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Numerical methods of solving inverse problems

Sergey Kabanikhin

Sobolev Institute of Mathematics

Institute of Computational Mathematics and Mathematical Geophysics

Novosibirsk State University

Novosibirsk, Russia

e-mail: kabanikh@math.nsc.ru

The speed of the bullet (the convergence rate of the regularization algorithm) strongly depends on the properties of the desired subject (smoothness, source-wise representation etc).
Jozef Brodsky, Nobel Prize Winner

- Iterative and direct methods
- Example of acoustic inverse problem
- Usage of *a priori* information – V.V. Vasin , A.G. Yagola
- Linear algebraic systems – S.K. Godunov scheme
- Mathematics
- Applications
- Remark: well-known approaches are described in the well-known books by A.N.Tikhonov *et al*, V.K. Ivanov *et al*, M.M. Lavrentiev *et al*, H. Engl *et al*
- *We use some new iapproaches from the books and papers by:*
- *Vasin, Ageev-Vasin, Eremin-Vasin*
- *Yagola , Leonov*
- *Kabanikhin-Scherzer-Shishlenin-Vasin*
- *Kabanikhin-Schieck*

Ill-Posed and Inverse Problems

Differential, Integral, Operator Equations

$$Aq = f$$

$$A(q + \delta q) - Aq = A'(q)\,\delta q + o(\|\delta q\|)$$

Variational Form

$$J(q) = \langle Aq - f, Aq - f \rangle \to \min$$

$$J'(q) = 2\,[A'(q)]^*\,(Aq - f)$$

Direct methods:

- Linearization: $A'(q_0)\,q_1 = f_1$
- Finite-Difference Scheme Inversion
- Gelfand-Levitan method
- Boundary Control method
- Singular Value Decomposition

Iterative methods:

Landweber iteration

$$q_{n+1} = q_n - \alpha\,[A'(q_n)]^*(Aq_n - f)$$

- Gradient methods

$$q_{n+1} = q_n - \alpha_n J'(q_n)$$

Newton-Kantorovich

$$q_{n+1} = q_n - [A'(q_n)]^{-1}(Aq_n - f)$$

Levenberg-Marquardt

Inverse problems in mathematics

- **It can be said that specialists in inverse and ill-posed problems study the properties of and regularization methods for unstable problems.**
- **In other words, they develop and study stable methods for approximating unstable mappings.**
- **In terms of linear algebra, this means developing approximate methods of finding normal pseudo-solutions to systems of linear algebraic equations with rectangular, degenerate, or ill-conditioned matrices.**
- **In functional analysis, the main example of ill-posed problems is represented by an operator equation $Aq = f$, where A is a compact (completely continuous) operator.**
- **In some recent publications, certain problems of mathematical statistics are viewed as inverse problems of probability theory.**
- **From the point of view of information theory, the theory of inverse and ill-posed problems deals with the properties of maps from compact sets with high epsilon-entropy to tables with low epsilon-entropy.**

Historical Perspective

- It is well known that many mathematical concepts and problem formulations are products of studying **physical phenomena**.
- This is certainly true for the theory of inverse and ill-posed problems.
- **Plato's philosophical allegory about echo and shadows on the cave walls** (i.e., the data of an inverse problem) being the only reality available to human cognition was a precursor to **Aristotle's solution to the problem of reconstructing the shape of the Earth from its shadow on the moon** (projective geometry).
- The introduction of the **physical concept of instantaneous speed** led **Isaac Newton to the discovery of the derivative**, and the instability (ill-posedness) of the problem of numerical differentiation of an approximate function is still a subject of present-day research.
- **Lord Rayleigh's research in acoustics** led him to the question of whether it is possible **to determine the density of a non-uniform string from its sound** (the inverse problem of acoustics), which brought about the development of seismic prospecting on one hand, and the theory of spectral inverse problems on the other hand.
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- The study of the motions of celestial objects and the problem of estimating unknown parameters based on measurement results that **contain random errors** led **Legendre and Gauss to overdetermined systems of algebraic equations and to the method of least squares**.
- **Cauchy** proposed the steepest descent method for finding the minimum of a multivariate function.
- In 1948, **L.V. Kantorovich generalized**, developed, and **applied** these ideas to operator equations in Hilbert spaces. At present, the steepest descent method together with the conjugate gradient method are among the most popular methods for solving ill-posed problems. It should be noted that **Kantorovich was the first to point out** that if the **problem is ill-posed**, then **the method he proposed converges only with respect to the objective functional.**

