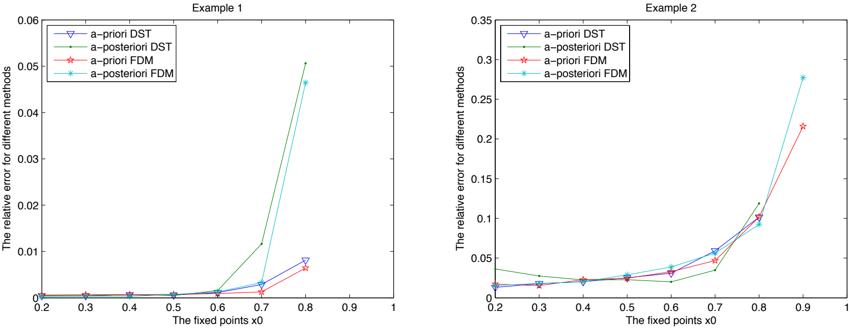

Figure: The relative errors for different fixed points x_0 with the same noisy level $\epsilon = 10^{-3}$ for Example 1 (Left) and Example 2 (Right).

Numerical results

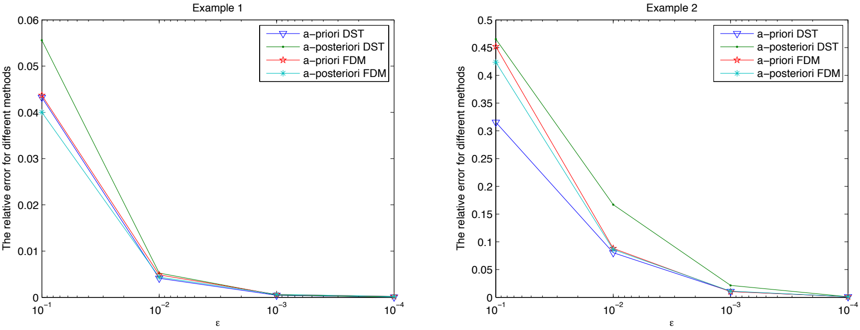

Figure: The relative errors for different noisy levels at $x_0 = 0.2$ for Example 1 (Left) and Example 2 (Right).

Left-Preconditioned GMRES

ALGORITHM 9.4. GMRES with Left Preconditioning

1. Compute $r_0 = M^{-1}(b - Ax_0)$, $\beta = \|r_0\|_2$, and $v_1 = r_0/\beta$
2. For $j = 1, \ldots, m$, Do
3. Compute $w := M^{-1}Av_j$
4. For $i = 1, \ldots, j$, Do
5. $h_{i,j} := (w, v_i)$
6. $w := w - h_{i,j}v_i$
7. EndDo
8. Compute $h_{j+1,j} = \|w\|_2$ and $v_{j+1} = w/h_{j+1,j}$
9. EndDo
10. Define $V_m := [v_1, \ldots, v_m]$, $\bar{H}_m = \{h_{i,j}\}_{1 \le i \le j+1, 1 \le j \le m}$
11. Compute $y_m = \text{argmin}_y \|\beta e_1 - \bar{H}_m y\|_2$ and $x_m = x_0 + V_m y_m$
12. If satisfied Stop, else set $x_0 := x_m$ and go to 1

Yousef Saad, Iterative Methods for Sparse Linear Systems, 2nd ed.

Two essential requirements for a GOOD preconditioner:

(1) $Mz = b$ can be solved quickly and stably.

(2) M is close to A (the eigenvalues of $M^{-1}A$ are clustered near 1).

R.H. Chan and G. Strang, Toeplitz equations by conjugate gradients with circulant preconditioner, SIAM J. Sci. Stat. Comput., (10)1989, 104–119.

