Approximate Solution to Inverse Problems for Elliptic Equations[1]

Hans–Jürgen Reinhardt, Mathias Charton[2]

Coworkers: Dinh Nho Hào[3]**, Houde Han**[4]

[2] Universität Siegen, Department of Mathematics
D-57068 Siegen, Germany

[3] Hanoi Institute of Mathematics, Hanoi, Vietnam

[4] Tshingua University, Beijing, China

Workshop IP–TA 2010

Warsaw, February 2010

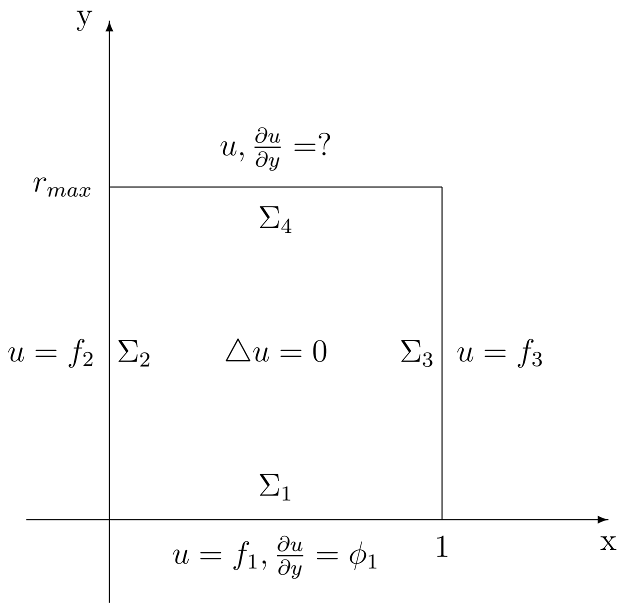
[1]"Method of Lines Approximations to Cauchy Problems for Elliptic Equations in two Dimensions".
In: Comput. Methods Applied Math. **9** (2009), 1 - 31.

CONTENTS

- PROBLEM SETTINGS
- METHOD OF LINES APPROXIMATION
- CONVERGENCE AND ERROR ESTIMATES
  (in case of bounded solutions)
  - Stability
  - Approximability
  - Error estimates
  - Optimal Convergence
- MORE GENERAL SITUATION: $\mathrm{div}\big(a(x)\nabla u\big) = 0$
- NUMERICAL RESULTS

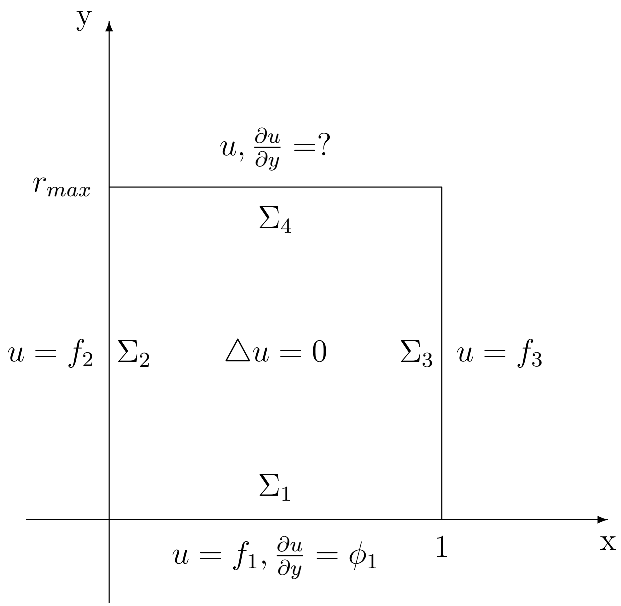
- **PROBLEM SETTINGS**

Cauchy–Problem
for Laplace's Equation (CPLE)



- **PROBLEM SETTINGS**

Cauchy–Problem for Laplace's Equation (CPLE)



Hadamard Example (1923)

$$\Delta u = 0 ,$$

$$u(x,0) = 0 ,$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{1}{n}\sin(n\pi x) =: g_n , \ x \in (0,1)$$

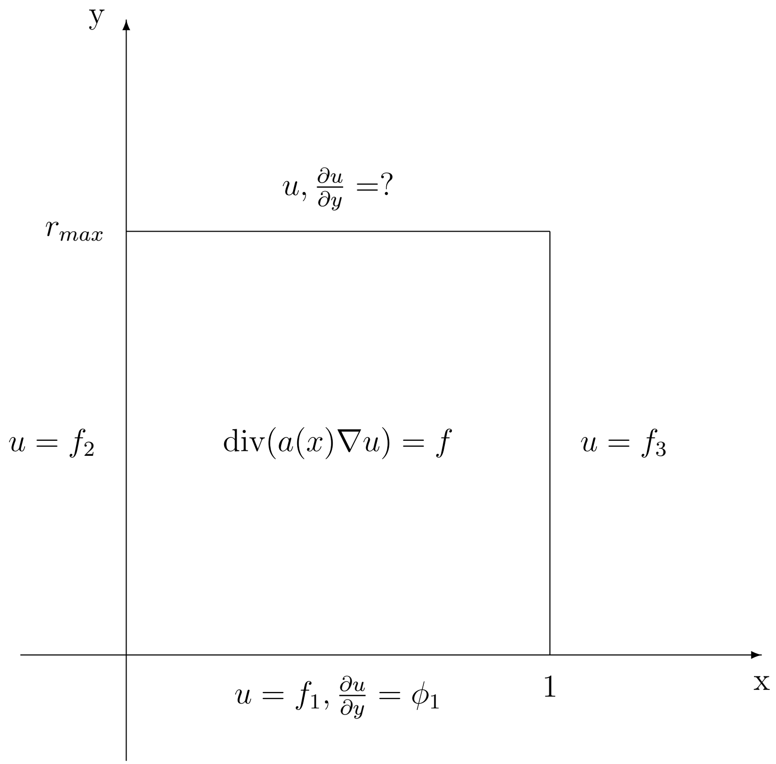
$$u(0,y) = u(1,y) = 0 , \ 0 \le y \le 1$$

$$\Longrightarrow \ u_n(x,y) = (n\pi)^{-2}\sin(n\pi x)\sinh(n\pi y) , \quad (x,y) \in [0,1] \times [0,1]$$

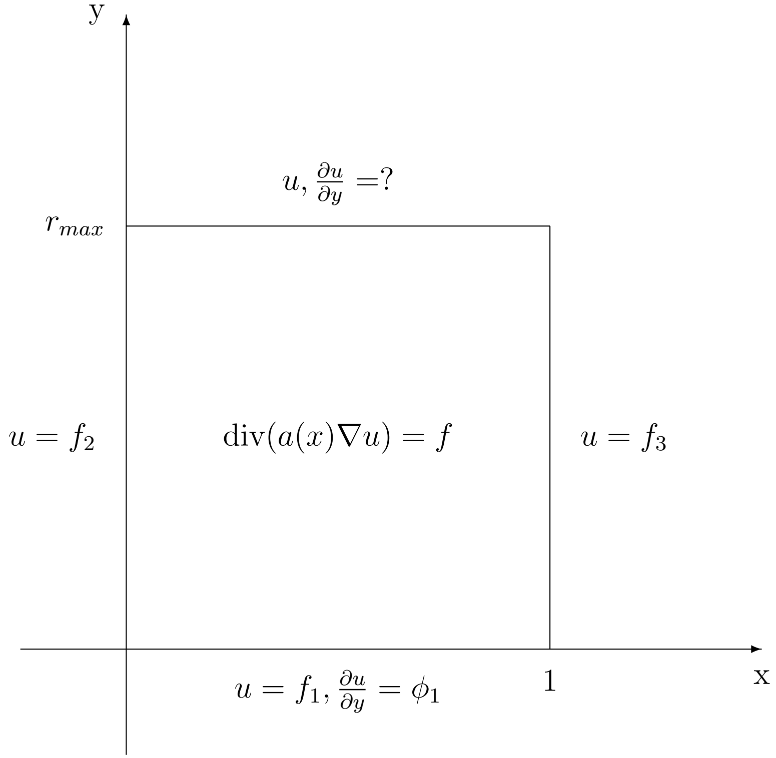
Illposed: $g_n \to 0$, but $u_n(x,y) \to \infty$ $(n \to \infty)$ for any $y > 0$

Lit.: Lavrentiev ('56), Payne ('60ff), Han ('82), Falk ('90), M. Kubo ('94), Kabanikhin + Karchevsky ('95), Fayazov + Lavrentiev ('95) and many others.

