On parameter choice in self-regularization and standard regularization algorithms

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Problem and projection methods

- We consider an operator equation

$$Au = f,$$

where $A : H \to F$ is a linear bounded operator between real Hilbert spaces, $f \in \mathcal{R}(A) \Rightarrow \exists$ (minimum-norm) solution $u_* \in H$. Given are noisy data: $A_\eta \in \mathcal{L}(H, F)$ and $f_\delta \in F$ with $\|A_\eta - A\| \leq \eta$, $\|f_\delta - f\| \leq \delta$.

- Projection methods
  - $H_n \subset H$, $F_n \subset F$, $\dim H_n = \dim F_n < \infty$
  - P_n, Q_n orthoprojectors, $P_n \colon H \to H_n$, $Q_n \colon F \to F_n$
  - Find $u_n \in H_n$ for which $Q_n A_\eta u_n = Q_n f_\delta$.
  - Self-regularization: a proper choice of $n = n(\delta, \eta)$ guarantees $\|u_{n(\delta,\eta)} - u_*\| \to 0$ as $\delta \to 0$, $\eta \to 0$.

Minimal error method: $H_n = A_\eta^* F_n$

- If $A_\eta = A$, $f_\delta = f$, then $u_n = P_n u_*$.
- Let
  - $\mathcal{N}(A) = \{0\}$, $\mathcal{N}(A^*) = \{0\}$,
  - $\|z - Q_n z\| \to 0$ $(n \to \infty,\ \forall z \in F)$,
  - $F_n \subset F_{n+1}$ $(n \geq 1)$.
- If $\eta \sup_{z_n \in F_n} \|z_n\| / \|A^* z_n\| \leq 1$, then the equation $Q_n A_\eta u_n = Q_n f_\delta$ has unique solution $u_n = A_\eta^* z_n$.
- **ME-rule** (monotone error rule): find $n_{\mathrm{ME}} = n(\delta, \eta)$ as the first $n \in \mathbb{N}$ such that

$$\frac{(z_n - z_{n+1}, f_\delta)}{\|z_n - z_{n+1}\|} \leq 2(\delta + M\eta), \quad M \geq \|u_*\|.$$

  Then
  - $\|u_n - u_*\| \leq \|u_{n-1} - u_*\|$ $(n = 1, 2, \ldots, n_{\mathrm{ME}})$
  - $\|u_{n_{\mathbf{ME}}(\delta,\eta)} - u_*\| \to 0$ as $\delta \to 0$, $\eta \to 0$

Least squares method: $F_n = A_\eta H_n$

- Let
  - $\mathcal{N}(A) = \{0\}$
  - $\|u - P_n u\| \to 0$ as $n \to \infty$ $(\forall u \in H)$
  - $\exists m \in \mathbb{N}$: $(\kappa_n + \kappa_{n+1})^{1/m} \|(I - P_n)|A|^{1/m}\| \le \text{const}$ $(n \ge 1)$, where $|A| = (A^*A)^{1/2}$ and $\kappa_n \equiv \sup\limits_{w_n \in H_n} \dfrac{\|w_n\|}{\|Aw_n\|}$.
- If $\eta\kappa_n < 1$, then $Q_n A_\eta u_n = Q_n f_\delta$ has a unique solution $u_n \in H_n$.
- If $n_{\mathrm{D}} = n(\delta, \eta)$ is chosen by **discrepancy principle**: n_{D} is the first $n \in \mathbb{N}$ for which $\|Au_n - f_\delta\| \le C(\delta + \|u_n\|\eta)$ with $C > 1$, then

$$\|u_{n_D(\delta,\eta)} - u_*\| \to 0 \quad \text{as} \quad \delta \to 0,\ \eta \to 0.$$

Galerkin method: $F_n = H_n$

- Let
  - $H = F,\ A = A^* > 0,\ A_\eta = A_\eta^* \geq 0$
  - $\|u - P_n u\| \to 0$ as $n \to \infty$ $(\forall u \in H)$
  - $\exists m \in \mathbb{N}$: $(\kappa_n + \kappa_{n+1})^{1/m} \|(I - P_n)A^{1/m}\| \leq \text{const}$ $(n \geq 1)$, where