$$A = \begin{pmatrix} -I & 0 & \dots & \dots & -\alpha h_x I \\ T & -I & 0 & \dots & \\ -I & T & -I & \dots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & -I & T & -I \end{pmatrix}$$

with

$$T = \begin{pmatrix} 2(1+r) & -r & \dots & 0 \\ -r & 2(1+r) & -r & \\ \ddots & \ddots & \ddots & \\ 0 & \dots & -r & 2(1+r) \end{pmatrix}$$

$$M = \begin{pmatrix} -I & 0 & \dots & \dots & -\alpha h_x I \\ C & -I & 0 & \dots & \\ -I & C & -I & \dots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & -I & C & -I \end{pmatrix}$$

with

$$C = \begin{pmatrix} 2(1+r) & -r & \dots & -r \\ -r & 2(1+r) & -r & \\ \ddots & \ddots & \ddots & \\ -r & \dots & -r & 2(1+r) \end{pmatrix}$$

$$\begin{pmatrix} F^* & 0 & \dots & \dots & 0 \\ 0 & F^* & 0 & \dots & \\ 0 & 0 & F^* & \dots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 0 & F^* \end{pmatrix} M \begin{pmatrix} F & 0 & \dots & \dots & 0 \\ 0 & F & 0 & \dots & \\ 0 & 0 & F & \dots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & 0 & F \end{pmatrix}$$

$$= \begin{pmatrix} -I & 0 & \dots & \dots & -\alpha h_x I \\ F^*CF & -I & 0 & \dots & \\ -I & F^*CF & -I & \dots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \dots & -I & F^*CF & -I \end{pmatrix}$$

Arbitrary Case

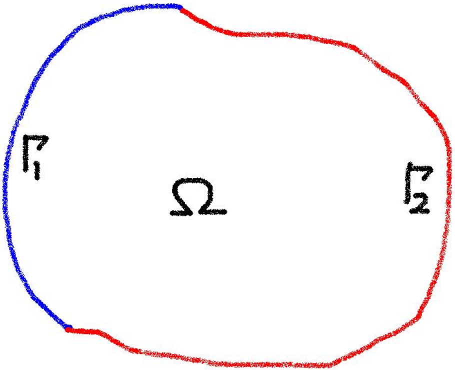

Figure: Arbitrary domain

I. Problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, & (x,t) \in (0,1) \times (0,T), \\ u(0,t) = 0, & t \in [0,T], \\ u_x(0,t) = u_x(1,t), & t \in [0,T], \\ u(x,T) = \varphi_1(x), & x \in [0,1]. \end{cases}$$

The approximate quasi-boundary value condition:

$$\alpha u(x,0) + (1-\alpha) u(x,T) = \varphi_1(x), \quad x \in [0,1].$$

M. Denche and K. Bessila, Quasi-boundary Value Method for Non-well Posed Problem for a Parabolic Equation with Integral Boundary Condition, Math. Probl. Eng., 7(2001), 129–145.

II. Problem:

$$\begin{cases} u_t(x,t) + \mathcal{L}u(x,t) = 0, & x \in (0,1), (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,T) = \varphi_2(x), & x \in \Omega, \end{cases}$$

The approximate quasi-boundary value conditions:
(1).

$$u(x,T) - \alpha u_t(x,0) = \varphi_2(x), \quad x \in \Omega.$$

M. Denche and K. Bessila, A modified quasi-boundary value method for ill-posed problems, J. Math. Anal. Appl., 301(2005), 419–426.

(2).

$$u(x,T) + \alpha u(x,0) = \varphi_2(x), \quad x \in \Omega.$$

K.A. Ames and L.E. Payne, Asymptotic for tow regularizations of the Cauchy problem for the backward heat equation, Math. Models Methods Appl. Sci., 8(1998), 187–202.

K.A. Ames, L.E. Payne and P.W. Schaefer, Energy and pointwise bounds in some non-standard parabolic problems, Proc. Roy. Soc. Edinburgh Sect. A, 134(2004), 1–9.

D.N. Hào, V.D. Nguyen and H. Sahli, A non-local boundary value problem method for parabolic equations backward in time, J. Math. Anal. Appl., 345(2008), 805–815.