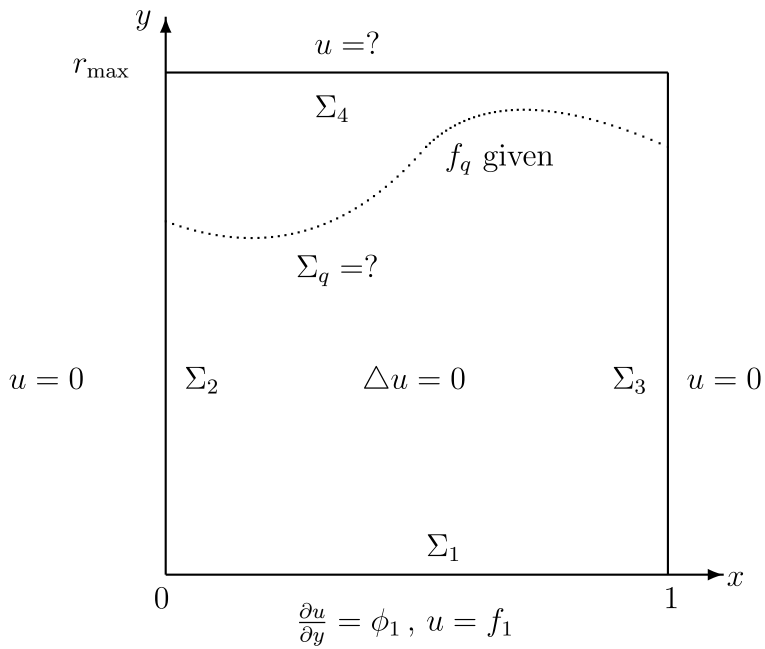
Cauchy–Problem for more general elliptic equations



Cauchy–Problem for more general elliptic equations



Shape Optimization Problem



Remarks

- All three problems are illposed.
- f, f_1, f_2, f_3 can be set to zero in the Cauchy-Problem for more general elliptic equations. Note: $a = a(x)$.
- The shape optimization problem needs the solution of the Cauchy–Problem beforehand.

Remarks

– All three problems are illposed.

– f, f_1, f_2, f_3 can be set to zero in the Cauchy-Problem for more general elliptic equations. Note: $a = a(x)$.

– The shape optimization problem needs the solution of the Cauchy–Problem beforehand.

Applications

– Medicine: Electrocardiology

– Geology: Gravimetric search of resources

– Steel production: Thickness of furnace wall

– Stationary Inverse Heat Conduction Problems

• METHOD OF LINES APPROXIMATION

$x_i = ih\,,\; i = 0, \ldots, N$

$\triangle_h u(x_i, y) = \frac{\partial^2 u}{\partial y^2}(x_i, y) + \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$

$u(x_i, .) \approx u_i\,,\; \frac{\partial^2 u}{\partial y^2}(x_i, .) \approx u_i''\,,$

$U = (u_1, \ldots, u_{N-1})$

$\mathbf{\triangle_h u = 0 \Longleftrightarrow U'' + AU = 0}$

with

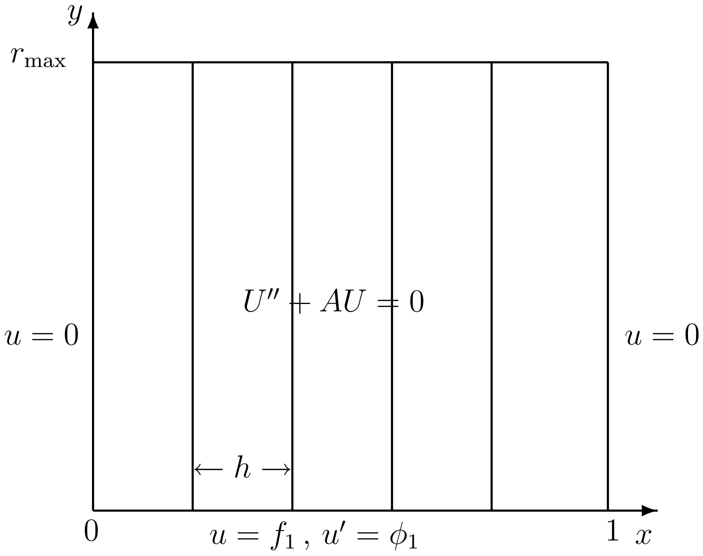
$$A := \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{pmatrix}$$

$\in \mathbb{R}^{N-1,N-1}$

Boundary Conditions:

$u_i(0) = f_1(x_i)\,,\; u_i'(0) = \phi_1(x_i)\,,$

$i = 1, \ldots, N-1$



- **METHOD OF LINES APPROXIMATION**

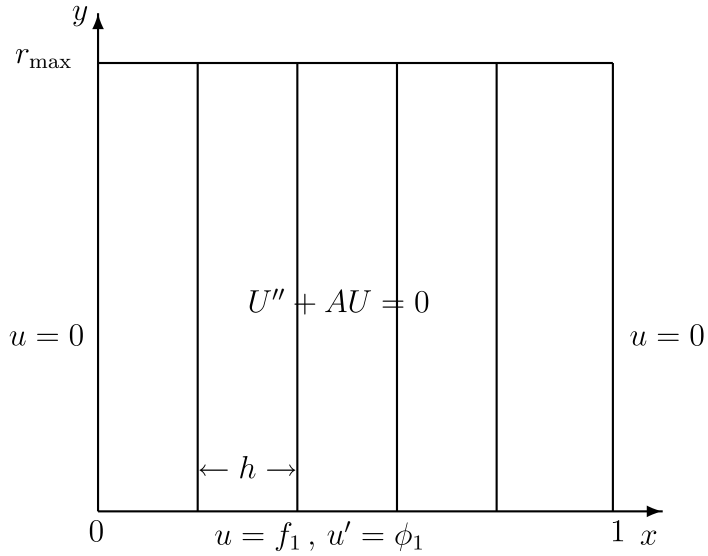
$x_i = ih\,,\ i = 0, \ldots, N$
$\triangle_h u(x_i, y) = \frac{\partial^2 u}{\partial y^2}(x_i, y) + \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}$
$u(x_i, .) \approx u_i\,,\ \frac{\partial^2 u}{\partial y^2}(x_i, .) \approx u_i''\,,$
$U = (u_1, \ldots, u_{N-1})$
$\triangle_{\mathbf{h}} \mathbf{u} = \mathbf{0} \Longleftrightarrow \mathbf{U}'' + \mathbf{AU} = \mathbf{0}$

with

$$A := \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & -2 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{pmatrix}$$

$\in \mathbb{R}^{N-1,N-1}$

Boundary Conditions:
$u_i(0) = f_1(x_i)\,,\ u_i'(0) = \phi_1(x_i)\,,$
$i = 1, \ldots, N-1$



The above system can be decoupled, since eigenvalues and eigenvectors of A are known.

METHOD OF LINES:

$$
\begin{aligned}
& U'' + AU = 0 \\
\iff \quad & WU'' + WAU = 0 \\
\iff \quad & WU'' + \underbrace{WAW^{-1}}_{=D} WU = 0 \\
\iff \quad & (WU)'' + D(WU) = 0 \\
\overset{V:=WU}{\iff} \quad & V'' + DV = 0\,,\ V = (v_1, \ldots, v_{N-1})^\top \\
\iff \quad & v_i'' + \lambda_i v_i = 0\,,\ i = 1, \ldots, N-1
\end{aligned}
$$

$\lambda_i =$ eigenvalues , $D =$ diag (λ_i) ,
$W = (w_1| \ldots |w_{N-1})$ eigenvectors of A (orthogonal)

METHOD OF LINES:

$$
\begin{aligned}
& U'' + AU = 0 \\
\iff\quad & WU'' + WAU = 0 \\
\iff\quad & WU'' + \underbrace{WAW^{-1}}_{=D} WU = 0 \\
\iff\quad & (WU)'' + D(WU) = 0 \\
\overset{V:=WU}{\iff}\quad & V'' + DV = 0\,,\; V = (v_1, \ldots, v_{N-1})^\top \\
\iff\quad & v_i'' + \lambda_i v_i = 0\,,\; i = 1, \ldots, N-1
\end{aligned}
$$

$\lambda_i =$ eigenvalues , $D =$ diag (λ_i) ,
$W = (w_1 | \ldots | w_{N-1})$ eigenvectors of A (orthogonal)

$$
\Longrightarrow \quad v_k(y) = \xi_k \exp(\sqrt{-\lambda_k} y) + \eta_k \exp(-\sqrt{-\lambda_k} y)\,,\; k = 1, \ldots, N-1
$$

$$
\Longrightarrow \xi_k = \sqrt{\frac{h}{2}} \sum_{j=1}^{N-1} \left(\sin(kjh\pi) f_1(x_j) + \frac{h}{2} \frac{\sin(khj\pi)}{\sin\left(kh\frac{\pi}{2}\right)} \phi_1(x_j) \right)
$$

$$
\eta_k = \sqrt{\frac{h}{2}} \sum_{j=1}^{N-1} \left(\sin(kjh\pi) f_1(x_j) - \frac{h}{2} \frac{\sin(khj\pi)}{\sin\left(kh\frac{\pi}{2}\right)} \phi_1(x_j) \right)
$$

$\Longrightarrow$ Solution $U = (U_1, \ldots, U_{N-1})$ of $U'' + AU = 0$:

$$
\begin{aligned}
u_i(y) &= (WV)_i(y) \\
&= 2h \cdot \sum_{k=1}^{N-1} \left(\sin(ikh\pi) \left(\cosh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi) f_1(x_j) \right.\right. \\
&\quad \left.\left. + \frac{h}{2\sin\left(kh\frac{\pi}{2}\right)} \sinh(\sqrt{-\lambda_k} y) \sum_{j=1}^{N-1} \sin(kjh\pi) \phi_1(x_j) \right) \right)
\end{aligned}
$$

Remarks:

1) The CPLG is solved by $u(x,y) = \sum_{k=1}^{\infty} g_k(x,y)$ with

$$g_k(x,y) \;=\; 2\sin(k\pi x)\left((f_1(.), \sin(k\pi .))_{L_2} \cosh(k\pi y) + \frac{(\phi_1(.), \sin(k\pi .))_{L_2}}{k\pi} \sinh(k\pi y) \right)$$

provided the series converges.

