Wave polynomials in inverse problems

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Schedule

* Introduction

* Wave polynomials

* Solving problems by means of Trefftz functions

* Thermoelasticity problems

* Conclusions

History

- **1926** Erich Trefftz, *Ein Gegenstük zum Ritz'schen Verfahren*. Proceedings 2nd International Congres of Applied Mechanics, Zurich, pp.131–137.

- **1978** Herrera I., Sabina F., *Connectivity as an alternative to boundary integral equations: Construction of bases*, Appl. Math. Phys. Sc., No. 75/5, 2059–2063.

- **Other authors:** Kupradze, Jirousek, Leon, Zienkiewicz, Zieliński, Kołodziej, Qin, Li, Lu, Hu, Cheng.

Continuous time:

- **1956** P.C.Rosenbloom, D.V. Widder – 1D Heat polynomials,

- **1998–2000** M.J.Ciałkowski, K.Grysa, S.Hożejowska, L.Hożejowski – Direct and inverse problems for heat conduction (1D–3D, several coordinates systems),

- **2000** M.J.Ciałkowski, A.Frąckowiak – Trefftz funtions for Laplace'a, Poisson'a, Helmholtza and 1D wave equations, Trefftz functions as FEM base functions,

- **2002–2005** A.Maciąg, J.Wauer – 2D and 3D wave polynomials,

- **2006–** A.Maciąg – Wave polynomials in thermoelasticity problems, T–functions for beam and plate vibrations problems, several variants of FEMT.

Main idea

The key idea of the method is to determine functions satisfying a given differential equation (T–functions) and to fit the linear combination of them to the governing initial and boundary conditions (usually The Least Square Method).

Advantages of the method

- Approximate solution (a linear combination of the solving functions) satisfies the equation identically and depends continuously on all space variables and time.
- The method is flexible in terms of given boundary and initial conditions (discrete, missing).
- The usage of T–functions as base functions in FEM allows to use big time–space elements.

Wave polynomials

Let us consider a wave equation in $\tilde{\Omega} = \Omega \times (0, t)$, where $\Omega \subset R^n$ and t denotes time:

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

For two-dimensional wave equation we take $P_{000} = 1, Q_{000} = 0$. Then the recurrent formulas for wave polynomials are:

$$P_{(n-k)k0} = \frac{1}{n}(-xQ_{(n-k-1)k0} - yQ_{(n-k)(k-1)0} - vtQ_{(n-k-2)k1} - vtQ_{(n-k)(k-2)1}),$$

$$P_{(n-k-1)k1} = \frac{1}{n}(-xQ_{(n-k-2)k1} - yQ_{(n-k-1)(k-1)1} - vtQ_{(n-k-1)k0}),$$

$$Q_{(n-k)k0} = \frac{1}{n}(xP_{(n-k-1)k0} + yP_{(n-k)(k-1)0} + vtP_{(n-k-2)k1} + vtP_{(n-k)(k-2)1}),$$

$$Q_{(n-k-1)k1} = \frac{1}{n}(xP_{(n-k-2)k1} + yP_{(n-k-1)(k-1)1} + vtP_{(n-k-1)k0}).$$

e.g.

$$\begin{aligned} P_{000} &= 1, \quad Q_{100} = x, \quad Q_{010} = y, \quad Q_{001} = t, P_{200} = -\frac{x^2}{2} - \frac{t^2}{2}, \\ P_{110} &= -xy, \quad P_{101} = -xt, \quad P_{011} = -yt, P_{020} = -\frac{y^2}{2} - \frac{t^2}{2}, \quad \ldots, \end{aligned}$$

For three-dimensional wave equation:

$$\begin{aligned} P_{(n-k-l)kl0} = & -\tfrac{1}{n}(xQ_{(n-k-l-1)kl0} + yQ_{(n-k-l)(k-1)l0} + zQ_{(n-k-l)k(l-1)0} + \\ & + vtQ_{(n-k-l-2)kl1} + vtQ_{(n-k-l)(k-2)l1} + vtQ_{(n-k-l)k(l-2)1}), \end{aligned}$$

$$\begin{aligned} P_{(n-k-l-1)kl1} = & -\tfrac{1}{n}(xQ_{(n-k-l-2)kl1} + yQ_{(n-k-l-1)(k-1)l1} + \\ & + zQ_{(n-k-l-1)k(l-1)1} + vtQ_{(n-k-l-1)kl0}), \end{aligned}$$

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$$\begin{aligned} Q_{(n-k-l-1)kl1} = & \tfrac{1}{n}(xP_{(n-k-l-2)kl1} + yP_{(n-k-l-1)(k-1)l1} + \\ & + zP_{(n-k-l-1)k(l-1)1} + vtP_{(n-k-l-1)kl0}). \end{aligned}$$

T–functions method

As an approximation of the wave equation we take:

$$u \approx w = \sum_{n=1}^{N} c_n V_n.$$

Because all polynomials V_n satisfy the wave equation, the linear combination w satisfies this equation as well. The coefficients c_n are chosen so that the error of fulfilling given boundary and initial conditions corresponding to the wave equation is minimized.

