Lipschitz continuity of Frechet gradients of cost functionals related to inverse problems for parabolic and hyperbolic PDEs

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Contents

1. The quasisolution approach, introduced by Ivanov, Vasin, Tanaka [32], and the weak solution theory [34, 40] for PDFs is used to obtain some useful apriori estimates *quasisolution-weak solution approach* (KWA) for inverse problems
2. The adjoint problem approach introduced by DuChateau [14-16] for ICPs related to parabolic PDEs is developed for various ICPs and ISPs related to parabolic and hyperbolic PDEs (see, Hasanov [22-29]
3. The new semigroup approach is proposed for abstract evolution equations. A general representation of ISPs with measured final data is derived. This permits one to find out non-uniqueness and uniqueness features of the solution of ISPs
4. Some publications
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6. Thanks

New Results

1. Explicit formulas for gradients of cost functionals corresponding to IPs for hyperbolic and parabolic PDEs.
2. Lipschitz continuity of the gradients: this permits one to construct a gradient type iteration process for the sequence of approximate quasisolutions of corresponding the inverse problems.
3. Characterization of the non-uniqueness/uniqueness cases using KWA and semigroup approach.
4. Monotonicity properties of gradient type iteration processes as well as input-output data for various ISPs.

1.1. Simultaneous Determination of Source Terms $\langle F(x,t), f(t)\rangle$ in a Parabolic PDE: Final Data Observation

$$
(\mathbf{ISP}) \left\{ \begin{array}{l} (\mathbf{DP}) \left\{ \begin{array}{l} u_t = (k(x)u_x)_x + F(x,t), \;\; (x,t) \in \Omega_T := \{0 < x < l, 0 < t \le T\}; \\ u(x,0) = \mu_0(x), \;\; x \in (0,l), \\ u_x(0,t) = 0, \;\; -k(l)u_x(l,t) = \nu[u(l,t) - f(t)], \;\; t \in (0,T], \end{array} \right. \\ \underline{\mu_T(x) = u(x,T)} \qquad\qquad (\textbf{Measured output data}). \end{array} \right. \tag{1}
$$

The values $\nu = 0$ and $\nu = +\infty$ of the heat transfer coefficient $\nu > 0$ correspond to the Neumann type $u_x(l,t) = 0$ and Dirichlet type $u(l,t) = 0$ boundary conditions at $x = l$, correspondingly.

1.2. The weak solution of the (DP)

$$\int_0^l u(x,T)v(x,T)dx - \int_0^l \mu_0(s)v(x,0)dx - \int\int_{\Omega_T}(uv_t - ku_x v_x)dxdt -$$
$$-\nu\int_0^T [u(l,t)-f(t)]v(l,t)dt = \int\int_{\Omega_T} F(x,t)v(x,t)dxdt, \quad \forall v \in H^{1,0}(\Omega_T),$$

Let $u = u(x,t;w)$ be the solution of the (DP), $w \in W := \mathcal{F} \times \mathbf{f}$ is the set of admissible unknown sources $F(x,t)$ and $f(t)$. If this function also satisfies the additional condition $\mu_T(x) = u(x,T;w)$, then it must satisfy the nonlinear functional equation $u(x,t;w)|_{t=T} = \mu_T(x), \quad x \in (0,l)$. Heovewer, this equation may not fulfilled due to measurement errors in $\mu_T(x)$. For this reason we define a quasi-solution of the (ISP), as a solution of the minimization problem for the cost functional:

$$J(w_*) = \inf_{w\in W} J(w), \quad J(w) = \int_0^l [u(x,T;w) - \mu_T(x)]^2 dx. \qquad (2)$$

1.3. The Relationship Between the (DP) and the Adjoint Problem

Lemma 1.1. *Let $w := \{F; f\}, w + \Delta w := \{F + \Delta F; f + \Delta f\} \in W$ be given elements. If $u = u(x, t; w) \in H^{1,0}(\Omega_T)$ is the corresponding solution of the (DP) and $\psi(x, t; w) \in H^{1,0}(\Omega_T)$ is the solution of the backward parabolic problem*

$$\underline{Adjoint\ \ Problem} \begin{cases} \psi_t = -(k(x)\psi_x)_x, & (x,t) \in \Omega_T, \\ \psi(x,T) = 2[u(x,T;w) - \mu_T(x)], & x \in (0,l), \\ \psi_x(0,t) = 0, \ -k(l)\psi_x(l,t) = \nu\psi(l,t), & t \in (0,T], \end{cases} \quad (3)$$

then for all $w \in W$ the following integral identity holds:

$$2\int_0^l [u(x,T;w) - \mu_T(x)]\Delta u(x,T;w)dx = \int\int_{\Omega_T} \psi(x,t;w)\Delta F(x,t)dxdt + \nu\int_0^T \psi(l,t;w)\Delta f(t)dt.$$