	Well – Posed Problems	Ill – Posed and Inverse Problems
Arithmetic	$q = a + b \qquad q = a - b$	$q = a \cdot b, \quad a \ge b \qquad q = \frac{a}{b}, \quad b << a$
Algebra		$Aq = f \qquad A$ is ill-conditioned $\det A$ is very small
Analysis	$f(x) = f(0) + \int_0^x q(\xi)\, d\xi$	$q(x) = f'(x)$
Differential Equations	$u''(x) - q(x)u(x) = \lambda u(x),$ $u(0) + hu'(0) = 0,$ $u(1) + Hu'(1) = 0$	Inverse Sturm-Liouville problem $\{\lambda_n, \|u_n\|^2\} \to q(x)$
Integral Geometry		$\int_{\Gamma(x,y)} q(\xi,\eta)ds = f(x,y)$
Integral Equations	$q(x) = f(x) + \int_0^x K(x,\xi)q(\xi)d\xi$	$0 = f(x) + \int_0^x K(x,\xi)q(\xi)d\xi$
Elliptic Equations	$\Delta u = 0$ Dirichlet, Newman, Robin (mixed)	$\Delta u = 0$ Cauchy, Initial-boundary. Part of the boundary
Parabolic Equations	$\Delta u = u_t$ Cauchy, Initial-boundary at $t = 0$	$\Delta u = -u_t$ $u\|_{t=0} = f(x,y)$ $u_t = \Delta u$ $u\|_{x=0} = f_1(y,t), \quad u_x\|_{x=0} = f_2(y,t)$
Hyperbolic Equations	Cauchy Initial-boundary	Dirichlet, Newman, Cauchy. Time-like part
Coefficient Inverse Problems		$L_q u = 0 \qquad L_q$ – elliptic operator of the second order $L_q u = u_t$ $L_q u = u_{tt}$

Least Squares Method

1806 – A.M. Legendre — Nouvelles methodes pour la determination des orbites des cometes. Paris, Courcier

1809 – C.F. Gauss — Theoria motus corporum coelestium in sectionibus conicis. Solem ambientium, Hamburgi

1948 – L.V. Kantorovich — Functional analysis and applied mathematics. Uspehi Mat. Nauk, V.3, №6, pp. 89–187

1963 – A.N. Tikhonov — Regularization of incorrectly posed problems. Russian Math. Doklady, V. 153, №1, pp. 49–52

$$Aq = f$$

$$J(q) = \langle Aq - f, Aq - f \rangle \rightarrow \min$$

$$J'(q) = 2[A']^{*}(Aq - f)$$

$$A(q + \delta q) - Aq = A'(q)\delta q + o(\|\delta q\|)$$

Gradient Methods

$$q_{n+1} = q_n - \alpha_n [A']^{*}(Aq_n - f)$$

Newton-Kantorovich Method

$$q_{n+1} = q_n - [A']^{-1}(Aq_n - f)$$

Theorem (on the singular value decomposition of a compact operator.) If Q and F are separable Hilbert spaces and $A : Q \to F$ is a compact linear operator, then there exist orthonormal sequences of functions $\{v_n\} \subset Q$ (right singular vectors), $\{u_n\} \subset F$ (left singular vectors), and a nonincreasing sequence of nonnegative numbers $\{\sigma_n\}$ (singular values) such that

$$\begin{aligned} Av_n &= \sigma_n u_n, \\ A^* u_n &= \sigma_n v_n, \\ \overline{\mathrm{span}\,\{v_n\}} &= \overline{R(A^*)} = N(A)^\perp, \\ \overline{\mathrm{span}\,\{u_n\}} &= \overline{R(A)} = N(A^*)^\perp, \end{aligned}$$

and the set $\{\sigma_n\}$ has no nonzero limit points.

The sequence $\{u_n\}$ is a complete orthonormal system of eigenvectors of the operator AA^* such that

$$Av_n = \sigma_n u_n, \qquad A^* u_n = \sigma_n v_n, \qquad n \in \mathbb{N}$$

and the following decompositions hold:

$$Aq = \sum_n \sigma_n \langle q, v_n \rangle u_n, \qquad A^* f = \sum_n \sigma_n \langle f, u_n \rangle v_n.$$

Let $A : Q \to F$ be a compact linear operator, where Q and F are separable Hilbert spaces (i.e., Hilbert spaces with countable bases). A system $\{\sigma_n, u_n, v_n\}$ with $n \in \mathbb{N}$, $\sigma_n \geq 0$, $u_n \in F$, and $v_n \in Q$ will be called the *singular system of the operator* A if the following conditions hold:

- the sequence $\{\sigma_n\}$ consists of nonnegative numbers such that $\{\sigma_n^2\}$ is the sequence of eigenvalues of the operator A^*A arranged in the descending order with respect to multiplicity;
- the sequence $\{v_n\}$ consists of the eigenvectors of the operator A^*A corresponding to $\{\sigma_n^2\}$ (and is orthogonal and complete, $\overline{R(A^*)} = \overline{R(A^*A)}$);
- the sequence $\{u_n\}$ is defined in terms of $\{v_n\}$ as $u_n = Av_n/\|Av_n\|$.

Let Q_f^{p} denote the set of all pseudo-solutions to the equation $Aq = f$ for a fixed $f \in F$.
Then

$$Q_f^{\text{p}} = \{q_{\text{p}} : \|Aq_{\text{p}} - f\| = \inf_{q \in Q} \|Aq - f\|\} = \{q : Aq = P_{\overline{R(A)}} f\},$$

which implies that Q_f^{p} is nonempty if and only if $f \in R(A) \oplus N(A^*)$. In this case, Q_f^{p} is a convex closed set, and it contains the element of minimal length, q_{np} — a pseudo-solution of minimal norm (Nashed and Votruba, 1976), also called the *normal pseudo-solution to the equation* $Aq = f$ (with respect to the zero element).