Remarks:

1) The CPLG is solved by $u(x,y) = \sum_{k=1}^{\infty} g_k(x,y)$ with

$$g_k(x,y) \;=\; 2\sin(k\pi x)\left((f_1(.),\sin(k\pi .))_{L_2}\cosh(k\pi y) + \frac{(\phi_1(.),\sin(k\pi .))_{L_2}}{k\pi}\sinh(k\pi y)\right)$$

provided the series converges.

2) The method of lines approximation is still illposed on every line.

Remarks:

1) The CPLG is solved by $u(x,y) = \sum_{k=1}^{\infty} g_k(x,y)$ with

$$g_k(x,y) \;=\; 2\sin(k\pi x)\left((f_1(.),\sin(k\pi .))_{L_2}\cosh(k\pi y) + \frac{(\phi_1(.),\sin(k\pi .))_{L_2}}{k\pi}\sinh(k\pi y)\right)$$

provided the series converges.

2) The method of lines approximation is still illposed on every line.

3) The CPLE is conditionally well-posed if data $f_1, \phi_1 \in D_M$ (data with *bounded frequencies*), where

$$D_M \;=\; \left\{ \phi \in C^1(0,1) \,\middle|\, \phi(0) = \phi(1) = 0\,,\; \int_0^1 sin(k\pi t)\phi(t)\,dt = 0,\; k > M \right\}$$

Solution:

$$u(x,y) \;=\; \sum_{k=1}^{M}\left(2\sin(k\pi x)\left((f_1(.),\sin(k\pi .))_{L_2}\cosh(k\pi y) + \frac{(\phi_1(.),\sin(k\pi .))_{L_2}}{k\pi}\sinh(k\pi y)\right)\right)$$

4) Data spaces

$$
\begin{aligned}
D: \quad &= \left\{\phi \in C^1[0,1] \,\Big|\, \phi(0) = \phi(1) = 0\right\} \\
D_M: \, &= \left\{\phi \in D \,\Big|\, \int\limits_0^1 \sin(k\pi t)\phi(t)\,dt = 0, k > M\right\} \\
D_M^h: \, &= \left\{\Phi \in \mathbb{R}^{N-1} \,\Big|\, \sum_{j=1}^{n-1} \sin(k\pi jh)\Phi_j = 0,\; N > k > M\right\} \quad (N > M)
\end{aligned}
$$

where $\Phi := (\phi(h), \ldots, \phi\big((N-1)h\big)^\top,\; \phi \in D.$

Projection $P_M : \mathbb{R}^{N-1} \longrightarrow D_M^h$

$$
(P_M\Phi)_j = \sum_{k=1}^{M} \left(2h \sum_{\ell=1}^{N-1} \Phi_\ell \sin(k\pi\ell h)\right) \sin(k\pi jh)
$$

4) Data spaces

$$D :\quad = \left\{\phi \in C^1[0,1] \,\middle|\, \phi(0) = \phi(1) = 0\right\}$$

$$D_M : \; = \left\{\phi \in D \,\middle|\, \int_0^1 \sin(k\pi t)\phi(t)\,dt = 0, k > M\right\}$$

$$D_M^h : \; = \left\{\Phi \in \mathbb{R}^{N-1} \,\middle|\, \sum_{j=1}^{n-1} \sin(k\pi jh)\Phi_j = 0,\ N > k > M\right\} \quad (N > M)$$

where $\Phi := (\phi(h), \ldots, \phi\big((N-1)h\big)^\top,\ \phi \in D.$

Projection $P_M : \mathbb{R}^{N-1} \longrightarrow D_M^h$

$$(P_M\Phi)_j = \sum_{k=1}^{M} \left(2h \sum_{\ell=1}^{N-1} \Phi_\ell \sin(k\pi\ell h)\right) \sin(k\pi jh)$$

$\Longrightarrow$ Fourier coeff. of $\phi \in D_M$ are finite sums,

$$\int_0^1 \phi(t)\sin(k\pi t)\,dt \; = \; h \sum_{j=1}^{N-1} \sin(k\pi jh)\Phi_j$$

$\forall\ M, N, k \in \mathbb{N},\ M < N, k < N, f \in D_M\,.$

- **CONVERGENCE AND ERROR ESTIMATES in case of bounded solutions**

  - **Stability Theorem:** If $u \in C^2\left(\mathrm{int}(\Omega)\right) \cap C\left(\Omega\right)$ s.t.

$$\begin{aligned} \triangle u(x,y) &= 0 \quad \text{in} \quad \mathrm{int}(\Omega) \\ u &= 0 \qquad \text{on} \quad \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \; (i.e.\; f_1 = f_2 = f_3 = 0) \\ \frac{\partial u}{\partial y}(x) &= \phi_1(x) \quad \text{on} \quad \Sigma_1 \\ \|u\|_{L_2(\Sigma_4)} &\leq E\,. \qquad\qquad (*) \end{aligned}$$

Then

$$\|u(.,y)\|_{L_2} \;\leq\; R_1 \|\phi_1\|_{L_2}^{1-\frac{y}{r_{max}}} E^{\frac{y}{r_{max}}}$$

for all $y \in [0, r_{max}]$, with $R_1 = \max(r_{\max}, 1)$.

- **CONVERGENCE AND ERROR ESTIMATES in case of bounded solutions**
  - **Stability Theorem:** If $u \in C^2(\mathrm{int}(\Omega)) \cap C(\Omega)$ s.t.

$$
\begin{aligned}
\triangle u(x,y) &= 0 \quad \text{in} \quad \mathrm{int}(\Omega) \\
u &= 0 \qquad \text{on} \quad \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \; (i.e.\; f_1 = f_2 = f_3 = 0) \\
\frac{\partial u}{\partial y}(x) &= \phi_1(x) \quad \text{on} \quad \Sigma_1 \\
\|u\|_{L_2(\Sigma_4)} &\le E\,. \qquad\qquad\qquad (*)
\end{aligned}
$$

Then

$$\|u(.,y)\|_{L_2} \;\le\; R_1 \|\phi_1\|_{L_2}^{1-\frac{y}{r_{max}}} E^{\frac{y}{r_{max}}}$$

for all $y \in [0, r_{max}]$, with $R_1 = \max(r_{\max}, 1)$.

Remarks:

1) The proof uses *logarithmic convexity* of $s(y) = \left\|u(\cdot,y)\right\|_{L_2}^2 / y^2$, i.e. $ln(s)$ convex.

2) CPLE is conditionally wellposed in this case.

– **Approximability:**

Projection (orth.) $P_M : D \longrightarrow D_M$ satisfies

$$\|\phi_1 - P_M\phi_1\|_{L_2(\Sigma_1)} \leq \frac{E}{r_{max}^2(1-\exp(-4\pi r_{max}))} \cdot \frac{M}{\exp(M\pi r_{max})}\,,$$

provided

$$(*): \ \|u\|_{L_2(\Sigma_u)} \leq E\,,$$

holds.

Assume $f_1 = 0$, $\|\phi_1 - \phi_1^\varepsilon\|_\infty = O(\varepsilon)$. Then ...

– **Error estimates:**

$$
\begin{aligned}
&\|(u - \overline{(u_\varepsilon^*)_h})(.,y)\|_{L_2} \\
&\leq \; C_1(r_{max})E \cdot \left(\frac{M}{\exp(M\pi r_{max})}\right)^{1-\frac{y}{r_{max}}} \\
&+ \; C_2(y)\frac{2}{\pi^2}\frac{\sinh(M\pi y)}{M}\varepsilon \\
&+ \; \frac{M^4\pi^3 y}{12}\exp(M\pi y)\|(\phi_1)_\varepsilon^*\|_{L_1}h^2 \quad \forall y \in [0, r_{\max}) \,.
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{(u_\varepsilon^*)_h}(x,y) \; &= \; \sum_{k=1}^{M}\left(2h\sum_{j=1}^{N-1}\sin(k\pi jh)u_{j,\varepsilon}^*(y)\right)\sin(k\pi x) \\
&= \; \text{continuation of } \left(u_{1,\varepsilon}^*(y), \ldots, u_{N-1,\varepsilon}^*(y)\right)^\top \text{ in } D_M \\
u_{i,\varepsilon}^*(y) \; &= \; \text{solution on i-th line with data } P_M\phi_1^\varepsilon \,.
\end{aligned}
$$