  $\kappa_n \equiv \sup\limits_{w_n \in H_n} \dfrac{\|w_n\|}{\|Aw_n\|}$.
- If $\eta\kappa_n < 1$, then $Q_n A_\eta u_n = Q_n f_\delta$ has a unique solution $u_n \in H_n$.
- If $n_{\mathrm{D}} = n(\delta, \eta)$ is chosen by the **discrepancy principle** with C large enough, then

$$\|u_{n_{\mathrm{D}}(\delta,\eta)} - u_*\| \to 0 \quad \text{as} \quad \delta \to 0,\ \eta \to 0.$$

Collocation method [Nashed, Wahba 1974]

- Consider integral equation

$$(Au)(t) \equiv \int_0^1 K(t,s)u(s)ds = f(t) \quad (0 \le t \le 1)$$

$$A: \ L_2(0,1) \to L_2(0,1), \quad \mathcal{N}(A) = \{0\}, \quad f \in C[0,1]$$

- Let the kernel satisfy
    - $\int_0^1 |K(t,s)|^2 ds \le \text{const} \quad (0 \le t \le 1),$
    - $\int_0^1 \left|K(t',s) - K(t,s)\right|^2 ds \to 0$ as $t' \to t \quad (0 \le t, \ t' \le 1)$
- Let the node set $\{t_i \in [0,1], t_i \neq t_j$ for $i \neq j, i,j \in I\}$ satisfy
    - $\{K(t_i,s), i \in I\}$ is a linear independent system
    - the indexsets I_n satisfy $I_n \subset I_{n+1} \subset \ldots \subset I \quad (n \ge 1)$
    - $\Delta_n = \sup_{t\in[0,1]} \inf_{i \in I_n} |t - t_i| \to 0$, as $n \to \infty$.

Collocation method (continuation)

- Given $f_\delta \in C[0,1]$, $\left|f_\delta(t_i) - f(t_i)\right| \le \delta_i$ $(i \in I)$, $K_\eta(t,s)$, $\max_{0\le s\le 1}|K_\eta(t_i,s) - K(t_i,s)| \le \eta_i$ $(i \in I)$, $\Delta_i = \delta_i + M\eta_i$, $\max_{0\le s\le 1}|u_*(s)| \le M$.
- Approximate solution $u_n = \sum_{i\in I_n} c_i^{(n)} K_\eta(t_i,s)$, where $\{c_i^{(n)}\}$ is the solution of the system

$$\sum_{i\in I_n} c_i^{(n)} \int_0^1 K_\eta(t_i,s)K_\eta(t_j,s)ds = f_\delta(t_j) \quad (j \in I_n).$$

- **ME-rule** for choice of discretization level $n_{\mathrm{ME}} = n(\delta,\eta)$: n_{ME} is the first $n = 1, 2, \ldots,$ for which

$$\|u_{n+1}\|^2 - \|u_n\|^2 \le \sum_{i\in I_{n+1}\setminus I_n} \left|c_i^{(n+1)}\right|\Delta_i + \sum_{i\in I_n}\left|c_i^{(n)} - c_i^{(n+1)}\right|\Delta_i$$

  Then
  - $\|u_n - u_*\| \le \|u_{n-1} - u_*\|$ $(n = 1, 2, \ldots, n_{\mathrm{ME}})$
  - $\|u_{n_{\mathbf{ME}}(\delta,\eta)} - u_*\| \to 0$, as $\lim_{n\to\infty}\sum_{i\in I_n}\Delta_i^2 = 0$.

Standard regularization methods

- In the following $A_\eta = A$ but we consider three cases of knowledge about noise level for $\|f_\delta - f\|$:
  - Case 1: exact noise level δ: $\|f_\delta - f\| \leq \delta$
  - Case 2: no information about $\|f_\delta - f\|$
  - Case 3: approximate noise level δ: $\lim \|f_\delta - f\|/\delta \leq C$ as $\delta \to 0$, with unknown constant C
- We consider standard regularization methods for discretized or non-discretized problems.