The function $\Delta u := \Delta u(x, t; w)$ is the solution of the following parabolic problem

$$\begin{cases} \Delta u_t = (k(x)\Delta u_x)_x + \Delta F(x,t), & (x,t) \in \Omega_T, \\ \Delta u(x,0) = 0, & x \in (0,l), \\ \Delta u_x(0,t) = 0, \ -k(l)\Delta u_x(l,t) = \nu[\Delta u(l,t) - \Delta f(t)], & t \in (0,T]. \end{cases} \quad (4)$$

1.4. The Fréchet Gradient of the Cost Functional

The integral identity with the first variation formula of the cost functional, i.e. the system

$$\left\{\begin{array}{l} 2\int_0^l [u(x,T;w)-\mu_T(x)]\Delta u(x,T;w)dx = \int\int_{\Omega_T}\psi(x,t;w)\Delta F dxdt \\ +\nu\int_0^T \psi(l,t;w)\Delta f(t)dt \\ \Delta J(w) := J(w+\Delta w)-J(w) = 2\int_0^l [u(x,T;w)-\mu_T(x)]\Delta u(x,T;w)dx \\ +\int_0^l [\Delta u(x,T;w)]^2 dx \end{array}\right.$$

implies:

$$\left\{\begin{array}{l} \Delta J(w) = (J'(w),\Delta w)_W + \int_0^l [\Delta u(x,T;w)]^2 dx; \\ (J'(w),\Delta w)_W := \int\int_{\Omega_T}\psi(x,t;w)\Delta F dxdt + \nu\int_0^T \psi(l,t;w)\Delta f(t)dt. \end{array}\right.$$

This, with the estimate

$$\int_0^l [\Delta u(x,T;w)]^2 dx \le \frac{c_0}{\varepsilon}\|\Delta w\|_W^2,\ c_0 = \max\{1;\nu\},\ \varepsilon = \min\left\{\frac{l^2}{k_*};\ \frac{2\nu}{\nu+2l}\right\},$$

$$k_* = \min_{[0,l]} k(x) > 0,$$

yields the **gradient formula**: $\underline{J'(w) = \{\psi(x,t;w);\nu\psi(l,t;w)\}}$.

1.5. Lipschitz Continuity of the Gradient

Lemma 1.2. *Then the cost functional $J(u)$ is of Hölder class $C^{1,1}(W)$ and*

$$\|J'(w+\Delta w) - J'(w)\|_W \leq L\|\Delta w\|_W, \quad \forall w, w+\Delta w \in W,$$

where

$$\|J'(w+\Delta w) - J'(w)\|_W^2 := \int\int_{\Omega_T} (\Delta\psi(x,t;w))^2\, dxdt + \nu^2 \int_0^T (\Delta\psi(l,t;w))^2\, dt,$$

and the Lipschitz constant $L = 2\sqrt{(c_0/\varepsilon)(l^2/k_* + l/\nu + \nu/2)} > 0$ is defined via the input parametes $c_0,\ \varepsilon > 0$.

Lemma 1.3. *Let W be a closed convex set in a Hilbert space and $J(w) \in C^{1,1}(W)$. Then*

$$|J(w_1) - J(w_2) - (J'(w_2), w_1 - w_2)| \leq \frac{1}{2} L\|w_1 - w_2\|_0^2, \quad \forall w_1, w_2 \in W, \quad L > 0.$$

1.6. Application to the Gradient Type Iteration (Minimization) Scheme. Monotonicity of the Scheme

Any gradient method for the above minimization problem (2) requires an estimation of the iteration parameter $\alpha_n > 0$ in the iteration process

$$w^{(n+1)} = w^{(n)} - \alpha_n J'(w^{(n)}), \quad n = 0, 1, 2, ..., \tag{5}$$

where $w^{(0)} \in W$ is a given initial iteration. Choice of the parameter $\alpha_n > 0$ defines various gradient methods, although in many situation an estimation of this parameter is a difficult problem. However, in the case of Lipschitz continuity of the gradient $J'(w)$ the parameter α_n can be estimated via the Lipschitz constant as follows: $0 < \delta_0 \le \alpha_n \le 2/(L + 2\delta_1)$, where $\delta_0, \delta_1 > 0$ are arbitrary parameters. Applying Lemma 1.3. we get the following