– Error estimates:

$$\|(u - \overline{(u_\varepsilon^*)_h})(.,y)\|_{L_2}$$

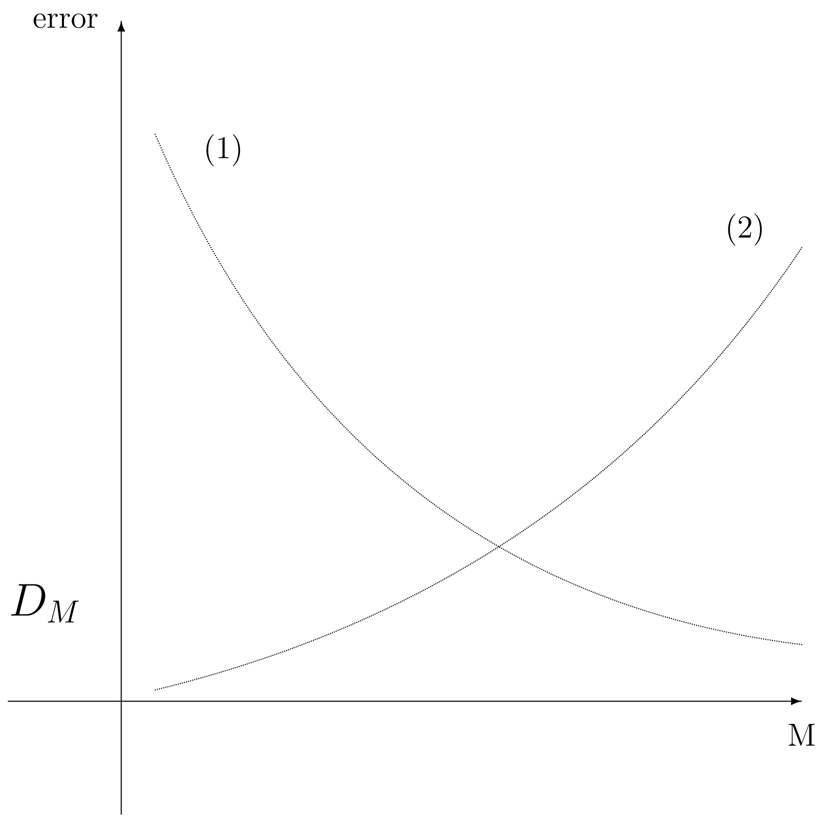
$$\le \; C_1(r_{max})E \cdot \left(\frac{M}{\exp(M\pi r_{max})}\right)^{1-\frac{y}{r_{max}}}$$

$$+ \; C_2(y)\frac{2}{\pi^2}\frac{\sinh(M\pi y)}{M}\varepsilon$$

$$+ \; \frac{M^4\pi^3 y}{12}\exp(M\pi y)\|(\phi_1)_\varepsilon^*\|_{L_1}h^2 \quad \forall y \in [0, r_{\max})\,.$$

where

$$\begin{aligned} \overline{(u_\varepsilon^*)_h}(x,y) &= \sum_{k=1}^{M}\left(2h\sum_{j=1}^{N-1}\sin(k\pi jh)u_{j,\varepsilon}^*(y)\right)\sin(k\pi x) \\ &= \text{continuation of } \left(u_{1,\varepsilon}^*(y),\ldots,u_{N-1,\varepsilon}^*(y)\right)^\top \text{ in } D_M \\ u_{i,\varepsilon}^*(y) &= \text{solution on i-th line with data } P_M\phi_1^\varepsilon\,. \end{aligned}$$



(1) projection error, (2) data error; ε, h fixed

– Optimal Convergence:

If $M = \left\lceil \dfrac{\ln 1/\varepsilon}{\pi r_{\max}} \right\rceil$ and $h = \sqrt{\varepsilon}$, then

$$\begin{aligned}
\|(u - \overline{(u_\varepsilon^*)_h})(.,y)\|_{L_2} \;\le\;& C_1(r_{max})E \cdot \left(\frac{\varepsilon \cdot \ln\left(\frac{1}{\varepsilon}\right)}{\pi r_{max}} + \varepsilon\right)^{1-\frac{y}{r_{max}}} \\
+\;& 2C_2(y)\exp(\pi y) r_{max} \frac{\varepsilon^{1-\frac{y}{r_{max}}}}{\ln\left(\frac{1}{\varepsilon}\right)} \\
+\;& \frac{y}{12\pi}\|(\phi_1)_\varepsilon^*\|_{L_1} \frac{\left(\ln\left(\frac{1}{\varepsilon}\right) + \pi r_{max}\right)^4 \cdot \exp(\pi y) \cdot \varepsilon^{1-\frac{y}{r_{max}}}}{r_{max}^4} \qquad \longrightarrow 0 \ (\varepsilon \to 0)
\end{aligned}$$

- **MORE GENERAL SITUATION:** $div\big(a(x)\nabla u\big) = 0$

  (Assumptions: $0 < r_a \leq a(x) \leq R_a, \big|a'(x)\big| \leq R'_a$)

  Same program:

  – Method of lines approximation
    Difficulty: Eigenvalues, -vectors are not explicitly known

  – Analyse discrete Sturm-Liouville-Eigenvalue Problem
    (convergence of eigenvalues and eigenvectors)

  – Use logarithmic convexity etc.

- **MORE GENERAL SITUATION:** $div\big(a(x)\nabla u\big) = 0$
  (Assumptions: $0 < r_a \le a(x) \le R_a, \big|a'(x)\big| \le R'_a$)

  Same program:

  - Method of lines approximation
    Difficulty: Eigenvalues, -vectors are not explicitly known
  - Analyse discrete Sturm-Liouville-Eigenvalue Problem
    (convergence of eigenvalues and eigenvectors)
  - Use logarithmic convexity etc.

  Similar results as for CPLE w.r.t. $(v_h, w_h)_{0,a} = h \sum_{j\in\mathbb{Z}} a(x_j)v_h(x_j)w_h(x_j)$;

  use projection P_M onto D_M also w.r.t. $(\cdot,\cdot)_{0,a}$

Convergence and Error Estimates:

For $M = \left\lceil \ln(1/\varepsilon)/(\pi r_{\max}\sqrt{\frac{R_a}{r_a}}) \right\rceil$, $h = \sqrt{\varepsilon}$

$$\begin{aligned}
\|u - \overline{u^*_{\varepsilon,h}}\|_a \;&\leq\; CE\left(\frac{\varepsilon^{\frac{r_a}{R_a}}\ln\left(\frac{1}{\varepsilon}\right)}{\pi r_{max}\sqrt{\frac{R_a}{r_a}}} + \varepsilon^{\frac{r_a}{R_a}}\right)^{1-\frac{y}{r_{max}}} \\
&+\; \frac{R_a^2 r_{max} C(y)\exp\left(\sqrt{\frac{R_a}{r_a}}\pi y\right)}{r_a}\cdot\frac{\varepsilon^{1-\frac{y}{r_{max}}}}{\ln\left(\frac{1}{\varepsilon}\right)} \\
&+\; C\varepsilon^{1-\frac{y}{r_{max}}}\ln\left(\frac{1}{\varepsilon}\right) \longrightarrow 0\,(\varepsilon \to 0)
\end{aligned}$$

- **Numerical results for Cauchy Problem (Hadamard's example):** $a = \text{const.}$