For example, the problem $Aq = f$ is said to be weakly ill-posed if $\sigma_n = O(n^{-\gamma})$ for some $\gamma \in \mathbb{R}_+$, and strongly ill-posed otherwise (for instance, if $\sigma_n = O(\mathrm{e}^{-n})$).
The singular value decomposition of the operator A can be used to construct a regularization method for the problem $Aq = f$ based on the projections of

$$q_{\delta n} = \sum_{j=1}^{n} \frac{\langle f_{\delta n}, u_j \rangle}{\sigma_j} \, v_j,$$

and prove that

$$\|q_{\delta n} - q_{\mathrm{np}}\| = O\Big(\sigma_{n+1} + \frac{\delta}{\sigma_n}\Big), \qquad n \to \infty.$$

Forward (Direct) Problem

$$(1) \quad c^{-2}(x,y)v_{tt} = \Delta v - \nabla \ln \rho(x,y) \cdot \nabla v,$$

$$y \in R^{n-1}, \quad x > 0, \quad t > 0;$$

$$(2) \quad v|_{t<0} \equiv 0;$$

$$(3) \quad v_x(+0,y,t) = h(y) \cdot \delta(t), \quad y \in R^n, t \in R$$

$0 < c_0 \leq c(x,y)$ (c_0 = const) - velocity;

$0 < \rho_0 \leq \rho(x,y)$ (ρ_0 = const) - density;

$v(x,y,t)$ - exceeded pressure.

Inverse Problem: find coefficients of equation (1) using additional information:

$$(4) \quad v(+0,y,t) = f(y,t), \quad y \in R^n, t \in R.$$

Projection Method

$$u_{tt} = u_{xx} + u_{yy} - \nabla \ln \rho(x,y) \nabla u, \quad x > 0, t > 0$$

$$u\,|_{t<0} \equiv 0;$$

$$u_x\big|_{x=0} = h(y)\,\delta(t)$$

$$u\,|_{x=0} = f(y,t)$$

$$u(x,y,t) = S(x,y)\,\theta(t-x) + \tilde{u}(x,y,t)$$

$$\frac{S_x(x,y)}{S(x,y)} = \frac{1}{2}\frac{\rho_x(x,y)}{\rho(x,y)}$$

$$u(x,y,t) = \sum_j u_j(x,t)\,e^{ijy}; \qquad \rho(x,y) = \sum_j \rho_j(x)\,e^{ijy}$$

$$V_{tt} = V_{xx} - A(x)V_x - B(x)V \qquad V = \left(v_{-N}, v_{-N+1}, \ldots, v_0, \ldots, v_N\right)$$

$$V\,|_{t<0} \equiv O$$

$$V_x\big|_{x=0} = H\,\delta(t)$$

$$V\big|_{x=0} = F(t)$$

A(x) and *B(x)* depend on $\rho_j(x)$, $j = -N, -N+1, \ldots, 0, \ldots, N$

Gel'fand-Levitan-Krein method – statement of the problem

The sequence of forward initial boundary value problems
$k \in Z = \{0, \pm 1, \pm 2, \ldots\}$ or $k \in Z^n$.

$$u_{tt}^k = u_{xx}^k + \Delta_y u^k - \nabla \ln \rho(x,y) \nabla u^k, \qquad x > 0, \quad t > 0;$$

$$u^k \,|_{t<0} \equiv 0;$$

$$u_x^k \Big|_{x=+0} = \delta(t) \exp\{iky\}.$$

$$\rho(x,y) = \sum_m \rho_m(x) \exp\{iky\}$$

<u>Inverse problem</u>: find $\rho(x,y)>0$ using the traces of the solutions:

$$u^k(+0, y, t) = f^k(y,t).$$

The necessary condition of existence of solution to the inverse problem:

$$f^k(y,+0) = -\exp\{iky\}, \qquad k \in Z.$$

Gel'fand-Levitan-Krein Method

We use the sequence of Green's functions, which solves

$$w_{tt}^m = w_{xx}^m + \Delta_y w^m - \nabla \ln \rho(x,y) \nabla w^m, \quad x > 0, t \in R$$

$$w^m(0,y,t) = \delta(t)\exp\{imy\}, \quad w_x^m(0,y,t) = 0$$

For every $m \in Z$

$$w^m(x,y,t) = S^m(x,y)\left[\delta(x+t) + \delta(x-t)\right] + \tilde{w}^m(x,y,t)$$

$$S^m(x,y) = \frac{1}{2}\sqrt{\frac{\rho(x,y)}{\rho(0,y)}}\exp\{imy\}$$

The sequence $\{w^m\}$ is some kind of a bridge between the original problem and $\{\Phi^m(x,t)\}$ - the solution of the system of Gel'fand-Levitan equations (Kabanikhin, 1988) :

$$2\Phi^k(x,t) - \sum_m \int_{-x}^{x} \frac{\partial}{\partial t} f_m^k(t-s)\,\Phi^m(x,s)\,ds = -\int_{-\pi}^{\pi} \frac{e^{iky}}{\rho(0,y)}dy, \quad |t| < x, \quad k \in Z$$

$$\Phi^m(x,t) = \int_{-\pi}^{\pi}\int_{o}^{x} \frac{w^m(\xi,y,t)}{\rho(\xi,y)}d\xi\,dy \qquad \Phi^m(x,x-0) = \int_{-\pi}^{\pi} \frac{e^{imy}}{2\sqrt{\rho(x,y)\rho(0,y)}}dy$$