The method is convergent – more polynomials leads to better approximation.

Membrane's vibrations

Consider the free vibrations of a square membrane described by equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (x, y) \in (0,1) \times (0,1), \quad t > 0,$$

and

initial conditions: $u(x, y, 0) = u_0(x, y) = \frac{1}{20}\sin(x)\sin(\pi y), \quad \frac{\partial u(x,y,0)}{\partial t} = 0,$

boundary conditions: $u(0, y, t) = u(x, 0, t) = u(x, 1, t) = 0.$

For simulation we take a condition on the boundary $x = 1$:

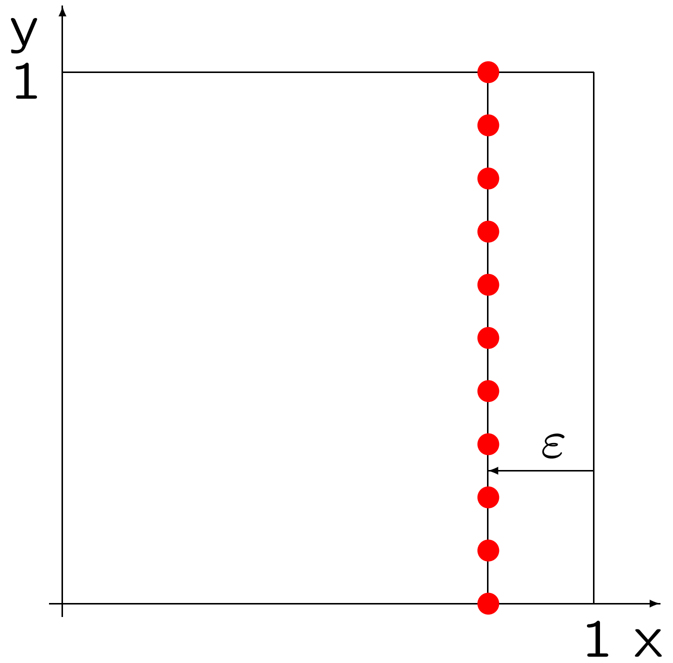
$$u(1, y, t) = \frac{1}{20}\sin(1)\sin(\pi y)\cos(t\sqrt{\pi^2 + 1}).$$

We assume that this condition is not known, but we know the internal responses of a membrane's deflection in distance ε from border $x = 1$:

$$u(1 - \varepsilon, 0.1 \cdot (k - 1), t) = u_k(t), \ k = 1, \ldots, 11.$$

Values of u_k for $k = 1 \ldots 11$ were simulated from exact solution:

$$u(x, y, t) = \frac{1}{20} \sin(x) \sin(\pi y) \cos(t\sqrt{\pi^2 + 1}).$$



If $\varepsilon > 0$ we have an inverse problem and we search for a solution in the whole domain but especially $u(1, y, t)$.

We check the quality of the identification of the boundary condition by calculating the average relative error in the whole time interval $(0, \Delta t)$, which is defined as:

$$E = \sqrt{\frac{\int_0^{\Delta t}[u(1,0.5,t) - w(1,0.5,t)]^2 \mathrm{d}t}{\int_0^{\Delta t}[u(1,0.5,t)]^2 \mathrm{d}t}} \cdot 100\%.$$

The values of error E in dependence on the distance ε and order N are shown in the table:

$N \backslash \varepsilon$	0	0.05	0.1	0.2	0.3	0.4	0.6
3	29.25	28.65	29.00	32.29	39.76	55.18	100.96
5	4.28	4.42	5.00	7.38	12.75	19.79	25.12
7	**0.33**	**0.36**	**0.38**	**0.97**	**2.27**	**2.50**	**6.00**

Disturbance and smoothing of data

Now we assume that the internal responses are given in discrete time points: $u_{kl} = u(1-\varepsilon, (k-1)\cdot 0.1, (l-1)\cdot 0.1)$ where $k, l = 1, \ldots, 11$. This data are disturbed according to the formula:

$$u^d_{kl} = u_{kl}(1 + \xi_{kl}),$$

where ξ_{kl} are random numbers of normal distribution ($N(0, 0.01)$).
In this case the relative error E has a value from circa 2000% for $\varepsilon = 0$ (direct problem) to circa 180000% for $\varepsilon = 0.8$.