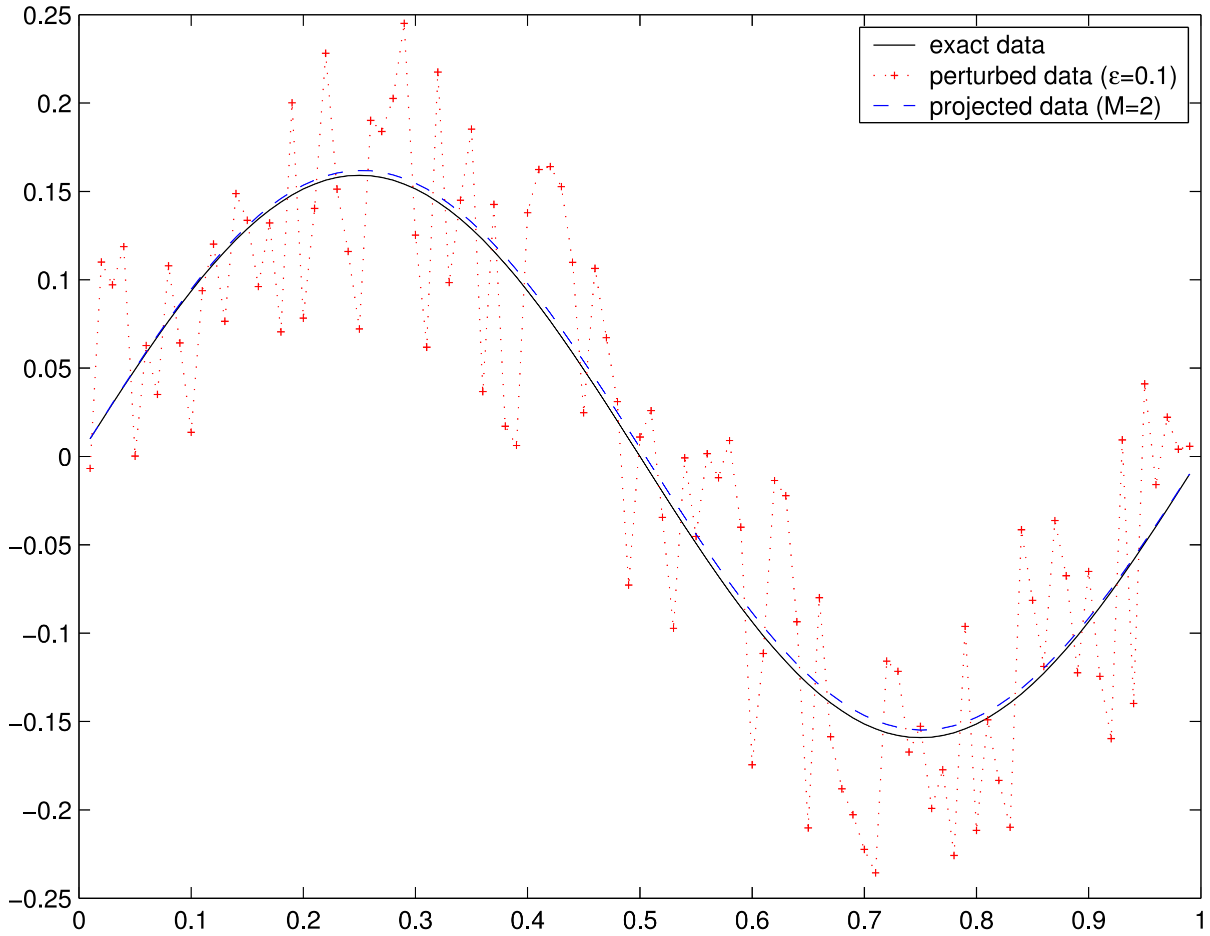


Figure 1: Projected data Φ_1 (from Hadamard Example) at $y = 0$ onto D_M with $\varepsilon = 0.1, h = \frac{1}{100}$, $m = 2$, $M = 2$

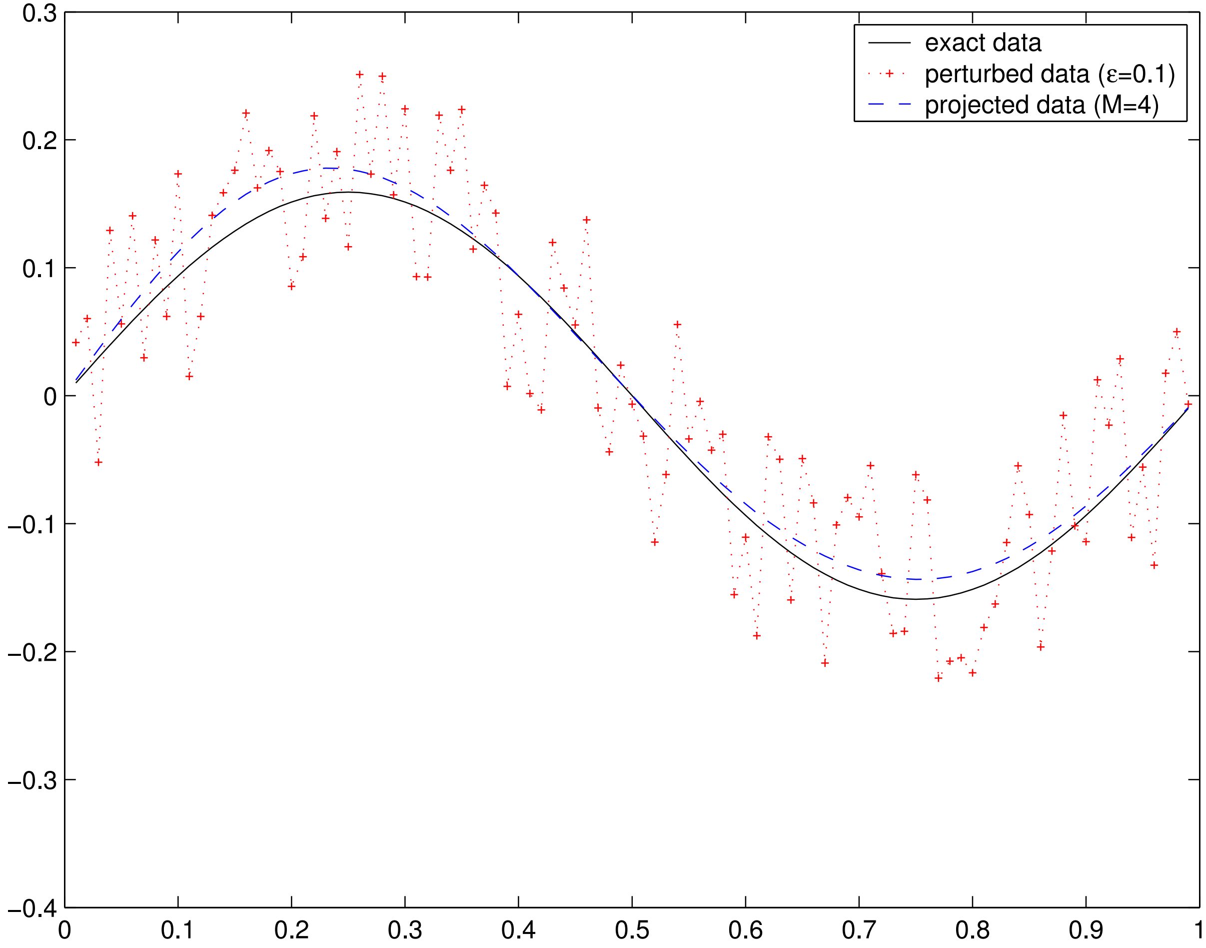


Figure 2: same as Fig. 1 with $M = 4$

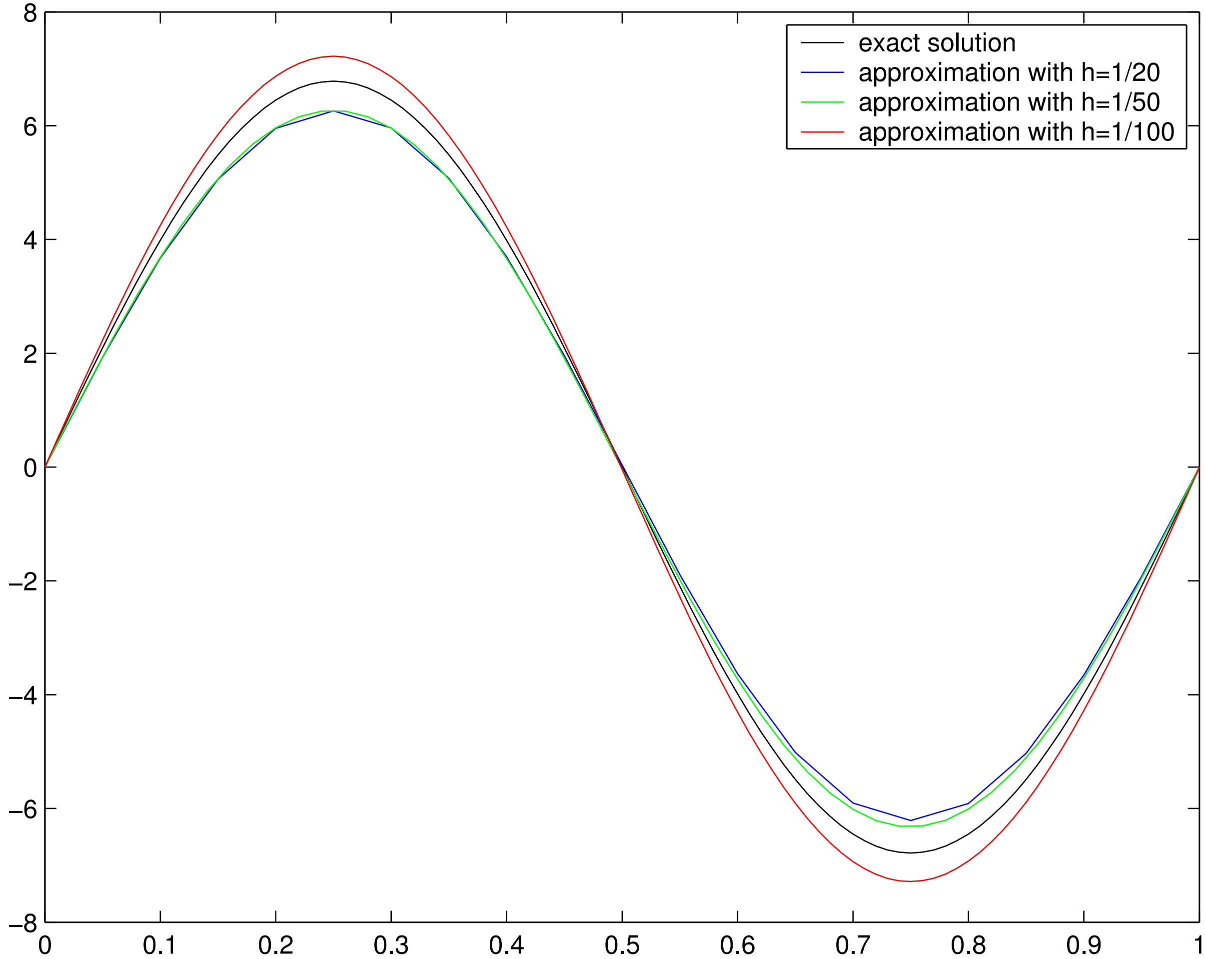


Figure 3: Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 0.1$, $m = 2$, $M = 2$ and different h's