Gel’fand-Levitan-Krein Method

The multidimensional analog of Gelfand-Levitan-Krein (GLK) equation

$$2\Phi^k(x,t) - \sum_m \int_{-x}^{x} \frac{\partial}{\partial t} f_m^k(t-s)\,\Phi^m(x,s)\,ds = -\int_{-\pi}^{\pi} \frac{e^{iky}}{\rho(0,y)}\,dy$$

$$|t| < x, \quad k \in Z$$

The solution of inverse problem can be obtained from the solution of Gelfand-Levitan-Krein equation by formula

$$\Phi^m(x, x-0) = \int_{-\pi}^{\pi} \frac{e^{imy}}{2\sqrt{\rho(x,y)\rho(0,y)}}\,dy$$

$$\frac{1}{\rho(x,y)} = \frac{1}{\pi}\sqrt{\rho(0,y)\,e^{iky}} \sum_m \Phi^m(x, x-0)\,e^{-imy}$$

Therefore in order to find solution $\rho(x,y)$ in the depth x_0 we solve GLK equation with the fixed parameter x_0 and then calculate $\rho(x_0, y)$.

Newton-Kantorovich Method 1D

$$q_{n+1} = q_n - \left[A'(q_n)\right]^{-1}\left(Aq_n - f\right)$$

Theorem. *Let $T > 0$ and for $f \in L_2(T)$ there exists solution q^* of equation $Aq = f$. Then exist $\delta^* > 0$ and $\omega \in (0,1)$ such that if $q^{(0)} \in B_{\delta^*}(q^*)$ then NK converges to the solution q^* and*

$$\| q^* - q^{(n)} \| \leq \omega^n \| q^* - q^{(0)} \|$$

1. Chose approximation $q_0(x)$.
2. Let $q_n(x)$ be known.
3. Solve the direct problem:

$$u_{n\,tt} = u_{n\,xx} - q_n(x) \cdot u_{n\,x}, \quad (x,t) \in \Delta(T)$$

$$u_n(x,x) = s_n(x); \quad u_{n\,x}(0,t) = 0$$

4. Find $A'(q_n)\,\mu_n = u_n(0,t) - f(t)$.
5. Solve linear inverse problem:

$$w_{n\,tt} = w_{n\,xx} - q_n(x) \cdot w_{n\,x} - \mu_n(x) \cdot u_{n\,x}, \quad (x,t) \in \Delta(T)$$

$$w_n(x,x) = \frac{u_n(x,x)}{2} \cdot \int_0^x \mu_n(\xi)\,d\xi; \quad w_{n\,x}(0,t) = 0$$

$$w_n(0,t) = u_n(0,t) - f(t)$$

6. Put $q_{n+1}(x) = q_n(x) - \mu_n(x)$

Landweber Iterations

$$q_{n+1} = q_n - \alpha\left[A'(q_n)\right]^{*}\left(Aq_n - f\right)$$

1. Chose the approximation $q_0(x)$
2. Let $q_n(x)$ be known
3. Solve the direct problem:

$$u_{n\,tt} = u_{n\,xx} - q_n(x)\,u_{n\,x}, \quad (x,t)\in\Delta(T)$$

$$u_n(x,x) = s_n(x); \quad u_{n\,x}(0,t) = 0$$

4. Find the discrepancy $\eta_n(t) = u_n(0,t) - f(t)$
5. Solve the adjoint problem:

$$\psi_{n\,tt} = \psi_{n\,xx} + \left(q_n(x)\cdot\psi_n\right)_x, \quad (x,t)\in\Delta(T)$$

$$\psi_n(x,2T-x) = 0$$

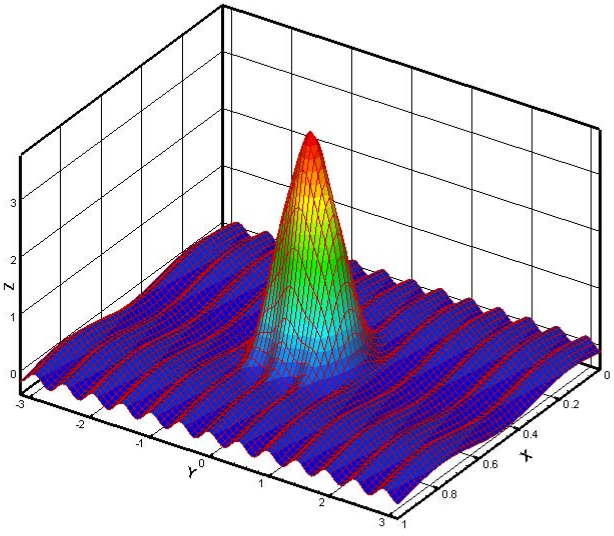
$$\psi_{n\,x}(0,t) + q(0)\cdot\psi(0,t) + 2\left[u_n(0,t) - f(t)\right] = 0$$