Lemma 1.4. *Let $w^{(n)} \in W$, $n = 0, 1, 2, ...$, be iterations defined by (5), with $\alpha_n = \alpha = const > 0$, $\forall n$. Then for all $n = 0, 1, 2, ...$,*

$$J(w^{(n)}) - J(w^{(n+1)}) \ge \frac{1}{2L} \|J'(w^{(n)})\|_0^2, \quad L > 0. \tag{6}$$

The optimal value of the iteration parameter $\alpha_n > 0$ is obtained here as $\underline{\alpha_* = 1/L}$.

Inequqlity means in particular the monotonicity of the scheme (5).

1.7. Convergence and the Rate of Convergence of the Iteration Scheme

Theorem 1.1 *Let W be a closed convex set and $J(w) \in C^{1,1}(W)$. If $\{w^{(n)}\} \subset W$ is the sequence of iterations defined by $w^{(n+1)} = w^{(n)} - \alpha_* J'(w^{(n)})$, $n = 0, 1, 2, ...,$, $\alpha_* = 1/L$, then $\{J(w^{(n)})\}$ is a monotone decreasing convergent sequence and*

$$\lim_{n\to\infty} \|J'(w^{(n)})\|_0 = 0.$$

Moreover, $\|w^{(n+1)} - w^{(n)}\|_0 \leq 2[J(w^{(n)}) - J(w^{(n+1)})], \quad n = 0, 1, 2, ...$.
Denote by now the limit of the sequence $\{J(w^{(n)})\}$ as $J(w_*) = \lim_{n\to\infty} J(w^{(n)})$, $w_* \in W$.

Theorem 1.2. (The rate of convergence) *For any initial data $w^{(0)} \in W$ the sequence of iterations $\{w^{(n)}\} \subset W$, given by (1.5), weakly converges in $H^0(\Omega_T) \times H^0[0,T]$ to a quasisolution $w_* \in W_*$ of the ISP. Moreover, for the rate of convergence of the sequence $\{J(w^{(n)})\}$ the following estimate holds:*

$$0 \leq J(w^{(n)}) - J(w_*) \leq 2Ld^2 n^{-1}, \;\; d > 0, \;\; n = 0, 1, 2, ...$$

$$(L > 0 \text{ is the above Lipschitz constant}).$$

2.1. Simultaneous Determination of Source Terms $\langle F(x,t), f(t)\rangle$ in a Hyperbolic PDE: Final Data Observation

$$
(\textbf{ISP}) \begin{cases} (\textbf{DP}) \begin{cases} u_{tt} = (k(x)u_x)_x + F(x,t), & (x,t) \in \Omega_T := (0,l) \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in (0,l), \\ -k(0)u_x(0,t) = f(t), \quad k(l)u_x(l,t) = 0, & t \in (0,T). \end{cases} \\ \underline{u(x,T) = \mu(x)}. \qquad\qquad (\textbf{Measured output data}). \end{cases} \tag{7}
$$

The inverse problems here consists of the determination of the pair $w := \{F(x,t); f(t)\}$ from the final state obzervation $\mu(x)$.
For a given pair $w := \{F(x,t); f(t)\}$ the hyperbolic problem (DP) will be refered as a direct (or forward) problem.

2.2. The Quasisolution Approach Based on Weak Solution Theory for PDFs

The weak (generalized) solution of the hyperbolic (DP): Find $u \in H^1(\Omega_T)$, with $u(x,0) = u_0(x)$, such that

$$\int_{\Omega_T} [-u_t v_t + k(x) u_x v_x] dxdt = \int_{\Omega_T} F(x,t) v(x,t) dxdt + \int_0^l u_1(x) v(x,0) dx + \int_0^T f(t) v(0,t) dt,$$

for all $v \in H^1(\Omega_T)$, with $v(x,T) = 0$. For the existence of a unique solution the well known conditions with respect to the functions $k(x),\ F(x,t),\ u_0(x),\ u_1(x),\ f(x)$ are required.