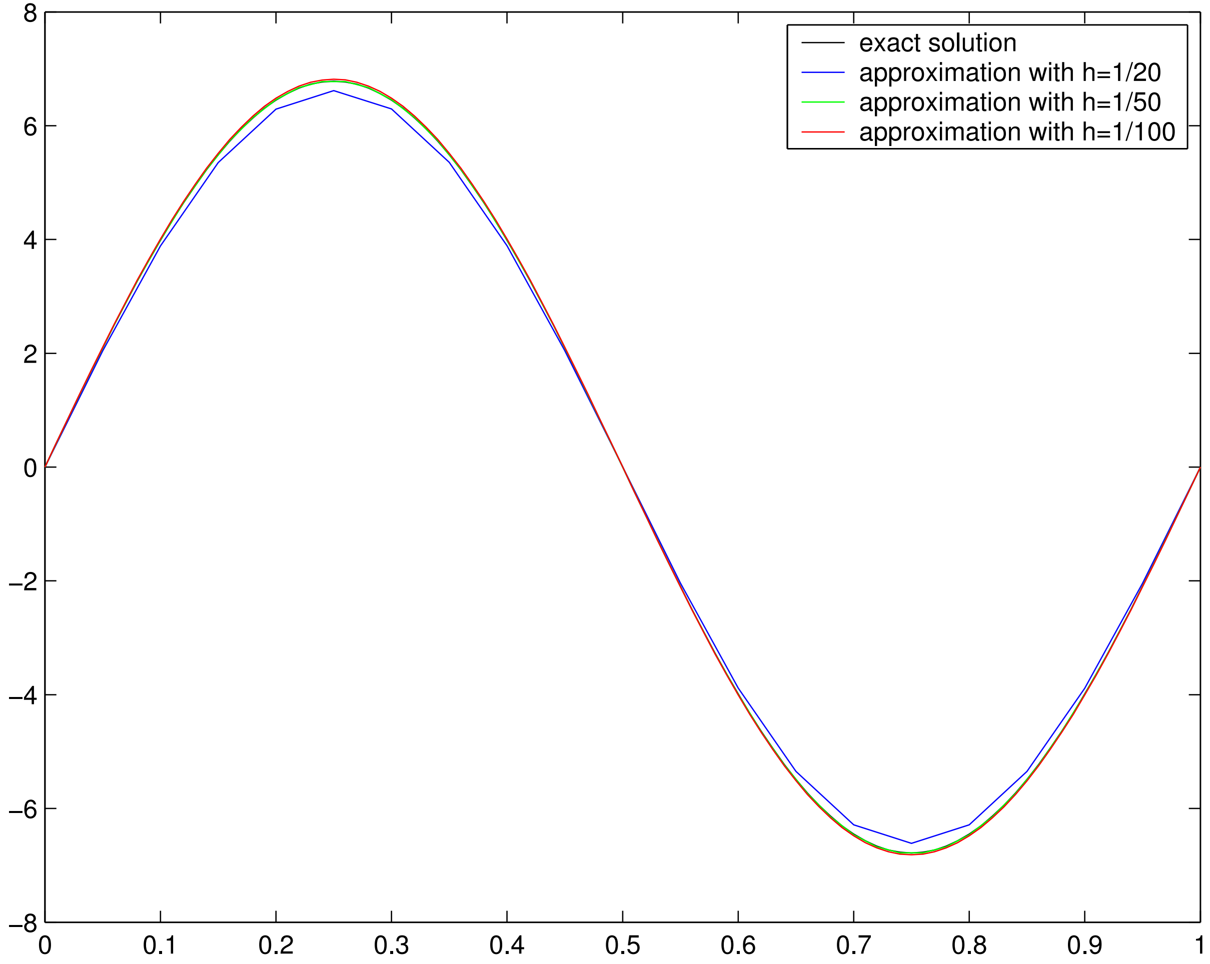


Figure 4: same as Fig. 3 with $\varepsilon = 10^{-2}$

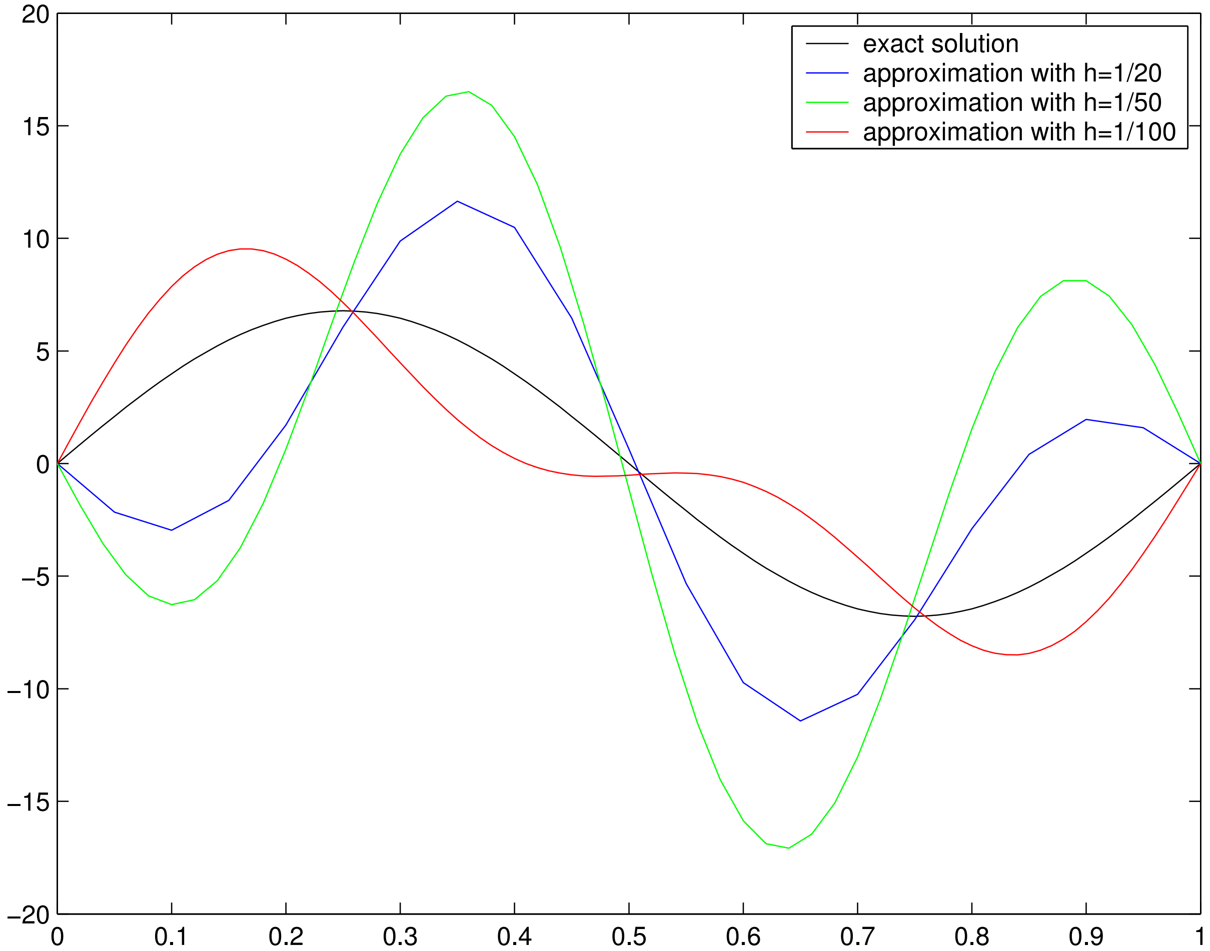


Figure 5: Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 10^{-2}$, $m = 2$, $M = 4$ and different h's