III. Problem:

$$\begin{cases} u_{xx}(x,y) - \mathcal{L}u(x,y) = 0, & x \in (0,x_0), y \in \Omega, \\ u(x,y) = 0, & x \in [0,x_0] \times \partial\Omega, \\ u(0,y) = \varphi_3(y), & y \in \Omega, \\ u_x(0,y) = 0, & y \in \Omega. \end{cases}$$

The approximate quasi-boundary value problem:

$$\begin{cases} u_{xx}(x,y) - \mathcal{L}u(x,y) = 0, & x \in (0,ax_0), y \in \Omega, a \geq 1, \\ u(x,y) = 0, & x \in [0,ax_0] \times \partial\Omega, \\ u(0,y) + \alpha u(ax_0,y) = \varphi_3(y), & y \in \Omega, \\ u_x(0,y) = 0, & y \in \Omega. \end{cases}$$

- M. Hanke and P.C. Hansen, Regularization methods for large-scale problems, Surveys Math. Industry, 3(1993), 253–315.
- P.N. Vabishchevich, Numerical solution of nonlocal elliptic problems, Izv. Vyssh. Uchebn. Zaved. Mat., 1983, 13–19 (in Russian).
- P.N. Vabishchevich and A.Y. Denisenko, Regularization of nonstationary problems for elliptic equations, J. Eng. Phys. Thermophys., 65(1993), 1195–1199.
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- D.N. Hào, N.V. Duc and D. Lesnic, A non-local boundary value problem method for the Cauchy problem for elliptic equations, Inverse Problems, 25(2009), 055002(27pp).

Phenomena

1.

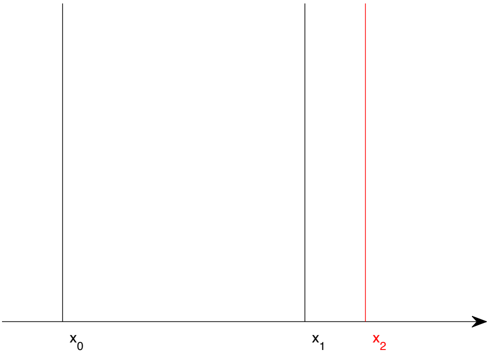

Figure: The positions of the conditions

2.

- **Problem.**

$$\begin{cases} \Delta u(x,y) = 0, & x \in (0,1), y \in \mathbb{R}^n, n \geq 1, \\ u(0,y) = \varphi_4(y), & y \in \mathbb{R}^n, \\ u_x(0,y) = 0, & y \in \mathbb{R}^n, \end{cases}$$

-

-

2.

- **Problem.**

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- **Failed.**

$$u(0,y) + \alpha u(1,y) = \varphi_4^\delta(y), \tag{3}$$

$$\hat{u}_x^{\alpha,\delta}(x,\xi) = \hat{\varphi_4^\delta}(\xi)\frac{|\xi|\sinh(|\xi|x)}{1+\alpha\cosh(|\xi|)}, \tag{4}$$

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2.

- **Problem.**

$$\begin{cases} \Delta u(x,y) = 0, & x \in (0,1), y \in \mathbb{R}^n, n \geq 1, \\ u(0,y) = \varphi_4(y), & y \in \mathbb{R}^n, \\ u_x(0,y) = 0, & y \in \mathbb{R}^n, \end{cases}$$

- **Failed.**

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- **Succeed.**

$$u(0,y) + \alpha u_x(1,y) = \varphi_4^\delta(y), \tag{5}$$

$$\hat{u}_x^{\alpha,\delta}(x,\xi) = \hat{\varphi_4^\delta}(\xi)\frac{|\xi|\sinh(|\xi|x)}{1+\alpha|\xi|\sinh(|\xi|)}, \tag{6}$$

3.

- **Operator equation.**

$$A(x)X(x) = Y, \quad \text{for} \quad x \in (x_0, x_1], \tag{7}$$

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$$(\beta_1(\alpha)A(x) + \beta_2(\alpha)B(x, x_2))X(x) = Y, \quad \text{for} \quad x \in (x_0, x_1], \tag{8}$$

where

$$\beta_1(\alpha) \to 1 \quad \text{and} \quad \beta_2(\alpha) \to 0, \quad \text{as} \quad \alpha \to 0. \tag{9}$$

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- **Lavrentiev regularization method.**

$$(A(x) + \alpha I)X(x) = Y, \quad \text{for} \quad x \in (x_0, x_1]. \tag{10}$$

A is self-adjoint.

Future work

- Theoretical aspects:
  Improve the present error estimates;
  The general domain.
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Future work

- Theoretical aspects:
  Improve the present error estimates;
  The general domain.
- Numerical aspects:
  Multiple dimensions;
  Variable coefficients;
  Arbitrary domain.

Suggestions & Comments