Introducing the cost functional $J(w) = \int_0^l [u(x,T;w) - \mu(x)]^2 dx$ and using the estimate

$$\|u\|_{H^1(\Omega_T)} + \|u_t\|_{H^0(\Omega_T)} \le C_0 \left(\|F\|_{H^1(\Omega_T)} + \|u_0\|_{H^1[0,l]} + \|u_1\|_{H^0[0,l]} + \|f\|_{H^{1/2}[0,T]} \right),$$

can be prove an existence of a quasi-solution of the considered inverse problem in the set of admissible unknown sources $W := \{w = \langle F(x,t),\ f(t) \rangle\}$, as a solution of the minimization problem:

$$J(w_*) = \inf_{w \in W} J(w). \tag{8}$$

2.3. The Integral Relationship Between Solutions of (DP) and the Corresponding Adjoint Problem

Lemma 2.1. *Let $\Delta u(x,t;w) := y(x,t;w+\Delta w) - u(x,t;w)$ be the (weak) solution of the hyperbolic problem*

$$\left\{\begin{array}{l} \Delta u_{tt} = (k(x)\Delta u_x)_x + \Delta F(x,t), \quad (x,t)\in\Omega_T, \\ \Delta u(x,0) = 0, \quad \Delta u_t(x,0) = 0, \quad x\in(0,l), \\ -k(0)\Delta u_x(0,t) = \Delta f(t), \;\; \Delta u_x(l,t) = 0, \;\; t\in(0,T). \end{array}\right. \tag{9}$$

and $\psi(x,t;p,q)\in H^1(\Omega_T)$ is the solution of the following backward hyperbolic problem:

$$\left\{\begin{array}{l} \psi_{tt} = (k(x)\psi_x)_x, \quad (x,t)\in\Omega_T, \\ \psi(x,T) = 2p(x), \quad \psi_t(x,T) = -2q(x), \quad x\in(0,l), \\ \psi_x(0,t) = 0, \;\; \psi_x(l,t) = 0, \;\; t\in(0,T), \end{array}\right. \tag{10}$$

Then for all $p\in H^1(0,l)$, $q\in H^0(0,l)$ the following integral identity holds:

$$2\int_0^l p(x)\Delta u_t(x,T;w)dx + 2\int_0^l q(x)\Delta u(x,T;w)dx$$

$$= \int\int_{\Omega_T} \psi(x,t;p,q)\Delta F(x,T)dxdt + \int_0^T \psi(0,t;p,q)\Delta f(t)dt, \quad \forall \Delta w\in W. \tag{11}$$

2.4. The First Consequence of the Integral Relationship (for Numerical Implementations)

Corollary 2.1. *Let $\psi(x,t;0,q)$, $\psi(x,t;p,0) \in H^1(\Omega_T)$ be solutions of the backward hyperbolic problem, corresponding to the given data $\langle p(x) \equiv 0,\ q(x) \neq 0 \rangle$, and $\langle p(x) \neq 0,\ q(x) \equiv 0 \rangle$, respectively. Then for all $\Delta w \in W$, $p \in H^1(0,l)$, $q \in H^0(0,l)$ the following integral identities hold:*

$$\begin{cases} 2\int_0^l q(x)\Delta u(x,T;w)dx = \int\int_{\Omega_T} \psi(x,t;0,q)\Delta F(x,t)dxdt + \int_0^T \psi(0,t;0,q)\Delta f(t)dt; \\ 2\int_0^l p(x)\Delta u_t(x,T;w)dx = \int\int_{\Omega_T} \psi(x,t;p,0)\Delta F(x,t)dxdt + \int_0^T \psi(0,t;p,0)\Delta f(t)dt. \end{cases}$$

Choosing the arbitrary functions $p(x)$, $q(x)$ by an appropriate way, one can construct an effective numerical algorithm, as it has been done for inverse coefficient problems (see, Hasanov, Duchateau).

2.5. The Second Consequence: for Monotonicity of Input-Output Mappings

Corollary 2.2. *Choose $p(x)$, $q(x)$ as $p(x) \equiv 0$, $q(x) := u(x, T; w) - \mu(x)$ (bf final state data):*

$$2\int_0^l [u(x,T;w) - \mu(x)]\Delta u(x,T;w)dx = \int\int_{\Omega_T} \psi(x,t)\Delta F(x,t)dxdt + \int_0^T \psi(0,t)\Delta f(t)dt;$$

$$\begin{cases} \psi_{tt} = (k(x)\psi_x)_x, \quad (x,t) \in \Omega_T, \\ \psi(x,T) = 0, \quad \psi_t(x,T) = -2[u(x,T;w) - \mu(x)], \quad x \in (0,l), \\ \psi_x(0,t) = 0, \;\; \psi_x(l,t) = 0, \;\; t \in (0,T). \end{cases}$$