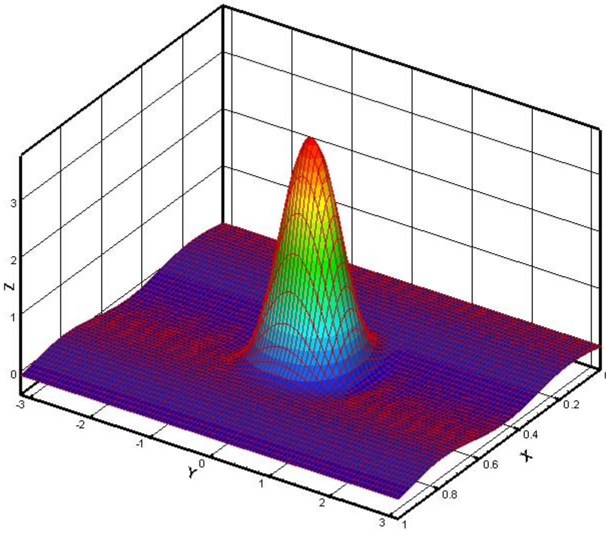
6. Find

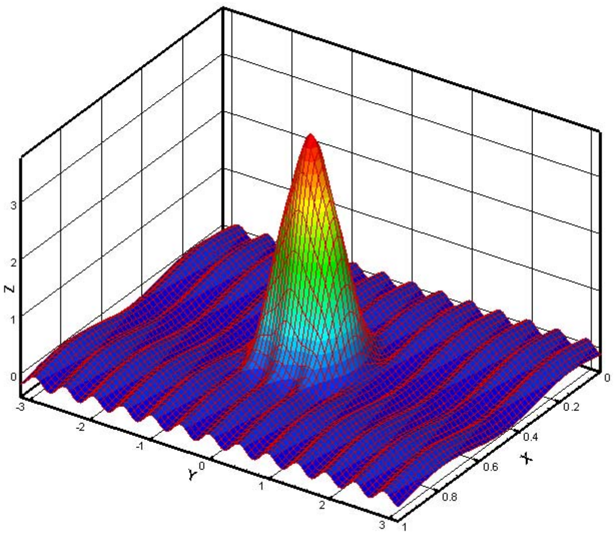
$$J'(q_n)(x) = -\int_x^{2T-x}\left[\psi_n u_{n\,x}\right](x,t)\,dt - \int_x^{T}\left[u_n\left(\psi_{n\,x} + \psi_{n\,t}\right)\right](t,t)\,dt - \frac{1}{2}\int_x^{T} q_n(t)\left[u_n\psi_n\right](t,t)\,dt$$

7. Put $q_{n+1}(x) = q_n(x) - \alpha\, J'(q_n)$

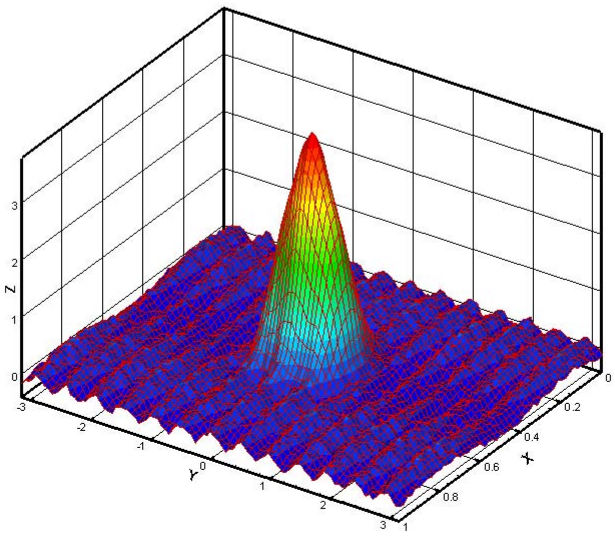
Numerical results – 2D-acoustics. MGL.
Exact, N=10, N=50