As an approximation of disturbed internal responses we take:

$$u_\varepsilon(1-\varepsilon, y, t) = \sum_{n=1}^{N} c_n V_n(1-\varepsilon, y, t),$$

V_n – wave polynomials. To chose coefficients c_n we minimize:

$$\sum_{k,l=1}^{11} [u_\varepsilon(1-\varepsilon, (k-1)\cdot 0.1, (l-1)\cdot 0.1) - u^d_{kl}]^2.$$

Then the values of the relative error E for the smoothed data are shown in the table:

ε	0	0.05	0.1	0.2	0.3	0.4	0.6
E	**1.0**	**0.54**	**1.26**	**1.04**	**1.26**	**3.72**	**23.36**

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If we take into account additional information concerning the velocity of the membrane in the same points (disturbed and smoothed by wave polynomials) we get:

ε	0	0.05	0.1	0.2	0.3	0.4	0.6
E	**0.32**	**0.31**	**0.32**	**0.42**	**0.5**	**0.6**	**3.48**

Thermoelasticity problem

Let us consider thermoelasticity equations:

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\text{grad div } \mathbf{u} = \rho\ddot{\mathbf{u}} + \gamma\text{grad T}$$

where $\mathbf{u}$ – displacement vector, ∇ – nabla operator, μ, λ – Lame constants, ρ – mass density, $\gamma = \frac{E}{1-2\nu}\alpha$, E – Young's modulus, ν – Poison's ratio, α – coefficient of thermal expansion.

The temperature field is described by the equation:

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \nabla^2 T,$$

where κ – coefficient of thermal diffusivity. Equations are completed by initial and boundary conditions or "internal responses". The relation between displacements and stresses is described by:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} - \gamma T\delta_{ij},$$

where $\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ – strain tensor, δ_{ij} – Kronecker delta.

The system of thermoelasticity equations can be simplified by substitution: $\mathbf{u} = \text{grad } \phi + \text{rot } \mathbf{\Psi}$ then we obtain:

$$(\nabla^2 - \frac{1}{v_1^2}\frac{\partial^2}{\partial t^2})\phi = mT, \quad (\nabla^2 - \frac{1}{v_2^2}\frac{\partial^2}{\partial t^2})\psi_i = 0, \qquad i = 1, 2, 3$$

where $v_1^2 = \frac{\lambda+2\mu}{\rho}$, $v_2^2 = \frac{\mu}{\rho}$, $m = \frac{\gamma}{c_1^2\rho}$.

We approximate the solution of the first equations by:

$$\phi \approx \widehat{\phi} = \sum_{n=1}^{N} c_n^0 V_n^0 + \phi_p.$$

For the second equation we take:

$$\psi_i \approx \widehat{\psi_i} = \sum_{n=1}^{N} c_n^i V_n^i, \quad i = 1, 2, 3$$

Here $V_n^i, i = 0, \ldots, 3$ are wave polynomials satisfying corresponding wave equation and ϕ_p is the particular solution.
Coefficients c_n^i are determined by given conditions.

Particular solution

The particular solution ϕ_p for $L\phi = mT$ is calculated as $L^{-1}(mT)$, where $L = \nabla^2 - \frac{1}{v_1^2}\frac{\partial^2}{\partial t^2}$.

The temperature distribution can be approximated by linear combination of heat polynomials (or Taylor series). Therefore, we have to know how to calculate the inverse operator L^{-1} for monomials. It is easy to prove that for two space variables we have three forms of L^{-1}:

$$L_1^{-1}(x^k y^l t^m) = \frac{1}{(k+2)(k+1)}(x^{k+2} y^l t^m - l(l-1)Z_{(k+2)(l-2)m} + \frac{m(m-1)}{v_1^2} Z_{(k+2)l(m-2)}),$$

$$L_2^{-1}(x^k y^l t^m) = \frac{1}{(l+2)(l+1)}(x^k y^{l+2} t^m - k(k-1)Z_{(k-2)(l+2)m} + \frac{m(m-1)}{v_1^2} Z_{k(l+2)(m-2)}),$$

$$L_3^{-1}(x^k y^l t^m) = \frac{v_1^2}{(m+2)(m+1)}(-x^k y^l t^{m+2} + k(k-1)Z_{(k-2)l(m+2)} + l(l-1)Z_{k(l-2)(m+2)}).$$