Corollary 2.3. *Choose $p(x)$, $q(x)$ as $p(x) := u_t(x,T;w) - \nu(x)$ (bf final speed data), $q(x) \equiv 0$, then*

$$2\int_0^l [u_t(x,T;w) - \nu(x)]\Delta u_t(x,T;w)dx = \int\int_{\Omega_T} \psi(x,t)\Delta F(x,t)dxdt + \int_0^T \psi(0,t)\Delta f(t)dt;$$

$$\begin{cases} \psi_{tt} = (k(x)\psi_x)_x, \quad (x,t) \in \Omega_T, \\ \psi(x,T) = 2[u_t(x,T;w) - \nu(x)], \quad \psi_t(x,T) = 0, \quad x \in (0,l), \\ \psi_x(0,t) = 0, \;\; \psi_x(l,t) = 0, \;\; t \in (0,T). \end{cases}$$

Remark 2.1. Note that the above integral identities hold also in the case, when the source functions F, f in the direct problem depend only on u, i.e. $F = F(u)$, $f = f(u)$.

2.6. Fréchet Differentiability of the Cost Functional $J(w)$

Lemma 2.2. *For the first variation of the cost functional the following estimate holds:*

$$J(w + \Delta w) - J(w) = (J'(w), \Delta w)_W + o(\|\Delta w\|_W^2).$$

Theorem 2.1. *In view of weak solution conditions the cost functional $J(w)$ is Fréchet differentiable on the set of admissible sources W: $J(w) \in C^1(W)$, and*

$$J'(w) = \{\psi(x, t; w); \psi(0, t; w)\}.$$

Corollary 2.4. *$w_* \in W_*$ is a strict solution of (ISP) if and only if $\psi(x, t; w_*) \equiv 0$, on Ω_T.*
The main distinguished feature of this theorem is that the gradient of the cost functional $J_i(w)$ is defined via the well-posed adjoint problem. In the "final speed measured output condition" case the similar result can be derived.

Remark 2.2. The above results remain true, in the case of the nonlinear source terms $F = F(u)$, $f = f(u)$.

2.7. Lipschitz Continuity of the Gradient

Lemma 2.3. *If $\Delta\psi(x,t;w) := \psi(x,t;w+\Delta w) - \psi(x,t;w) \in H^1(\Omega_T)$ is solution of the backward hyperbolic problem*

$$\begin{cases} \Delta\psi_{tt} = (k(x)\Delta\psi_x)_x, \quad (x,t) \in \Omega_T, \\ \Delta\psi(x,T) = 0, \quad \Delta\psi_t(x,T) = -2\Delta u(x,T;w), \quad x \in (0,l), \\ \Delta\psi_x(0,t) = 0, \ \ \Delta\psi_x(l,t) = 0, \ \ t \in (0,T], \end{cases}$$

Then the cost functional is of Hölder class, $J_(w) \in C^{1,1}(W)$ and

$$\left\| J'(w+\Delta w) - J'(w) \right\|_W \le L_T \left\| \Delta w \right\|_W, \ w, w+\Delta w \in W, \quad \underline{L_T = L_0 T^{3/2}}, \ L_0 > 0, \quad (12)$$

where $\left\| J_1'(w+\Delta w) - J_1'(w) \right\|_W^2 := \int\int_{\Omega_T} (\Delta\psi(x,t;w))^2 dxdt + \int_0^T (\Delta\psi(0,t;w))^2 dt.$

The Lipschitz constant $L_T := L_0 T^{3/2} > 0$, $L_0 >$ increases by increasing the value $T > 0$ of the final time, which leads to increase of the right hand side of the estimate (12).

This lemma permits one to apply some results of convex analysis for the functionals from $C^{1,1}(W)$.