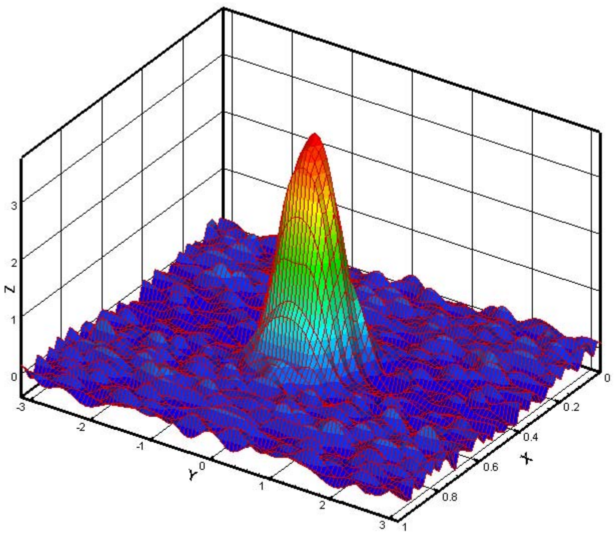


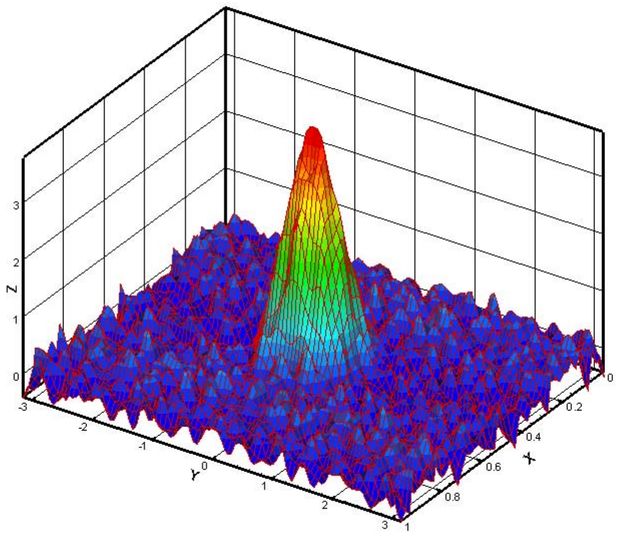




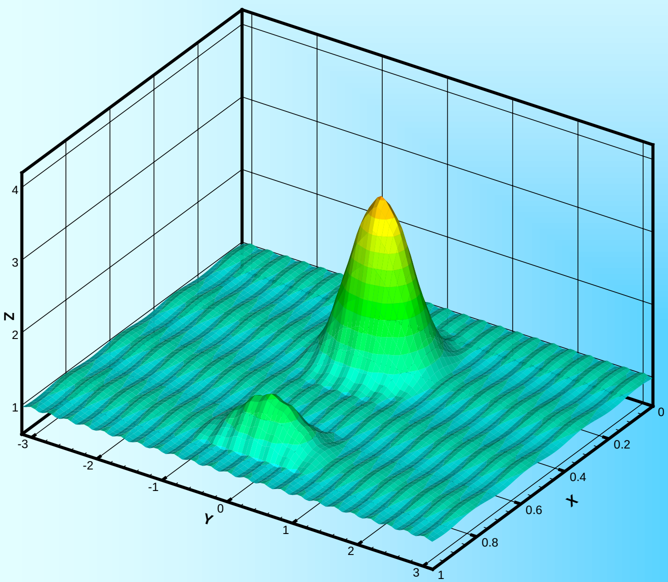
Numerical results – 2D acoustics. MGL.
Noise=0.05, N=10, N=50

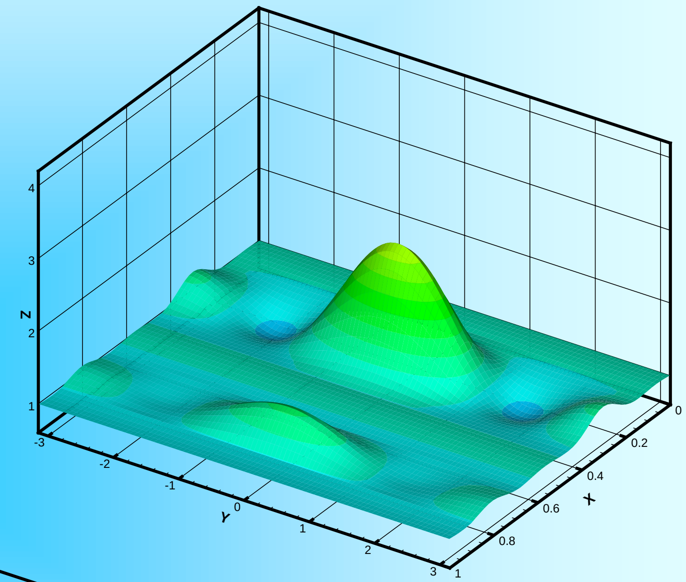


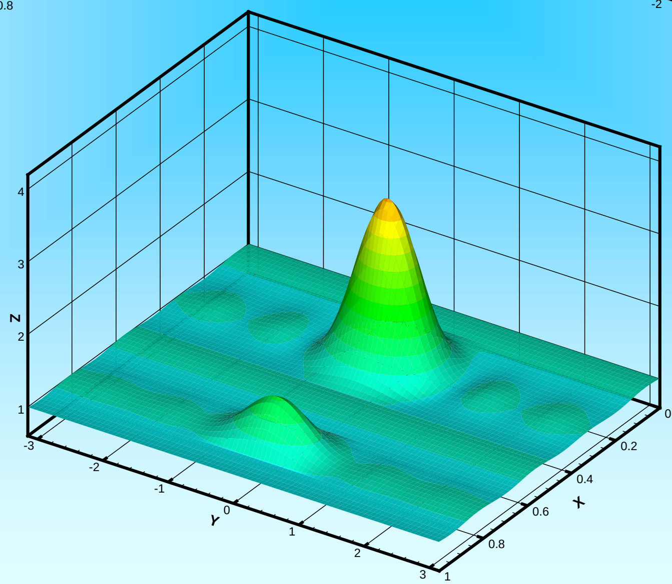




Numerical results – 2D-acoustics. MGL.
Exact, N=10, N=50







We consider as the simpliest example the following inverse problem: find function $q(x,y)$ in domain $(0,\ell)\times(-\pi,\pi)$, which satisfies the following conditions