Motivation of extrapolation for Tikhonov method

- Tikhonov approximation: $u_\alpha = (\alpha I + A^*A)^{-1}A^*f$. Extrapolated approximations are linear combinations of u_α with different α. Examples: $v_{2,\alpha} = 2u_{\alpha/2} - u_\alpha$, $v_{3,\alpha} = \frac{8}{3}u_{\alpha/4} - 2u_{\alpha/2} + \frac{1}{3}u_\alpha$.
- Error estimate: if $\|f_\delta - f\| \leq \delta$ and

$$u_* \in \mathcal{R}(|A|^p),$$

then for proper α

$$\begin{aligned} \|u_\alpha - u_*\| &\leq \text{const } \delta^{p/(p+1)} && (p \leq 2) \\ \|v_{2,\alpha} - u_*\| &\leq \text{const } \delta^{p/(p+1)} && (p \leq 4) \\ \|v_{3,\alpha} - u_*\| &\leq \text{const } \delta^{p/(p+1)} && (p \leq 6) \end{aligned}$$

Approximate solutions

- Landweber iteration method: $u_n = u_{n-1} - \mu A^*(Au_{n-1} - f_\delta)$, $n = 1, 2, \ldots, u_0 = 0$.
- Tikhonov method: $v_{1,\alpha} = u_\alpha = (\alpha I + A^*A)^{-1}A^*f$
- Extrapolated Tikhonov approximations, computed on grid $\alpha_i = q^i$, $i = 0, 1, \ldots$ ($q < 1$, we used $q = 0.9$):

$$v_{2,\alpha_i} = (1-q)^{-1}(u_{\alpha_{i+1}} - qu_{\alpha_i}).$$

In general

$$v_{n,\alpha_i} = \sum_{j=i}^{n+i-1} d_j u_{\alpha_j}, \qquad d_j = \prod_{k=i,k\neq j}^{n+i-1} \frac{1}{1-q^{j-k}}$$

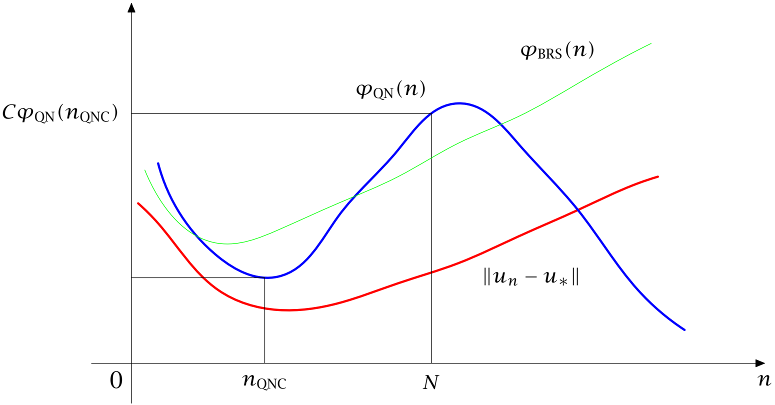
Regularization parameter may be n (approach 1) or α_i (approach 2).

Rules for Landweber approximation

The first rule is for noise information case 1, others for case 2.

- **Discrepancy principle:** choose $n = n_{\mathrm{D}}$ as the first $n \in \mathbb{N}$ for which $\|r_n\| \leq \delta$, where $r_n = Au_n - f_\delta$.
- **Rules HR, QN, BRS**: find n as the global minimizer of the functions $\sqrt{n}\,\|r_n\|$, $\|u_n - u_{2n}\|$, $\|r_n\|^2/\|A^* r_n\|$, respectively, $n \geq 1$.
- **Rules HRmC, QNmC, BRSC**: find n as the minimizer of $\psi(n)$ (HRmC: $\psi(n) = \sqrt{n}\,(\|r_n\| - \|r_{2n+100}\|)$,
  QNmC: $\psi(n) = \|u_n - u_{2n+100}\|$,
  BRSC: $\psi(n) = \|r_n\| \cdot (\|r_n\| - \|r_{2n+100}\|)/\|A^* r_n\|)$ in the interval $[1, N]$, where N is the smallest n, for which the value of $\psi(n)$ is C times (QNmC: $C = 20$, HRmC: $C = 50$, BRSC: $C = 15$) larger than its value at the minimum point.