2.8. The Monotone Iteration Scheme for the Hyperbolic (ISP)

Theorem 2.2. *The sequence $\{J(w^{(n)})\}$ is a monotone decreasing (convergent) one, for any initial data $w^{(0)} \subset W$. Moreover, the sequence of iterations $\{w^{(n)}\} \subset W$, defined by $w^{(n+1)} = w^{(n)} - \alpha_n J'(w^{(n)})$, weakly converges to a quasi-solution $w_* \in W_*$ of the problem (ISP), and for the rate of convergence the following estimate holds:*

$$0 \leq J_1(w^{(n)}) - J_1(w_*) \leq 2L_T d^2 n^{-1}, \quad n = 0, 1, 2, \ldots, \quad ¿0,\ L_T := L_0 T^{3/2} > 0,\ L_0 > . \quad (13)$$

The parameter $\varepsilon_w := 2L_T d^2 n^{-1} > 0$ on the right hand side of (13) can be defined as an accuracy of the iteration scheme $w^{(n+1)} = w^{(n)} - \alpha_n J'(w^{(n)})$. The right hand side of (13) shows that in the case of inverse problems with the large final time $T > 0$ observations one requires more iterations n in the scheme $w^{(n+1)} = w^{(n)} - \alpha_n J'(w^{(n)})$, to achieve the same accuracy $\varepsilon_w > 0$.

2.9. Unicity of the Solution of (ISP): A Sufficient Condition

Lemma 2.5. *For the cost functional $J(w) \in C^{1,1}(W)$ the following formula valid:*

$$(J_1'(w+\Delta w) - J_1'(w), \Delta w)_W = \int_0^l (\Delta u(x,T;w))^2 dx, \quad \forall w, \Delta w \in W.$$

This lemma implies, that in addition to convexity of the closed set $W \subset H^0(\Omega_T) \times H^0[0,T]$, the cost functional $J(w) \in C^{1,1}(W)$ is also convex one. Further, if the condition

$$\int_0^l (\Delta u(x,T;w))^2 dx > 0, \quad \forall w \in W \tag{14}$$

holds, then the cost functional is strictly convex. Using here the uniqueness theorem for strictly convex functionals defined on convex sets we may derive the following unicity

Lemma 2.6. *If the positivity condition (14) holds, then the problem (ISP) has at most one solution.*

Thus the above condition (14) can be considered as a sufficient condition for unicity of the solution of the problem (ISP). This condition shows that distinguishability of two input data $w_1, w_2 \in W$, can be characterized via trace at $t = T$ of the solution $\Delta u(x,t;w) = u(x,t;w_1) - u(x,t;w_2)$ of the hyperbolic (ISP). Specifically, if $\Delta u(x,T;w) = 0, \forall x \in (0,l)$, for $w_1 \neq w_2$, then the output data $\Delta u(x,T;w)$ can not distinguish the a'priori given two input data $w_1, w_2 \in W$.

2.10. Unicity of the Solution of (ISP): Non-Uniqueness Cases

Assuming $k(x) = 1$ in (DP), denote by $u(x, t; w)$, $u(x, t; w + \Delta w)$ the solutions of this problem, corresponding to the pairs $w = \{F, f\}$, $w + \Delta w = \{F + \Delta F, f + \Delta f\} \in W$ of arbitrary source functions. Then the function
$\Delta u(x, t; w) = u(x, t; w + \Delta w) - u(x, t; w)$ will be the solution of the hyperbolic problem with

$$\begin{cases} \Delta F(x,t) = (x-l)^2 h(t) - 2\int_0^t (t-\tau)h(\tau)d\tau, \\ \Delta f(t) = -2l\int_0^t (t-\tau)h(\tau)d\tau, \quad t \in (0,T), \end{cases} \mapsto \Delta u(x,t;w) = (x-l)^2 \int_0^t (t-\tau)h(\tau)d\tau$$

We define the synthetic final state observations $u(x, T; w)$, $u(x, T; w + \Delta)$, and then find from

$$\Delta u(x,T;w) = (x-l)^2 \int_0^T (T-t)h(t)dt, \quad x \in (0,l). \tag{15}$$

To analyze the non-uniqueness of the solution we need to show whether or not the condition

$$\Delta u(x,T;w) = 0, \quad \forall x \in (0,l) \tag{16}$$

holds, for the above arbitrary functions $F(x, t)$ and $f(t)$. Due to (15), this condition is equivalent to

$$\int_0^T (T-t)h(t)dt = 0. \tag{17}$$

2.11. Non-Uniqueness Cases: Examples

Denote by $\Delta\mu(x) := \mu_2(x) - \mu_1(x)$, and assume the synthetic (noise free) output data $\Delta\mu(x) = \Delta u(x, T; w)$ corresponds to the input data $\Delta w = \{\Delta F, \Delta f\} \in W$ (given by the above formulas). Let $h_\varepsilon(t) = at^n + b_\varepsilon T^n$, $n \in N_+$, $a \in R$, with $b_\varepsilon = -2a/[(n+1)(n+2)] + \varepsilon$, where $\varepsilon > 0$ is a small parameter (noise factor). Substituting this function in integral (17) we get:

$$\int_0^T (T-t)h_\varepsilon(t)dt = T^{n+2}\varepsilon, \quad \varepsilon > 0.$$