For three space variables we have four forms of L^{-1}:

$$Z_1(x^n y^k z^l t^m) = \frac{1}{(n+2)(n+1)}(x^{n+2} y^k z^l t^m + \frac{m(m-1)}{v_1^2} Z_{(n+2)kl(m-2)} - k(k-1) Z_{(n+2)(k-2)lm} - l(l-1) Z_{(n+2)k(l-2)m}),$$

$$Z_2(x^n y^k z^l t^m) = \frac{1}{(k+2)(k+1)}(x^n y^{k+2} z^l t^m + \frac{m(m-1)}{v_1^2} Z_{n(k+2)l(m-2)} - n(n-1) Z_{(n-2)(k+2)lm} - l(l-1) Z_{n(k+2)(l-2)m}),$$

$$Z_3(x^n y^k z^l t^m) = \frac{1}{(l+2)(l+1)}(x^n y^k z^{l+2} t^m + \frac{m(m-1)}{v_1^2} Z_{nk(l+2)(m-2)} - n(n-1) Z_{(n-2)k(l+2)m} - k(k-1) Z_{n(k-2)(l+2)m}),$$

$$Z_4(x^n y^k z^l t^m) = \frac{v_1^2}{(m+2)(m+1)}(-x^n y^k z^l t^{m+2} + n(n-1) Z_{(n-2)kl(m+2)} + k(k-1) Z_{n(k-2)l(m+2)} + l(l-1) Z_{nk(l-2)(m+2)}].$$

Test example

Consider the plane state of strain for $(x, y) \in (-1, 1) \times (-1, 1)$ when the strain tensor $\varepsilon_{ij} = \varepsilon_{ij}(x, y, t), (i, j = 1, 2)$ and $\varepsilon_{i3} = 0, (i = 1, 2, 3)$. Then the system of wave equations has a form:

$$(\nabla^2 - \frac{1}{v_1^2}\frac{\partial^2}{\partial t^2})\phi(x, y, t) = mT(x, y, t), \quad (\nabla^2 - \frac{1}{v_2^2}\frac{\partial^2}{\partial t^2})\psi(x, y, t) = 0.$$

Displacements and stresses are given as:

$$\begin{aligned}
\mathbf{u} &= [u_x(x, y, t), u_y(x, y, t)] = \\
&= \left[\frac{\partial \phi(x, y, t)}{\partial x} + \frac{\partial \psi(x, y, t)}{\partial y}, \frac{\partial \phi(x, y, t)}{\partial y} - \frac{\partial \psi(x, y, t)}{\partial x}\right], \\
\sigma_{xx} &= (2\mu + \lambda)\frac{\partial u_x}{\partial x} + \lambda\frac{\partial u_y}{\partial y} - \gamma T, \sigma_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \\
\sigma_{yy} &= \lambda\frac{\partial u_x}{\partial x} + (2\mu + \lambda)\frac{\partial u_y}{\partial y} - \gamma T, \sigma_{zz} = \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) - \gamma T.
\end{aligned}$$

Temperature and conditions: $T(x,y,t) = x^2/2 + y^2/2 + 2t$,

$$u_x(x,y,0) = \frac{mxy^2}{3}, \quad u_y(x,y,0) = \frac{mx^2y}{3},$$

$$\dot{u}_x(x,y,0) = \frac{2mx}{3} + \frac{\sqrt{2}c_2}{50000}\sin(x)\cos(y), \ \dot{u}_y(x,y,0) = \frac{2my}{3} - \frac{\sqrt{2}c_2}{50000}\cos(x)\sin(y),$$