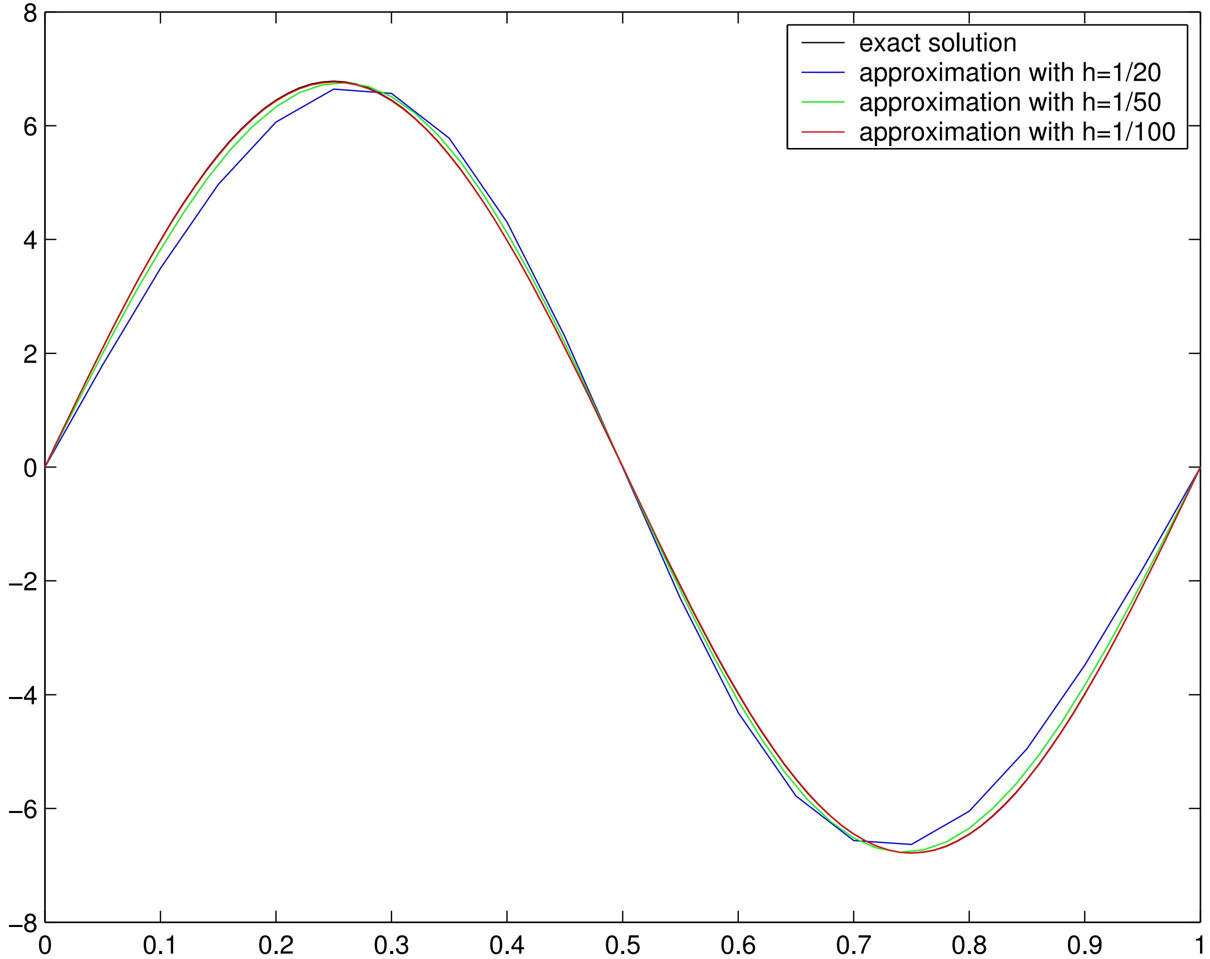


Figure 6: same as Fig. 5 with $\varepsilon = 10^{-4}$ (Note: $\varepsilon \approx 1/\exp$ M)

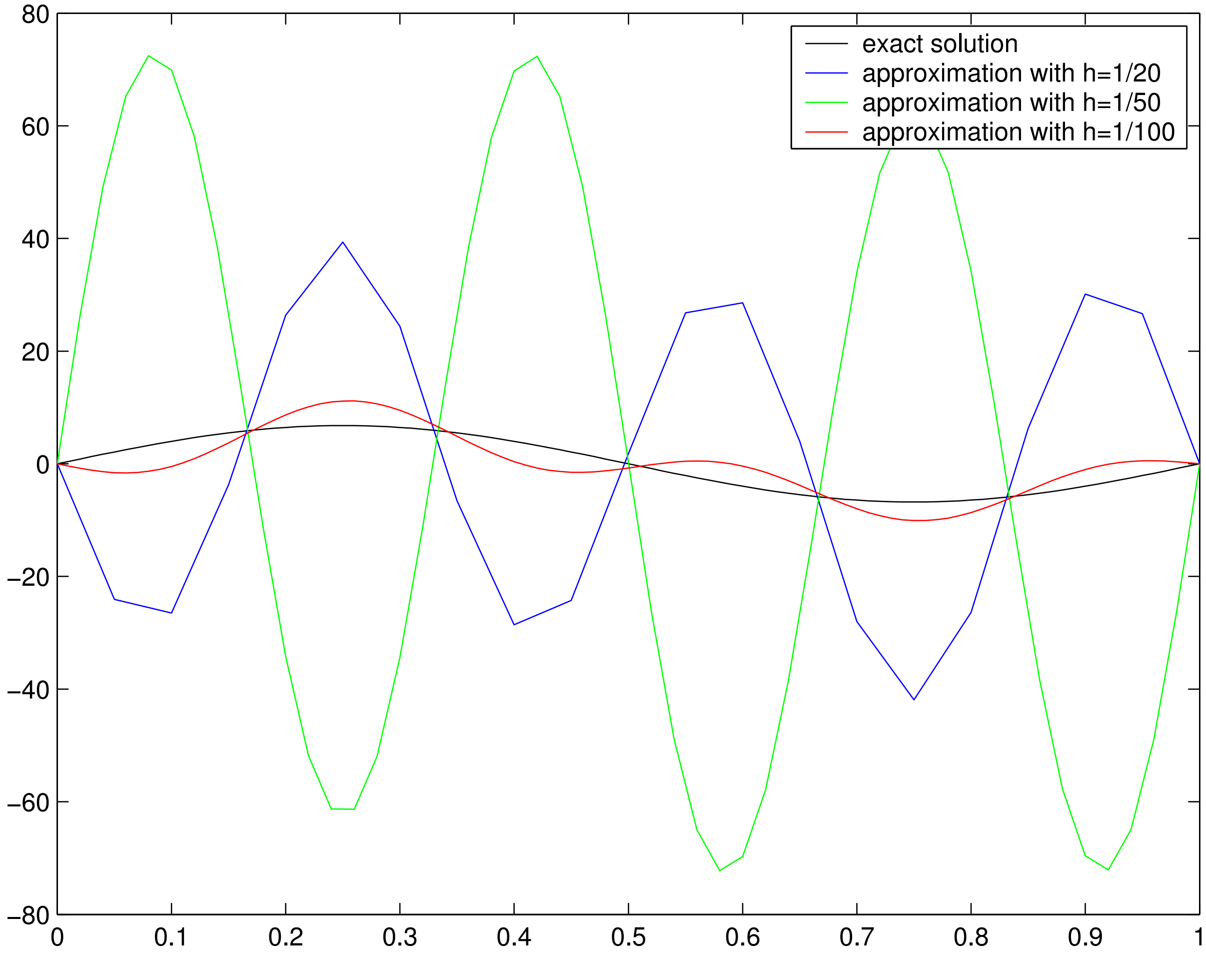


Figure 7: Solution and line method approximations of Hadamard Example at $y = 1$ for $\varepsilon = 10^{-4}$, $m = 2$, $M = 6$ and different h's