Typical behaviour of functions in heuristic rules



Rules for choice of parameters in v_{k,α_i} for exact noise level δ

- We compute v_{k,α_i} and $r_{k,\alpha_i} = Av_{k,\alpha_i} - f$ for $\alpha_0 = 1$, $\alpha_1 = q$, $\alpha_2 = q^2, \ldots$ until some condition is satisfied.
- Approach 1. Discrepancy principle: choose $k = k_{\mathrm{maxD}}$ as the first k with
$$d_{\mathrm{maxD}}(k) := \|r_{k,1}\| \leq \delta.$$
- Approach 2. We choose α_i in v_{k,α_i} ($k \in \{1,2\}$) by rules:
  - α_{D} is α_i with first i for which $\|r_{k,\alpha_i}\| \leq \delta$.
  - α_{ME} is α_i with first i for which
  $$(r_{k,\alpha_i}, r_{k+1,\alpha_i})/\|r_{k+1,\alpha_i}\| \leq \delta.$$
  - α_{R2} is α_i with first i for which
  $$d_{\mathrm{R2}}(\alpha_i) := \frac{\sqrt{\alpha_i}\,\|v_{k,\alpha_i} - v_{k+1,\alpha_i}\|^2 (1 + \alpha\|A\|^{-2})}{(v_{k,\alpha_i} - v_{k+1,\alpha_i}, v_{k+1,\alpha_i} - v_{k+2,\alpha_i})^{1/2}} \leq b_k \delta$$

  with $b_1 = 0.3$, $b_2 = 0.2$.

Post-estimation strategy for parameter choice in case of exact noise level δ

- In Landweber method: $n_{\mathrm{D}} \leq n_{\mathrm{opt}} + 1$, $n_{\mathrm{opt}} := \mathrm{argmin}\{\|u_n - u_*\|, n \geq 0\}$, $n_{\mathrm{De}} = \mathrm{round}(cn_{\mathrm{D}})$, $c = 2.3$.
- In v_{k,α_i} in approach 1 rule maxDe: find $k = \mathrm{maxD}$, replace α_i by nearest 0.9^n $(n = 0, 1, \dots)$ to $\alpha_i^{1.1}$.
- In Tikhonov method:
  - $\alpha_{\mathrm{ME}} \geq \alpha_{\mathrm{opt}} := \mathrm{argmin}\{\|u_\alpha - u_*\|, \alpha \geq 0\}$, computations suggested $\alpha_{\mathrm{MEe}} = \min(0.53\alpha_{\mathrm{ME}}, 0.6\alpha_{\mathrm{ME}}^{1.06})$, if $\|f_\delta - f\| = \delta$
  - $\alpha_{\mathrm{R2e}} = 0.5\alpha_{\mathrm{R2}}$ is good in case $\|f_\delta - f\| < \delta$
  - In both cases $\alpha_{\mathrm{Me}} = \min(\alpha_{\mathrm{MEe}}, \alpha_{\mathrm{R2e}})$ chooses the best of α_{MEe} and α_{R2e}.
  - Application of rule Me in extrapolated Tikhonov approximation with 2 terms resulted in rule 2Me.

Rule for choice of α in case of approximate noise level δ

Rule DM for (extrapolated) Tikhonov method

1) find $\underline{\alpha}$ as the first α_i for which $\sqrt{\alpha_i}\|v_{k,\alpha_i} - v_{k+1,\alpha_i}\| \leq c_1\delta$, $c_1 = \text{const}$ (we used $c_1 = 0.001 \ldots 0.02$);
2) find $\alpha_i = \operatorname{argmin} \varphi_{\mathrm{R2}}(\alpha_i)\, \alpha_i^{c_2}$ on $[\underline{\alpha}, 1]$, where $\varphi_{\mathrm{R2}}(\alpha) = d_{\mathrm{R2}}(\alpha)/\sqrt{\alpha}$. We used $c_2 = 0.03 \ldots 0.14$. If the first condition is not fulfilled up to $\alpha_i = 10^{-30}$, then $\underline{\alpha} = 10^{-30}$.

Convergence and convergence rate of $v_{k,\alpha}$

- Approach 1. Discrepancy principle gives $k = k_{\mathrm{maxD}}$ satisfying $\|v_{k,1} - u_*\| < \|v_{k-1,1} - u_*\|$ for $k = 1, 2, \ldots, k_{\mathrm{maxD}} - 1$, $\|v_{k_{\mathrm{maxD}},1} - u_*\| \to 0$ $(\delta \to 0)$. If $u_* \in \mathcal{R}(|A|^p)$, then $\|v_{k_{\mathrm{maxD}},1} - u_*\| \leq \mathrm{const}\, \delta^{p/(p+1)}$ for all $p > 0$.
- Approach 2. Convergence $v_{k,\alpha} \to u_*$ $(\delta \to 0)$ is guaranteed for choice of α by rules ME and R2 (and also by rule DM, if $\lim \|f_\delta - f\|/\delta \leq C$ as $\delta \to 0$). If $u_* \in \mathcal{R}(|A|^p)$, the rules ME and R2 (and DM if $c_1 \geq 0.24$) guarantee $\|v_{k,\alpha} - u_*\| \leq \mathrm{const}\, \delta^{p/(p+1)}$ $(p \leq 2k)$.
- If the parameter choice rule does not use δ, no convergence $v_{k,\alpha} \to u_*$ $(\delta \to 0)$ is guaranteed.