$$u_{tt} = u_{xx} + u_{yy} - q(x,y)u, \qquad (x,y,t)\in\Omega; \tag{2}$$

$$u\big|_{t<0} \equiv 0, \quad u_x\big|_{x=0} = \gamma\delta(t); \tag{3}$$

$$u\big|_{y=\pi} = u\big|_{y=-\pi}; \tag{4}$$

$$u(0,y,t) = f(y,t), \quad t\in(0,2\ell). \tag{5}$$

Here $\Omega = \{x,y,t \;:\; (x,t)\in\Delta(\ell), \quad y\in(-\pi,\pi)\}$ and $\Delta(\ell) = \{(x,t) : 0 < x < t < 2\ell - x\}$.
Approximate solution of the inverse problem will be found in the form of finite Fourier series (P_N-approximation):

$$q(x,y) \cong \sum_{n=-N}^{N} q_n(x)e^{iny}, \quad u(x,y,t) \cong \sum_{n=-N}^{N} u_n(x,t)e^{iny}. \tag{6}$$

Let us introduce: $U(x,t) = (u_{-N}, u_{-N+1}, \dots, u_0 \dots, u_N)$, $Q(x) = (q_{-N}, \dots, q_0 \dots, q_N)$ and consider inverse problem in vector form:

$$U_{tt} = U_{xx} - B(x)U, \quad (x,t) \in \Delta(\ell); \tag{7}$$

$$U_x|_{x=0} = 0, \quad t \in (0, 2\ell); \tag{8}$$

$$U(x,x) = S(x), \quad x \in (0, \ell); \tag{9}$$

$$U(0,t) = F(t), \quad t \in (0, 2\ell). \tag{10}$$

Here $B(x)$ is defined as follows $(n = -N, \dots, N)$:

$$\big[B(x)U\big]_n = n^2 u_n(x,t) + \sum_{|k| \le N, |k-n| \le N} q_{n-k}(x) u_k(x,t).$$

$S(x)$ and $F(x)$ are vector functions consisting of the Fourier coefficients of functions $-\gamma + \beta q(x,y)$ and $f(y,t)$ correspondingly. In inverse problem (7)–(10) we need to find the vector-function $Q(x)$ by known data $F(t)$.

Let us consider inverse problem (7)–(10) as nonlinear operator equation

$$A(Q) = F. \tag{11}$$

Properties of operator A have been investigated in [6].
Due to the uniqueness of the solution to the inverse problem (7)–(10) it is enough to find the minimum of cost functional

$$J(Q) = \|A(Q) - F\|^2_{L_2(0,2\ell)}. \tag{12}$$

Here

$$\|F\|^2_{L_2(2\ell)} = \sum_{|k|\leq N} \|f_k\|^2_{L_2(0,2\ell)}.$$

For solving minimization problem $J(q) \longrightarrow \inf$ we apply Landweber iteration

$$Q^{(n+1)} = Q^{(n)} - \alpha[A'(Q^{(n)})]^*(A(Q^{(n)} - F)), \quad n = 0, 1, \dots. \quad (13)$$

Landweber iteration method can be considered as optimization method with fixed descent parameter α_*. Indeed it is easy to show that

$$J'(Q) = 2\Big[A'(Q)\Big]^*(A(Q) - F).$$

Let us consider the following adjoint problem

$$\Psi_{tt} = \Psi_{xx} - [B(x)]^{*}\Psi, \quad (x,t) \in \Delta(\ell);$$

$$\Psi_{x}|_{x=0} = 2[U(0,t) - F(t)], \quad t \in (0, 2\ell);$$

$$\Psi(x, 2\ell - x) = 0, \quad x \in (0, \ell).$$

One can easily check that the component of the gradient of (12) is defined by formula

$$[J'(Q)]_m(x) = 2\beta(\psi_{m_x} + \psi_{m_t})(x,x) + \\ + \sum_{|k|\leq N, |k-m|\leq N} \int_{x}^{2\ell - x} u_{m-k}(x,t)\psi_m(x,t)\d{t}. \quad (14)$$

The scheme of Landweber iteration method

1. Let $Q^{(0)}(x)$ is initial guess.
2. Let $Q^{(n)}(x)$ be known, then solve the forward problem:

$$U_{tt}^{(n)} = U_{xx}^{(n)} - B^{(n)}(x)U^{(n)}, \quad (x,t) \in \Delta(\ell);$$

$$U_x^{(n)}\Big|_{x=0} = 0, \quad t \in (0, 2\ell);$$

$$U^{(n)}(x,x) = S^{(n)}(x), \quad x \in (0,\ell).$$

3. Find discrepancy $\eta^{(n)}(t) = U^{(n)}(0,t) - F(t)$ and its norm.
4. Solve the adjoint problem:

$$\Psi_{tt}^{(n)} = \Psi_{xx}^{(n)} - [B^{(n)}(x)]^{*}\Psi^{(n)}, \quad (x,t) \in \Delta(\ell);$$

$$\Psi_x^{(n)}|_{x=0} = 2[U^{(n)}(0,t) - F(t)], \quad t \in (0, 2\ell);$$

$$\Psi^{(n)}(x, 2\ell - x) = 0, \quad x \in (0,\ell).$$

   Define the gradient $[J'(Q^{(n)})](x)$ by formula (14)
5. Find $Q^{(n+1)}(x) = Q^{(n)}(x) - \alpha[J'(Q^{(n)})](x)$.