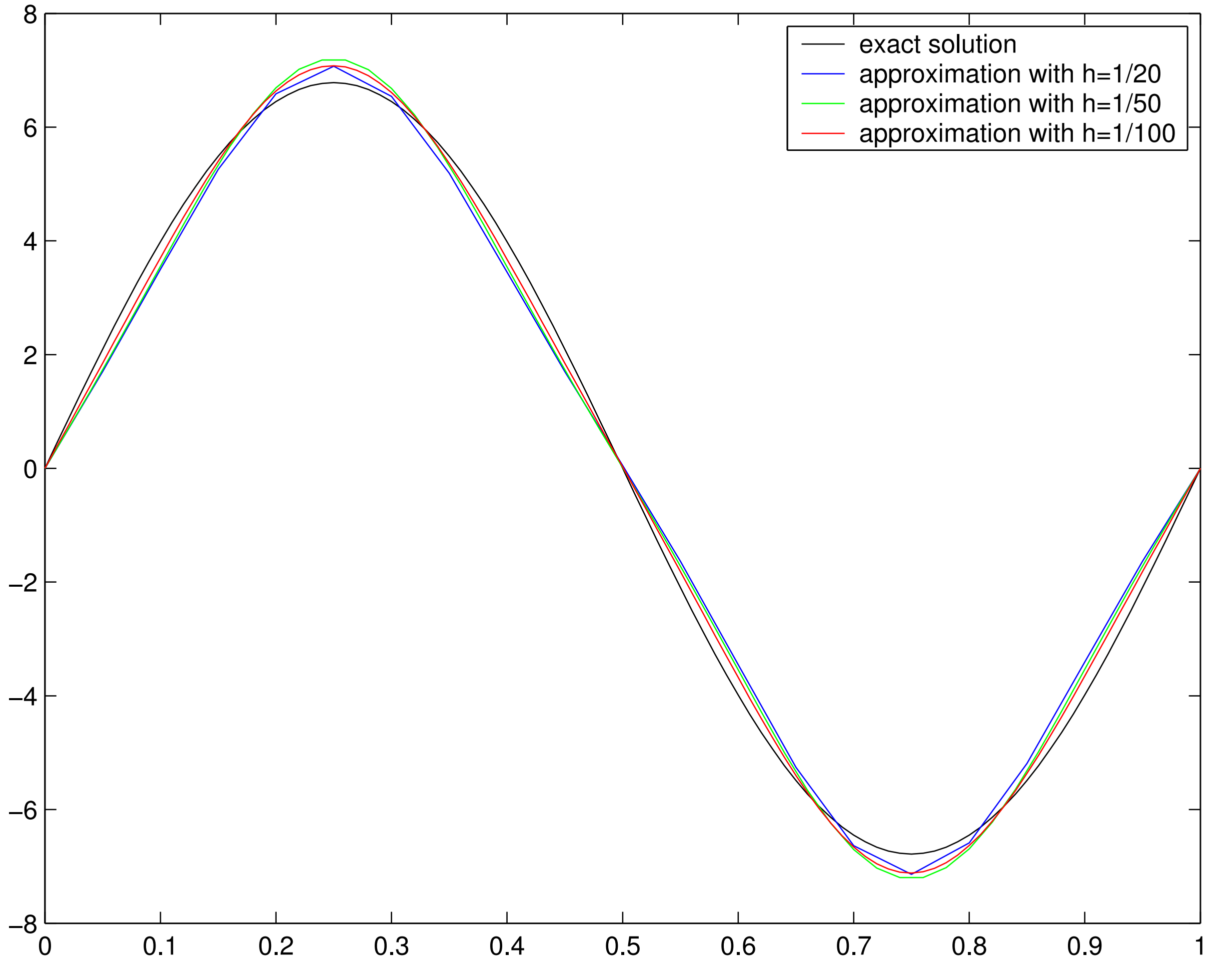


Figure 8: same as Fig. 7 with $\varepsilon = 10^{-6}$

- **Numerical results for Cauchy-Problem:** $a(x) = x + 1$

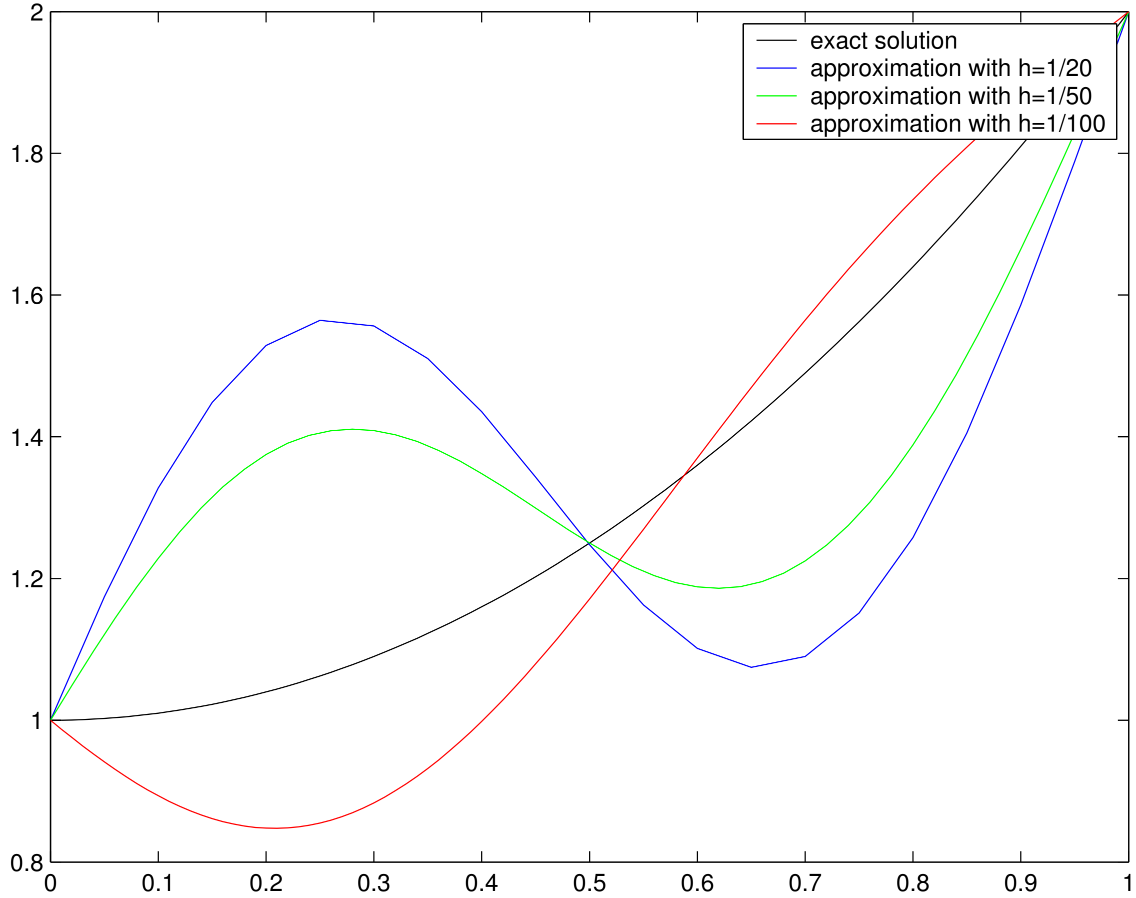


Figure 9: Solution $u = x^2 + y^2$ **and line method approximations in case of** $a(x) = x + 1$ **at** $y = 1$ **for** $\varepsilon = 10^{-1}$, $M = 2$, **different** h**'s (and** $\bar{h} = \frac{1}{200}$**)**

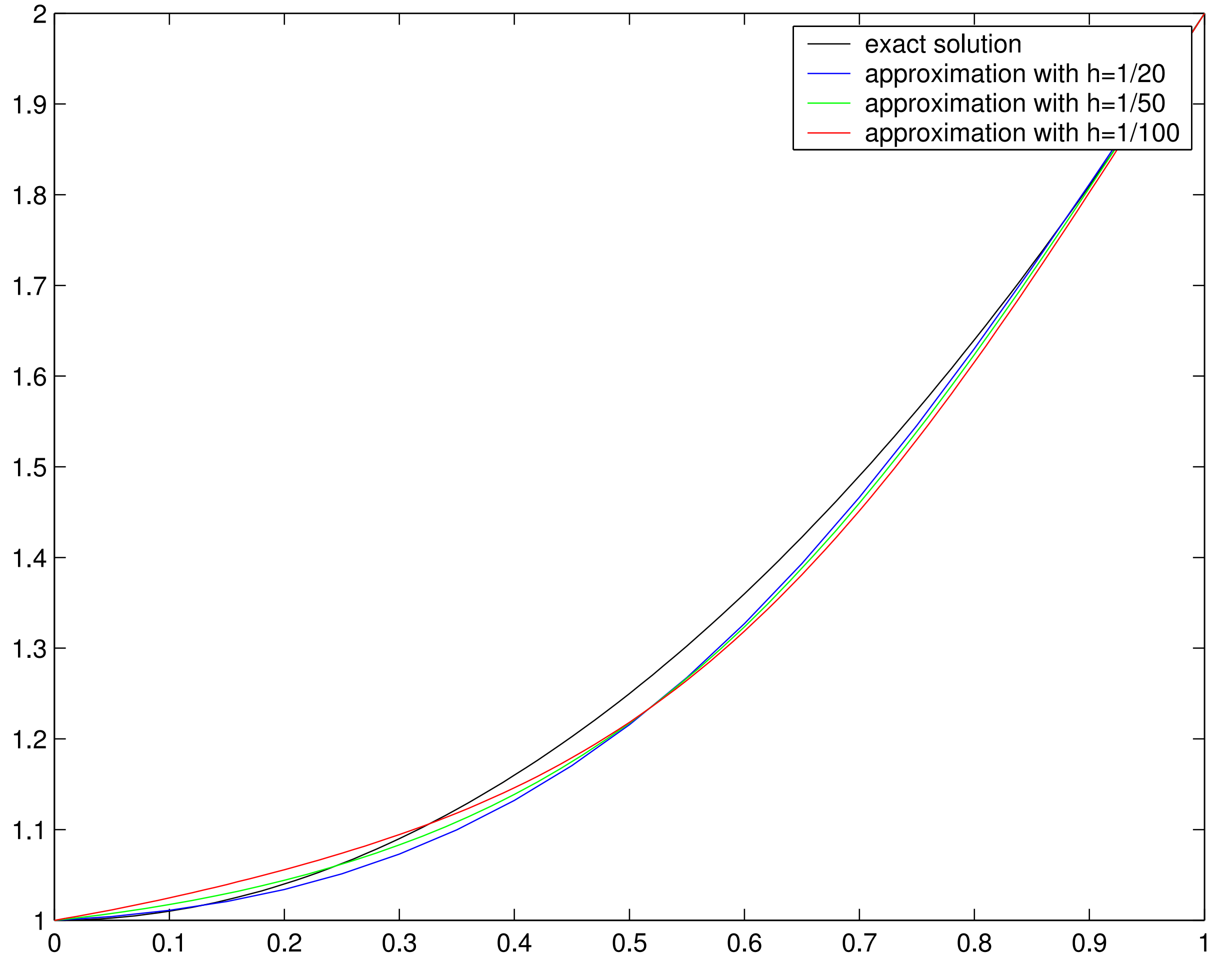


Figure 10: same as Fig. 9 with $\varepsilon = 10^{-3}$

Conclusions for Cauchy problem:

When the 2-d domain is a rectangle, or can be transformed to a rectangle, then

- the method of lines is a computable, efficient and convergent approximation scheme;
- the regularization parameter M (i.e. dimension of data space) can be chosen – and computed – in an optimal way depending on the magnitude ε of data perturbations, the bound E on the unknown part of the boundary, and the mesh width h.
- The general case $\nabla(a(x)\nabla u) = 0$ can be treated similarly.

e-mail: reinhardt@mathematik.uni-siegen.de