Heuristic rules in $v_{k,\alpha}$ not using noise level δ

- **Rules Q, HR, BRS**: find α_i as the global minimizer of $\psi(\alpha_i)$ (Q: $\psi(\alpha_i) = \varphi_{\mathrm{Q}}(\alpha_i) = \|v_{k,\alpha_i} - v_{k+1,\alpha_i}\|$,
  HR: $\psi(\alpha_i) = \varphi_{\mathrm{HR}}(\alpha_i) = (r_{k,\alpha_i}, r_{k+1,\alpha_i})/\sqrt{\alpha_i}$,
  BRS: $\psi(\alpha_i) = \varphi_{\mathrm{BRS}}(\alpha_i) = \|r_{k,\alpha_i}\|^2/(\alpha_i\|v_{k,\alpha_i}\|)$).
- **Rules QC, R2C and BRSC**: find α_i as the minimizer of $\psi(\alpha_i)$ (QC: $\psi(\alpha_i) = \|v_{k,\alpha_i} - v_{k+1,\alpha_i}\|$, R2C: $\psi(\alpha_i) = d_{\mathrm{R2}}(\alpha_i)$, BRSC: $\psi(\alpha_i) = \|r_{k,\alpha_i}\|^2/(\alpha_i\|v_{k,\alpha_i}\|)$) on the interval $[\underline{\alpha}, 1]$, where $\underline{\alpha}$ is the largest α_i, for which the value of $\psi(\alpha_i)$ is C times (QC, R2C: $C = 5$; BRSC: $C = 3$) larger than its value at the minimum point.

Heuristic rules in $v_{k,\alpha}$ not using noise level δ (continuation)

- **Rules DR21 and BRS1**: choose α_i as the largest local minimizer of functions $\|r_{k,\alpha}\|^{c_1}\varphi_{\mathrm{R2}}(\alpha)^{1-c_1}\alpha^{c_2}$ and $\varphi_{\mathrm{BRS}}(\alpha)\alpha^{c_3}$, respectively, where $c_1 = 0.25$, $c_2 = 0.33$, $c_3 = 0.5$ for $k = 1$, $c_1 = 0.9$, $c_2 = 0.4$, $c_3 = 0.58$ for $k = 2$.
- **Rule BR2**: choose α_i as the global minimizer of $\varphi_{\mathrm{BR2},\tau}(\alpha) = \varphi_{\mathrm{R2}}(\alpha)^{(\varphi_{\mathrm{R2}}(\alpha)/\varphi_{\mathrm{BRS}}(\alpha))^{\tau}}\varphi_{\mathrm{BRS}}(\alpha)^{1-(\varphi_{\mathrm{R2}}(\alpha)/\varphi_{\mathrm{BRS}}(\alpha))^{\tau}}$, $\tau = 0.05$.
- **Rule QHR**: choose α_i as the local minimizer of the function $\varphi_{\mathrm{Q}}(\alpha)(1 + \alpha\|A\|^{-2})$ such that the function $\varphi_{\mathrm{HR}}(\alpha)$ is minimal.

Test problems

- Test problems: 10 problems from P. C. Hansen's *Regularization tools*: baart, deriv2, foxgood, gravity, heat, i_laplace, phillips, shaw, spikes, wing
- Besides solution u_* also smoother solution $u_{*,p} = (A^*A)^{p/2}u_*$ with $f = Au_{*,p}$ was used.
- The problems were normalized, so that Euclidean norms of the operator and the right hand side were 1.
- For perturbed data we took $f_\delta = f + \Delta$, $\|\Delta\| = 0.5, 10^{-1}, \ldots, 10^{-6}$ with 10 different perturbations Δ generated by computer.