Taking into account this in the above formulas we find that the synthetic output data and its H^1-norm

$$\Delta\mu_\varepsilon(x) = (x-l)^2 T^{n+2}\varepsilon, \quad \|\Delta\mu_\varepsilon\|_{H^1[0,l]} = \left[\sqrt{\frac{l^5}{5}} + 2\sqrt{\frac{l^3}{3}}\right] T^{n+2}\varepsilon, \quad \varepsilon > 0$$

which corresponds to the input data

$$\begin{cases} \Delta F_\varepsilon(x,t) = (x-l)^2 h_\varepsilon(t) - 2t^{n+2}\varepsilon, & (x,t) \in \Omega_T, \\ \Delta f_\varepsilon(t) = -2l\varepsilon t^{n+2}, & t \in (0,T). \end{cases}$$

Evidently, the norm $\|\Delta F_\varepsilon\|_{H^0(\Omega_T)}$ does not tends to zero, although $\|\Delta\mu_\varepsilon\|_{H^1[0,l]} \to 0$ (and also $\|\Delta f_\varepsilon\|_{H^0[0,T]} \to 0$), as $\varepsilon \to 0$. This shows ill-conditionedness of the problem (ISP).

3.1. Problem Formulation

Let V be a seperable Hilbert space. For each $t \in (0, T]$ we define the positive defined symmetric continuous bilinear form (functional) $a(t; u, v) := (A(t)u, v)$ on V, generated by the linear self-adjoint operator $A(t) \in \mathcal{L}(V, V^{'})$, where $V^{'}$ is the dual of V. Assume further that H is a Hilbert space which is identified with its dual $H^{'}$, and the embedding $V \hookrightarrow H$ is dense and continuous. Hence $V \hookrightarrow H \hookrightarrow V^{'}$. Consider the abstract Cauchy problem

$$(ISP) \begin{cases} (DP) \begin{cases} u^{'}(t) = Au + F(t), & t \in (0, T], \\ u(0) = u_0, \end{cases} \\ u_T := u(t)|_{t=T}, \quad T > 0. \end{cases} \tag{18}$$

where $F \in H^0(0, T; V^{'})$ and $u_0 \in H$. The *abstract inverse source problem (AISP)* consists of determining the unknown source term $F \in \mathcal{F}$ in the Cauchy problem (18) from the final state observation (the measured output data) $u_T := u(t)|_{t=T}, \quad T > 0$.

3.2. The Input-Output Map in the Semigroup Approach

Denote by $u(t;F)$ the unique solution of (DP), for the given source term $F \in \mathcal{F}$. Introducing the input-output mapping $\Phi : \mathcal{F} \mapsto H$, $\Phi F := u(t;F)|_{t=T}$, we can reformulate the (AISP) as the operator equation $\Phi F = u_T, \quad F \in \mathcal{F}, \quad u_T \in H$. Thus the considered inverse problem can be reduced to the problem of invertibility of the input-output map.
To apply the semigroup method for the problem (AISP), we assume that $\{\mathcal{S}(t) : \ t \geq 0\}$ is a linear contruction semigroup consisting of the family of operators $\{\mathcal{S}(t)\}$, $t \geq 0$, on H, whose infinitesimal generator is an extension of the operator A. Then for each given $F \in \mathcal{F}$ the unique solution of (DP) can be represented as follows:

$$u(t) = \mathcal{S}(t)u_0 + \int_0^t \mathcal{S}(t-\tau)F(\tau)d\tau, \quad t \in (0,T].$$

Substituting $t = T$ in this representation and using the final data condition we have:

$$\int_0^T \mathcal{S}(T-\tau)F(\tau)d\tau = u_T - \mathcal{S}(T)u_0, \quad T > 0. \tag{19}$$

A solution of the integral equation (19) will be defined as a *mild solution* (or simply, solution) of the problem (AISP). If $\mathcal{F} \subset W^{1,1}([0,T];V')$ then this solution will be also a solution of the abstract Cauchy problem (36) (see, [16-17]).

3.3. The Main Representation for the Source Function

The principal tool in the analysis of solution of the problem (AISP) is the following representation.