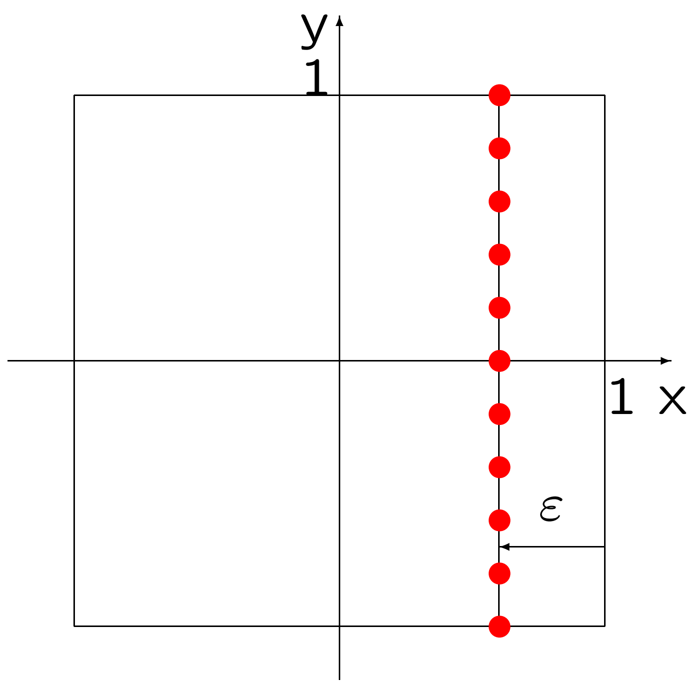
$$u_x(-1,y,t) = -\frac{m}{3}(2t + y^2 - \frac{c_1^2t^2}{2}) \pm \frac{\sin(1)}{50000}\cos(y)sin(\sqrt{2}c_2t),$$

$$u_x(x,-1,t) = u_x(x,1,t) = \frac{m}{3}(2xt + x - \frac{c_1^2xt^2}{2}) + \frac{\cos(1)}{50000}\sin(x)sin(\sqrt{2}c_2t),$$

$$u_y(-1,y,t) = \frac{m}{3}(2yt + y - \frac{c_1^2yt^2}{2}) - \frac{\cos(1)}{50000}\sin(y)sin(\sqrt{2}c_2t),$$

$$u_y(x,\pm 1,t) = \pm\frac{m}{3}(2t + x^2 - \frac{c_1^2t^2}{2}) \mp \frac{\sin(1)}{50000}\cos(x)sin(\sqrt{2}c_2t).$$

The continuous conditions $u_x(1,y,t)$ and $u_y(1,y,t)$ are not known. Instead of that we know the values of them in discrete points $(x-\varepsilon, -1+\frac{k}{5}, \frac{l\Delta t}{10})$, $k,l = 0,\ldots,10$ (internal responses).



Internal responses are simulated from the exact solution:

$$u_x = \frac{m}{3}(2xt + xy^2 - \frac{c_1^2 x t^2}{2}) + \frac{1}{50000}\sin(x)\cos(y) sin(\sqrt{2}c_2 t),$$

$$u_y = \frac{m}{3}(x^2 y + 2yt - \frac{c_1^2 y t^2}{2}) - \frac{1}{50000}\cos(x)\sin(y) sin(\sqrt{2}c_2 t).$$

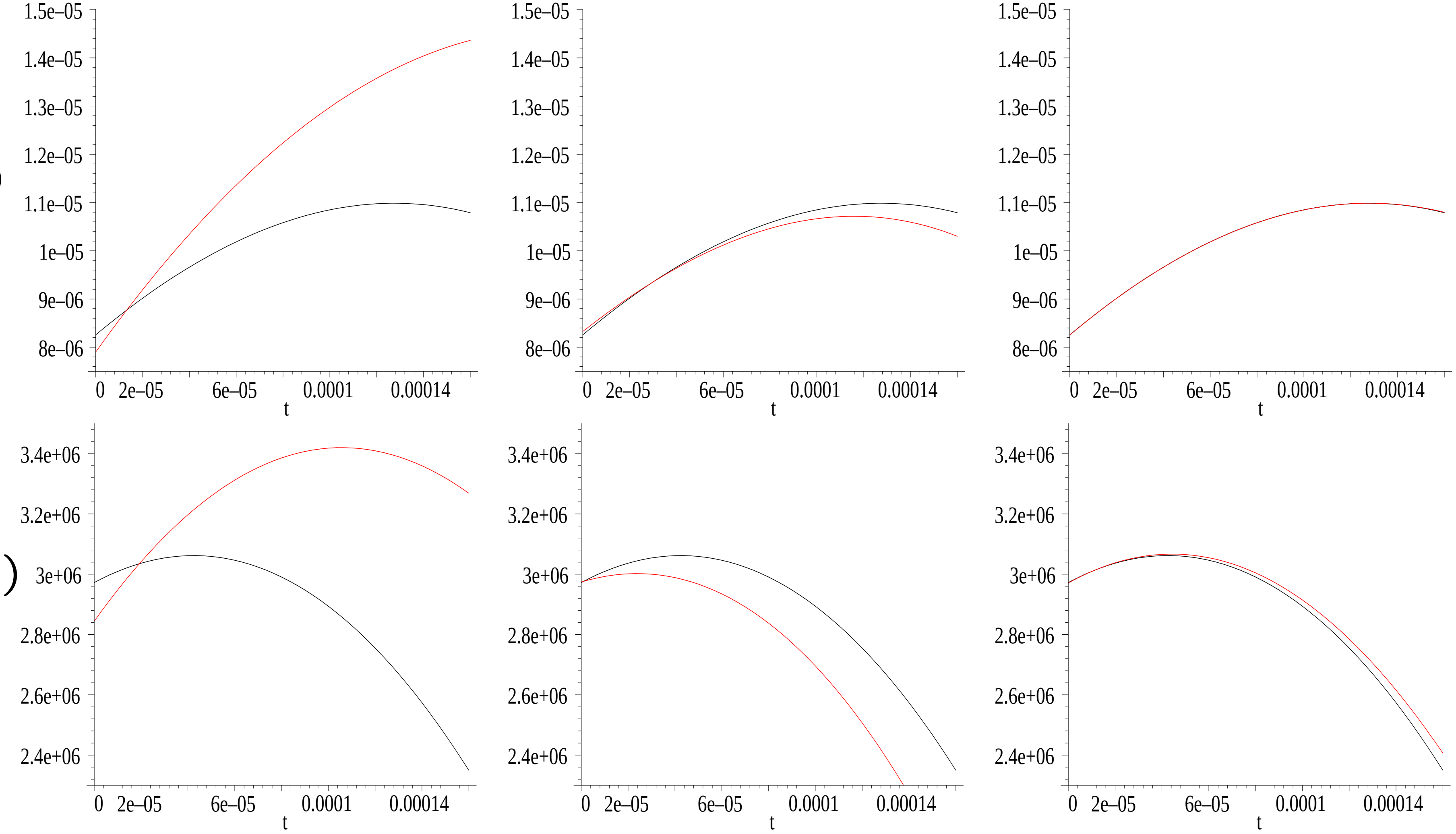
Approximate solution has a form: $\mathbf{u} \approx \hat{\mathbf{u}} = \mathrm{grad}\ \hat{\phi} + \mathrm{rot}\ \widehat{\mathbf{\Psi}}$, where