Data in Tables

- In the following tables we present averages of error ratios $\|u_\lambda - u_*\|/e_{\text{opt}}$, where $e_{\text{opt}} = \min\{\|u_\lambda - u_*\| : \lambda = \alpha \geq 0$ in Tikhonov method, $\lambda = n \in \mathbb{N}$ in Landweber method$\}$.
- We use $\delta = d\|f_\delta - f\|$ with $d = 1$, 2 in Tables 1, 3, 4 and many values in Tables 2, 7, 8, 9. Rules of Tables 5, 6 do not use δ. The column heading R,2 means that results for rule R are computed with $d = 2$ (with 2 times overestimated noise level).

Results for Tikhonov method in case $d = 1$ and $d = 2$.

p	MEe	MEe,2	R2e	R2e,2	Me	Me,2	2Me	2Me,2	maxDe	maxDe,2
0	1.15	2.01	1.39	1.59	1.16	1.58	1.24	1.51	1.17	2.25
0.25	1.56	3.37	2.63	3.04	1.57	2.55	1.79	2.40	1.63	3.82
0.5	1.56	3.92	2.02	2.59	1.58	2.59	1.74	2.42	1.60	4.65
0.75	1.35	3.66	1.57	2.08	1.36	2.08	1.41	1.83	1.28	4.47
1	1.18	3.34	1.33	1.65	1.19	1.65	1.15	1.41	0.99	4.00
1.5	1.13	2.93	1.14	1.28	1.14	1.28	0.83	0.88	0.71	3.34
2	1.12	2.75	1.11	1.17	1.13	1.17	0.69	0.67	0.52	2.67
4	1.12	2.72	1.11	1.14	1.13	1.14	0.55	0.48	0.30	1.78
8	1.12	2.69	1.11	1.14	1.13	1.14	0.54	0.47	0.26	1.47
mean	1.26	3.04	1.49	1.74	1.26	1.69	1.10	1.34	0.94	3.16

Results in Tikhonov method for rules D and R2e.

	$p \setminus d$	0.5	0.6	0.8	1	1.3	1.6	2	3	5
Rule D	0	$\gg 1$			1.19	1.82	2.05	2.23	2.56	3.06
	0.5				1.69	3.01	3.62	4.09	5.00	6.44
	1				1.81	2.28	2.76	3.14	4.01	5.69
	2				2.83	2.37	2.61	2.76	3.27	4.52
	4				3.01	2.45	2.70	2.80	3.26	4.49
	8				3.03	2.46	2.69	2.79	3.24	4.45
	mean				2.14	2.46	2.84	3.12	3.77	5.04
Rule R2e	0	3.84	1.75	1.36	1.39	1.46	1.52	1.59	1.76	2.09
	0.5	5.17	2.49	1.98	2.02	2.19	2.37	2.59	3.11	3.79
	1	4.73	1.93	1.29	1.33	1.41	1.51	1.65	2.01	2.74
	2	4.43	1.53	1.14	1.11	1.11	1.13	1.17	1.35	1.75
	4	4.07	1.54	1.14	1.11	1.10	1.11	1.14	1.29	1.64
	8	4.08	1.54	1.14	1.11	1.10	1.11	1.14	1.28	1.63
	mean	4.47	1.91	1.48	1.49	1.56	1.63	1.74	2.02	2.54

Results in Landweber method for $p = 0$.

Probl.	D	D,2	De	De,2	HR	HRmC	QN	QNmC	BRS	BRSC
1	1.46	2.47	1.40	2.36	2.71	2.50	1.80	1.75	2.60	2.03
2	1.30	1.91	1.05	1.43	976	1.95	976	1.59	976	1.62
3	1.92	6.36	1.76	5.00	7.75	2.98	5.01	3.63	6.29	3.61
4	1.42	2.84	1.16	2.13	2.70	1.83	1.27	1.61	2.90	1.40
5	1.22	1.89	1.05	1.53	1.67	7.39	1e+4	2.37	1.84	2.17
6	1.34	1.96	1.24	1.72	2.00	1.39	1.24	1.30	2.11	1.27
7	1.33	2.79	1.08	1.96	2e+5	1.44	2e+5	1.49	2e+5	1.39
8	1.40	2.36	1.29	2.03	2.58	2.11	1.60	1.56	2.51	1.55
9	1.02	1.05	1.02	1.05	1.07	1.07	1.05	1.04	1.07	1.05
10	1.19	1.37	1.18	1.35	1.55	1.41	1.48	1.41	1.49	1.41
mean	1.36	2.50	1.22	2.06	2e+4	2.41	2e+4	1.77	2e+4	1.75

Results in Landweber method for $p = 2$.