The usual convergence theorem can be proved as in [5, 6].

Theorem*(convergence of Landweber iteration). Let* $\ell > 0$ *and* $F \in L_2(\ell)$*. Suppose that there exists a solution* $Q_T \in L_2(\ell)$ *of the problem* $A(Q) = F$*. Then one can find such* $\nu_* \in (0,1)$*,* $\delta_* > 0$*,* $\alpha_* > 0$*, that if* $Q^{(0)} \in B(Q_T, \delta_*)$ *and* $\alpha \in (0, \alpha_*)$ *then the approximations* $Q^{(n)}$ *of Landweber iteration converge to the solution* Q_T *as* $n \to \infty$ *at the rate*

$$\|Q_T - Q^{(n)}\|^2_{L_2(\ell)} \leq \nu_*^n \delta_*^2.$$

Using constant r ($\|Q_T\|_{L_2(\ell)} \leq r$) we modify the Landweber iteration as follows. First, given the approximation $Q^{(n)}$ we calculate

$$\tilde{Q}^{(n+1)} = Q^{(n)} - \alpha \Big[A'(Q^{(n)}) \Big]^* \Big(A(Q^{(n)}) - F \Big).$$

Then we put

$$Q^{(n+1)} = \begin{cases} \tilde{Q}^{(n+1)}, & \text{if} \quad \|\tilde{Q}^{(n+1)}\|_{L_2(\ell)} < r; \\ \tilde{Q}^{(n+1)}\, r\, \|\tilde{Q}^{(n+1)}\|^{-1}_{L_2(\ell)}, & \text{if} \quad \|\tilde{Q}^{n+1}\|_{L_2(\ell)} \geq r. \end{cases}$$

Theorem*(convergence of modified algorithm). Suppose that there exists a solution $Q_T \in B(0,r) \bigcap L_2(\ell)$ of the problem $A(Q) = F$. Then one can find such $\nu_* \in (0,1)$, $C_* > 0$, $\alpha_* > 0$, that for any initial guess $Q^{(0)}$ and any $\alpha \in (0, \alpha_*)$ the approximations $Q^{(n)}$ of Landweber iteration converge to the solution Q_T as $n \to \infty$ at the rate*

$$\|Q_T - Q^{(n)}\|^2_{L_2(\ell)} \leq \nu_*^2 C_*^2.$$

- Ivanov V. K., Vasin V. V., Tanana V. P. *Theory of Linear Ill-Posed Problems and its Applications*. VSP, The Netherlands, 2002.
- Kabanikhin, S.I. *Projection-Difference Methods for Determining the Coefficients of Hyperbolic Equations.* (1988), Izd-vo SO Akad. Nauk SSSR, Novosibirsk (in Russian).
- Kabanikhin S.I., Satybaev A.D., Shishlenin M.A. *Direct Methods of Solving Inverse Hyperbolic Problems*. VSP, The Netherlands, 2004.
- Azamatov J. S., and Kabanikhin S. I., Nonlinear Volterra operator equations. L_2 – theory. *Journal of Inverse and Ill-Posed Problems* (1999) Vol.7(6), The Netherlands, Utrecht.

- S. I. Kabanikhin, R. Kowar, and O. Scherzer, On the Landweber iteration for the solution of parameter identification problem in a hyperbolic partial differential equation of second order. *JIIPP* (1998). Vol.6, No.5.
- Kabanikhin S.I., Scherzer O. and Shishlenin M.A. Iteration method for solving a two dimensional inverse problem for a hyperbolic equation. *JIIPP*, 11 (1), 2003.
- Kabanikhin S.I., Bakanov G.B. and Shishlenin M.A. Comparative analysis of methods of finite inversion scheme, Newton-Kantorovich and Landweber in inverse problem for hyperbolic equation. Novosibirsk, Novosibirsk State University, Preprint 12, 2001 (in Russian).

- V.G. Romanov and S.I. Kabanikhin, *Inverse Problems of Geoelectrics (Numerical Methods of Solution)*. Preprint No. 32. Inst. Math., Siberian Branch of the USSR Acad. Sci., Novosibirsk (1989) (in Russian).
- S. He and S.I. Kabanikhin, *An optimization approach to a three-dimensional acoustic inverse problemin in the time domain*, J. Math. Phys. 36, 8 (1995).
- V.G. Romanov, *Local solvability of some multidimensional inverse problems for equations of hyperbolic type*. Differential Equations, 25, 2 (1989) (in Russian).