$$\psi \approx \hat{\psi} = \sum_{n=1}^{N} c_n^1 V_n^1, \ \ \phi \approx \hat{\phi} = \sum_{n=1}^{N} c_n^0 V_n^0 + \phi_p.$$

The coefficients c_n^i are chosen so that the error of fulfilling given boundary and initial conditions is minimized:

$$\begin{aligned}
I \;=\; & \int_{-1}^{1}\int_{-1}^{1}\{\underbrace{[\hat{u}_x(x,y,0)-u_x(x,y,0)]^2+[\hat{u}_y(x,y,0)-u_y(x,y,0)]^2+\ldots}_{\text{given initial conditions}}\}\mathrm{d}y\mathrm{d}x \\
+ \; & \underbrace{\int_{-1}^{1}\int_{0}^{\Delta t}\{[\hat{u}_x(-1,y,t)-u_x(-1,y,t)]^2+[\hat{u}_y(-1,y,t)-u_y(-1,y,t)]^2\}\mathrm{d}t\mathrm{d}y+\ldots}_{\text{given boundary conditions}} \\
+ \; & \Delta t\sum_{k=0}^{10}\sum_{l=0}^{10}\{\underbrace{[\hat{u}_x(1-\varepsilon,-1+\frac{k}{5},\frac{l\Delta t}{10})-u_x(1-\varepsilon,-1+\frac{k}{5},\frac{l\Delta t}{10})]^2}_{\text{internal responses for } u_x}\} \\
+ \; & \Delta t\sum_{k=0}^{10}\sum_{l=0}^{10}\{\underbrace{[\hat{u}_y(1-\varepsilon,-1+\frac{k}{5},\frac{l\Delta t}{10})-u_y(1-\varepsilon,-1+\frac{k}{5},\frac{l\Delta t}{10})]^2}_{\text{internal responses for } u_y}\} \rightarrow Min
\end{aligned}$$