Probl.	D	D,2	De	De,2	HR	HRmC	QN	QNmC	BRS	BRSC
1	2.78	27.5	2.96	11.8	52.5	24.2	6.05	7.84	19.3	6.55
2	1.22	4.03	1.23	2.21	2e+4	1.29	2e+4	1.64	2e+4	1.39
3	1.71	28.3	2.91	8.25	38.6	8.04	6.67	8.25	18.2	7.94
4	1.50	6.22	1.26	2.29	3.38	2.99	1.49	3.02	4.88	1.48
5	1.19	3.56	1.18	1.83	2.34	1.26	1e+5	1.49	2.89	1.16
6	1.64	6.82	1.22	3.00	4.11	3.55	1.37	2.34	6.59	1.56
7	1.44	5.95	1.25	2.04	5e+5	1.59	5e+5	1.81	5e+5	1.32
8	1.60	7.55	1.27	3.01	3.89	8.20	1.93	2.47	5.50	2.22
9	1.78	10.4	1.38	3.94	6.76	5.59	1.57	1.76	7.92	1.65
10	2.39	31.0	2.74	8.34	33.9	11.9	9.62	9.15	19.4	8.99
mean	1.73	13.1	1.74	4.67	5e+4	6.86	6e+4	3.98	5e+4	3.43

Results in Tikhonov method by problems for heuristic rules, $p = 0$.

Probl.	Q	HR	BRS	R2C	QC	BRSC	Q1	DR21	BR2	QHR
1	1.56	2.76	2.60	1.84	1.89	2.60	2.29	2.03	1.81	2.79
2	1.87	960	960	1.10	1.14	1.35	1.06	1.50	1.97	1.21
3	2.18	8.16	5.24	2.11	2.18	5.24	1.80	2.50	2.72	9.56
4	1.13	2.81	2.08	1.11	1.13	2.08	1.40	1.06	1.06	1.10
5	1e+4	1.70	1.35	1.19	1.25	1.35	1.13	1.14	1.22	1.82
6	1.19	2.05	1.87	1.18	1.19	1.87	1.20	1.23	1.16	1.27
7	1.08	1e+5	1e+5	1.08	1.08	1.61	1.16	1.08	1.09	1.09
8	1.44	2.57	2.25	1.46	1.47	2.25	1.73	1.43	1.42	2.38
9	1.04	1.07	1.06	1.05	1.05	1.06	1.08	1.05	1.05	1.06
10	1.43	1.56	1.55	1.44	1.79	1.55	1.83	1.47	1.47	1.55
mean	1e+3	1e+4	1e+4	1.35	1.42	2.10	1.47	1.45	1.50	2.38

Results for 2-extrapolated Tikhonov approximation.

	$p = 0$					$p = 2$				
Probl.	QC	R2C	BRSC	DR21	BRS1	QC	R2C	BRSC	DR21	BRS1
1	1.90	1.89	2.63	2.18	2.69	1.10	1.17	2.44	0.68	1.51
2	1.62	1.58	1.36	1.59	1.36	0.61	0.67	1.07	0.62	0.73
3	2.43	2.38	7.09	2.99	4.15	0.82	0.86	1.90	0.53	1.11
4	1.11	1.09	2.23	1.15	1.68	0.64	0.69	0.90	0.60	0.61
5	1.29	1.26	1.42	1.83	1.17	0.72	0.78	0.97	0.74	0.73
6	1.19	1.18	1.85	1.28	1.50	0.70	0.76	1.12	0.65	0.75
7	1.09	1.08	1.78	1.07	1.33	0.54	0.58	0.75	0.54	0.55
8	1.53	1.52	2.44	1.48	1.94	0.81	0.87	1.28	0.66	0.84
9	1.05	1.05	1.07	1.05	1.07	0.82	0.87	1.99	0.75	1.22
10	1.42	1.42	1.88	1.51	1.86	1.19	1.28	3.07	1.02	1.85
mean	1.46	1.45	2.37	1.61	1.87	0.79	0.85	1.55	0.68	0.99

Results in Tikhonov method for Rule DM, $c_1 = 0.002$, $c_2 = 0.03$.