Theorem 3.1. *Let conditions $u_0, u_T \in \mathcal{D}(A)$ hold, and $g \in H^0(0,T;\mathcal{D}(A))$ be an arbitrary function. Then the function $\tilde{F}(t)$, $t \in (0,T]$ given by*

$$\tilde{F}(t) = A(I-\mathcal{S}(T))^{-1}(u_T - \mathcal{S}(T)u_0) - A(I-\mathcal{S}(T))^{-1}\int_0^T \mathcal{S}(T-\tau)g(\tau)d\tau + g(t) \quad (20)$$

is a solution of the problem (AISP).
Conversely, if $\tilde{F}(t) \in H^0(0,T;V')$ is any solution of problem (AISP), then there exists such a function $g \in H^0(0,T;\mathcal{D}(A))$ that this solution can be represented by formula (20).

3.4. Time Independent Source Term Case: Unique Representation

$$\begin{cases} u'(t) = Au + F_0, \quad t \in (0,T), \\ u(0) = u_0, \ \ u(T) = u_T. \end{cases} \tag{21}$$

$F_0 \in \mathcal{F}_0 \subset \mathcal{D}(A)$ time independent source term. Taking $F(t) = F_0$ in (19) we get:

$$\int_0^T \mathcal{S}(T-\tau)F_0 d\tau = u_T - \mathcal{S}(T)u_0, \quad T > 0.$$

With the identity $\int_0^t \mathcal{S}(\tau)v d\tau = A^{-1}(\mathcal{S}(t) - I)v, \ \ \forall v \in \mathcal{D}(A), \ \ t \in (0,T]$ this implies:

$$A^{-1}(\mathcal{S}(T) - I)F_0 = u_T - \mathcal{S}(T)u_0, \quad T > 0.$$

Hence in this case we have a unique solution.

Lemma 3.1. *Let $u_0, u_T \in \mathcal{D}(A)$ hold. Then the unique solution of the inverse problem (21) with the time independent source term $F_0 \in \mathcal{D}(A)$ can be representated by the formula*

$$F_0 = A(\mathcal{S}(T) - I)^{-1}(u_T - \mathcal{S}(T)u_0). \tag{22}$$

3.5. The Structure of Final Data (ISP)

Comparing this result with formula (41) we obtain the following

Corollary 3.1. *Let conditions of Theorem 3.1 and Lemma 3.1 hold. Then any solution of the problem (AISP) with time dependent source term $\tilde{F}$ is the sum of two elements:*

$$\tilde{F} = F_0 + F(t), \quad F_0 \in \mathcal{F}_0, \ \ F \in \mathcal{F}, \quad F_0 = A(\mathcal{S}(T) - I)^{-1}(u_T - \mathcal{S}(T)u_0) \qquad (23)$$

where F_0 is the unique solution of (ISP) with the time independent source term $F_0 \in \mathcal{D}(A)$, and $F \in \mathcal{F}$, given by

$$F(t) = g(t) - A(\mathcal{S}(T) - I)^{-1} \int_0^T \mathcal{S}(T-\tau)g(\tau)d\tau, \qquad (24)$$

with the arbitrary function $g \in H^0(0, T; \mathcal{D}(A))$, is a solution of the problem (AISP)

$$\begin{cases} u'(t) = Au + F(t), & t \in (0, T], \\ u(0) = 0, \ \ u(T) = 0, \end{cases}$$

with the homogeneous initial and final data.

Formulas (23)-(24) show structure of the final data inverse source problem solution.

Some publications

- ⋆ A. Hasanov, P. DuChateau, B. Pektas, *Journal of Inverse and Ill-Posed Problems*, **14(5)**(2006), pp.435-463.
- ⋆ A. Hasanov, *J. Math. Anal. Appl.*, **330**(2007), pp. 766-779.
- ⋆ A. Hasanov, A. Demir, A. Erdem, *J. Math. Anal. Appl.*, **335**(2007), pp. 1434-1451.
- ⋆ A. Hasanov, Zhenhai Liu, *Applied Mathematics Letters*, **21(6)** (2008), pp. 563-570.
- ⋆ A. Hasanov, D. Lesnic, *Inverse Problems in Science and Engineering*, **16** (2008), 413 - 424.
- ⋆ A. Hasanov, Yun Hua Ou, Zhenhai Liu, *Acta Mathematica Sinica*, **24(10)** (2008), pp. 1617-1624.
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MANY THANKS !

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