Convergence

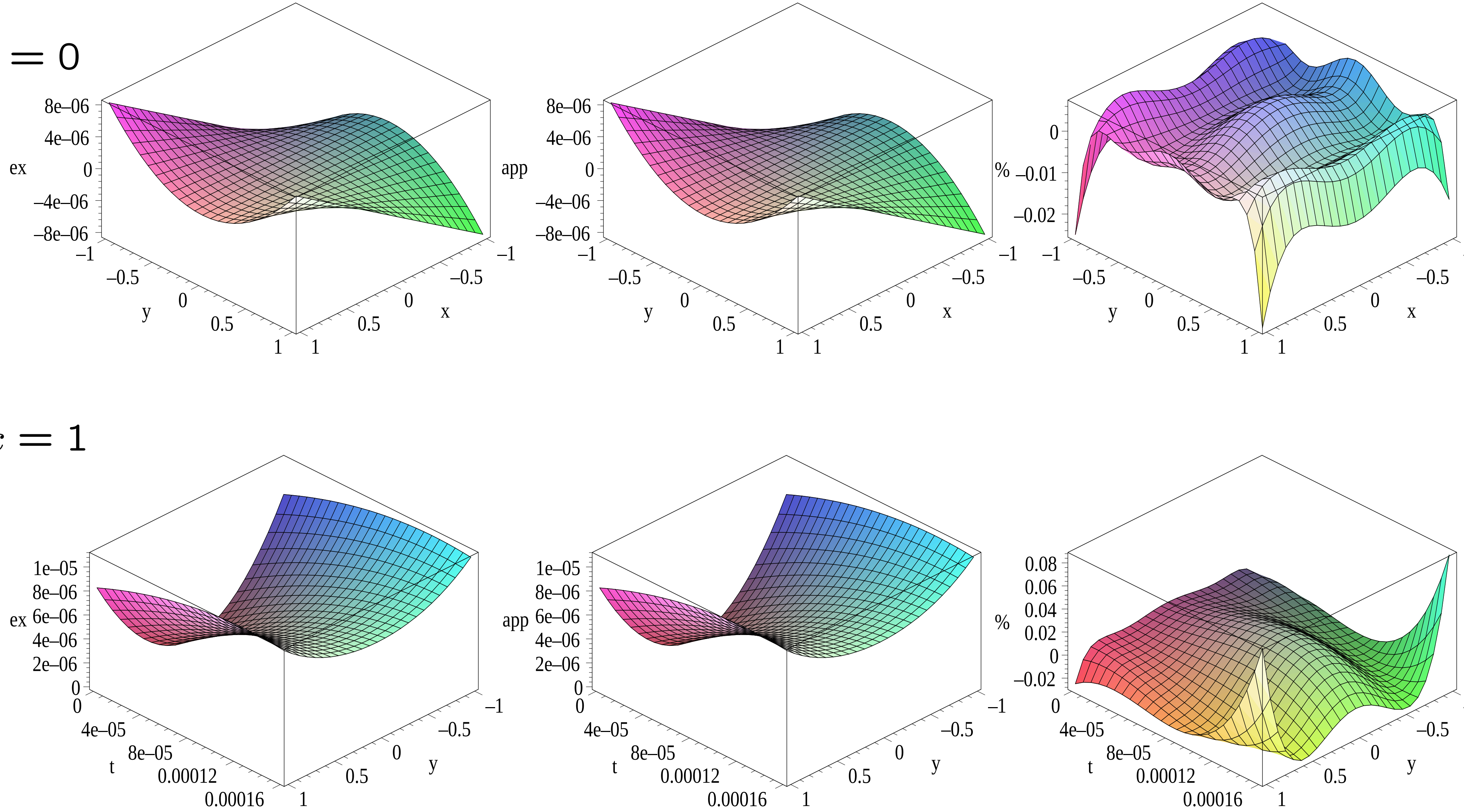


Order form 0 to 4 | Order form 0 to 5 | Order form 0 to 7

Displacement u_x ($\varepsilon = 0$)