	Case $p = 0$, values of d							Case $p = 2$, values of d						
Probl.	0.01	0.1	0.5	1	2	10	100	0.01	0.1	0.5	1	2	10	100
1	1.46	1.46	1.46	1.46	1.49	1.69	2.51	1.93	1.93	1.93	1.84	1.74	1.33	3.33
2	1.56	1.56	1.34	1.08	1.08	1.07	1.25	1.22	1.22	1.22	1.22	1.22	1.22	1.43
3	2.02	2.02	2.02	2.02	2.02	1.84	5.88	1.55	1.55	1.55	1.55	1.55	1.26	3.23
4	1.12	1.12	1.12	1.12	1.12	1.11	1.62	1.33	1.33	1.33	1.33	1.33	1.25	1.75
5	1.66	1.16	1.16	1.10	1.10	1.10	1.17	1.21	1.21	1.21	1.21	1.21	1.21	1.15
6	1.16	1.16	1.16	1.16	1.16	1.16	1.44	1.33	1.33	1.33	1.33	1.33	1.29	1.82
7	1.11	1.11	1.11	1.11	1.11	1.11	1.36	1.21	1.21	1.21	1.21	1.21	1.21	1.37
8	1.39	1.39	1.39	1.39	1.39	1.46	2.06	1.49	1.49	1.49	1.49	1.49	1.26	2.02
9	1.03	1.03	1.03	1.03	1.03	1.03	1.05	1.42	1.42	1.42	1.42	1.42	1.26	2.38
10	1.42	1.42	1.42	1.42	1.42	1.47	1.54	2.20	2.20	2.20	1.85	1.62	1.28	3.85
mean	1.39	1.34	1.32	1.29	1.29	1.30	1.99	1.49	1.49	1.49	1.44	1.41	1.26	2.23

Results in 2-extrapolated Tikhonov method for Rule DM, $c_1 = 0.002$, $c_2 = 0.03$.

	Case $p = 0$, values of d							Case $p = 2$, values of d						
Probl.	0.01	0.1	0.5	1	2	10	100	0.01	0.1	0.5	1	2	10	100
1	1.53	1.53	1.53	1.54	1.56	1.71	2.55	1.24	1.24	1.24	1.17	1.08	0.87	2.13
2	7.93	5.04	1.57	1.57	1.56	1.09	1.20	15.9	0.68	0.68	0.68	0.68	0.67	0.94
3	2.55	2.55	2.55	2.55	2.55	2.27	6.26	0.88	0.88	0.88	0.88	0.89	0.60	1.77
4	1.11	1.11	1.11	1.11	1.11	1.08	1.62	0.70	0.70	0.70	0.70	0.70	0.65	0.87
5	6.42	2.37	1.15	1.15	1.10	1.10	1.17	0.79	0.79	0.79	0.79	0.79	0.79	0.82
6	1.19	1.19	1.19	1.19	1.19	1.19	1.44	0.77	0.77	0.77	0.77	0.77	0.74	1.05
7	1.08	1.08	1.08	1.08	1.08	1.08	1.34	0.58	0.58	0.58	0.58	0.58	0.58	0.72
8	1.45	1.45	1.45	1.45	1.46	1.49	2.12	0.92	0.92	0.92	0.92	0.86	0.76	1.20
9	1.04	1.04	1.04	1.04	1.04	1.04	1.06	0.89	0.89	0.89	0.89	0.89	0.79	1.58
10	1.42	1.42	1.42	1.42	1.43	1.48	1.54	1.34	1.34	1.34	1.13	1.05	0.77	2.59
mean	2.57	1.88	1.41	1.41	1.41	1.35	2.03	2.40	0.88	0.88	0.85	0.83	0.72	1.37

Rule DM, $p = 0$

			values of d										
Nr	c_1	c_2	0.01	0.03	0.1	0.3	0.5	1	2	4	10	30	100
I	0.02	0.14	1.87	1.45	1.41	1.36	1.32	1.28	1.28	1.45	1.78	2.51	3.49
II	0.002	0.07	1.85	1.42	1.38	1.34	1.34	1.33	1.29	1.30	1.28	1.43	1.85
III	0.002	0.03	1.39	1.37	1.34	1.34	1.32	1.29	1.29	1.30	1.30	1.46	1.99
IV	0.001	0.03	1.44	1.39	1.36	1.34	1.34	1.32	1.29	1.29	1.31	1.37	1.58