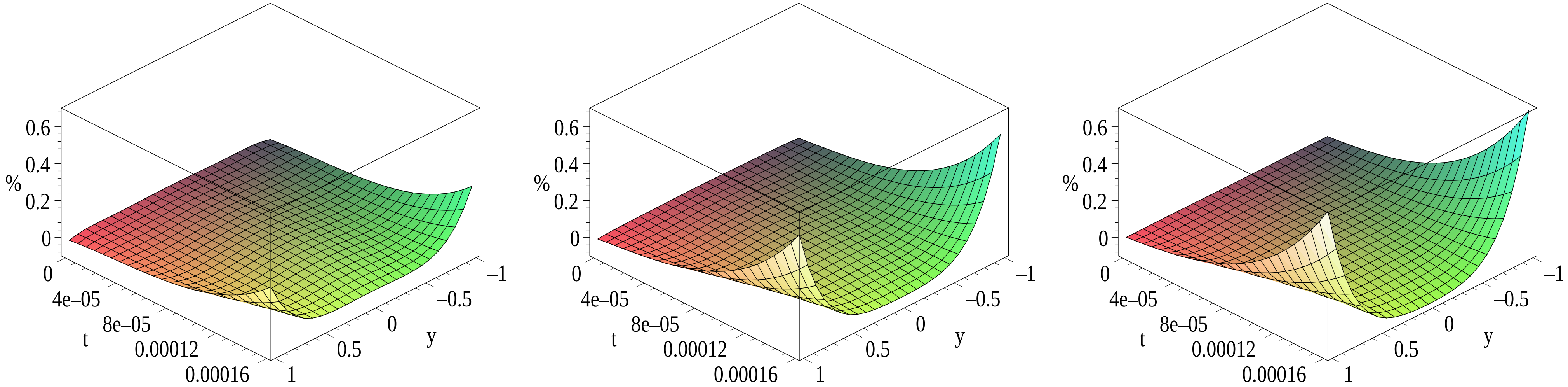
$t = 0$

$x = 1$

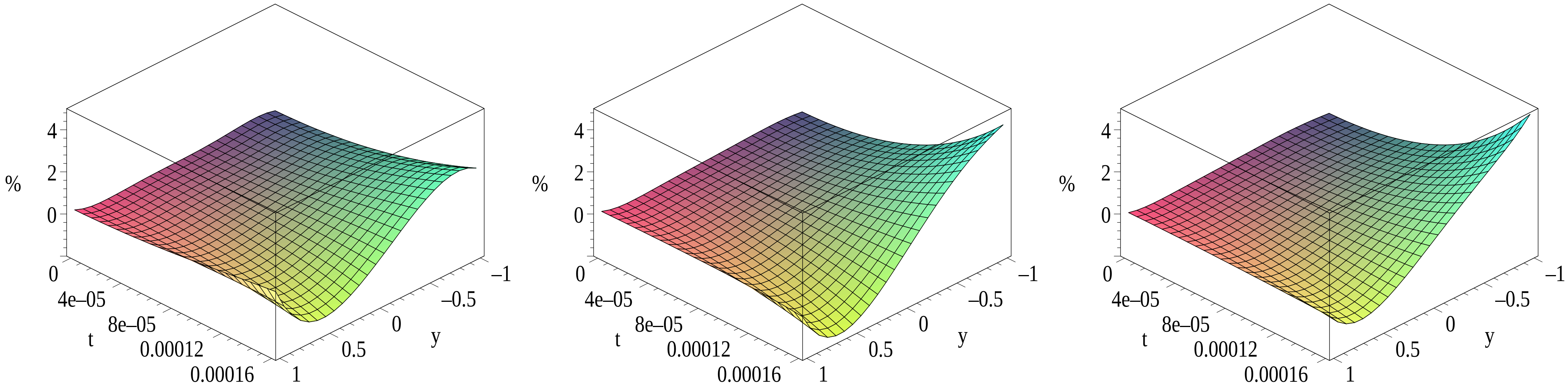


Exact | Approximation | Relative error [%]

The relative error for $u_x(1, y, t)$ (inverse problem)



The disturbed internal responses – normal distribution N(0,0.04)



$\varepsilon = 0.1$ $\varepsilon = 0.3$ $\varepsilon = 0.5$

The mean relative error of approximation of boundary condition $u_x(1, y, t)$ is defined as:

$$\mathsf{E} = \sqrt{\frac{\int_{-1}^{1}\int_{0}^{\Delta t}[\widehat{u}_x(1;y;t) - u_x(1;y;t)]^2 \mathrm{d}t\mathrm{d}y}{\int_{-1}^{1}\int_{0}^{\Delta t}[u_x(1;y;t)]^2 \mathrm{d}t\mathrm{d}y}} \cdot 100\%.$$

In approximation $\widehat{u}_x$ we take all wave polynomials up to order 7. The table shows the error E[%] in dependence on the distance ε:

ε	Smooth data	Noisy data
0	0.022	0.896
0.1	0.044	1.073
0.3	0.108	1.503
0.5	0.133	1.394

For noisy data the error is bigger but still stays very low even for big distance ε.

Conclusions

- A new, relatively simple method of solving the direct and inverse problems for wave equations was proposed.
- Thanks to this method we obtain an analytical solution which satisfies given equation and depends continuously on all variables in the whole domain.
- The error of approximation is small.
- Considered example confirms the theoretical result that the wave polynomials method is convergent – more polynomials in approximate solution leads to better results.
- The method is flexible according to initial and boundary conditions (discrete, missing).
- A new and very effective approach towards smoothing by using Trefftz functions is proposed here.

Thank you for your attention!