Multiparameter regularization in downward continuation of satellite data

Sergei V. Pereverzev

In cooperation with Shuai Lu and Ulrich Tautenhahn

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Johann Radon Institute for Computational and Applied Mathematics, Austria

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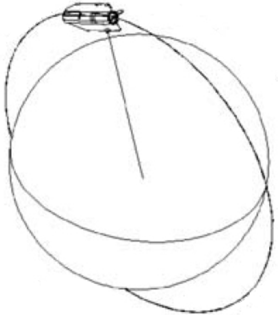
Downward continuation of satellite data

Given data $G(t)$ at $\Omega_r = \{t \in R^3 : |t| = r\}$, determine the disturbing gravity potential $U(t)$, harmonic outside the geocentric reference sphere Ω_R of the radius $R < r$.



Ω_R: Geocentric reference sphere, i.e., the Earth

Ω_r : Satellite orbit.

Observation data G(t):
derived from satellite gravimetry;

Gravity potential U(t):
harmonic outside the Ω_R

Downward continuation of satellite data

Upward continuation

$$\begin{cases} \Delta U(t) = 0 & \text{for} \quad |t| > R, \\ U(t) = F(t) & \text{for} \quad |t| = R, \end{cases}$$

where $F(t)$ is the disturbing gravity potential. Assume $U(t) = O(\frac{1}{|t|})$ and $|\nabla_t U(t)| = O(\frac{1}{|t|^2})$ for $t \to \infty$.

Abel-Poisson integral

$$U(t) \quad = \quad \frac{1}{4\pi R} \int_{\Omega_R} \frac{|t|^2 - R^2}{|t-\tau|^3} F(\tau) d\Omega_R(\tau).$$

Downward continuation of satellite data

$$A_{DWC}F(t) \quad := \quad \frac{1}{4\pi R}\int_{\Omega_R} \frac{r^2 - R^2}{(r^2 + R^2 - 2t\cdot\tau)^{3/2}} F(\tau)d\Omega_R(\tau) = G(t), \quad t \in \Omega_r.$$

SVD of $A = A_{DWC}$

$$A = \sum_{j=1}^{\infty} a_j u_j \langle v_j, \cdot\rangle_{L^2(\Omega_R)},$$

where

$$a_j = \left(\frac{R}{r}\right)^k, \quad u_j = u_j(t) = \frac{1}{r}Y_{k,i}(\frac{t}{r}), \quad v_j = v_j(\tau) = \frac{1}{R}Y_{k,i}(\frac{\tau}{R}),$$
$$j = i + k^2, \quad i = 1, 2, \ldots, 2k+1, \quad k = 0, 1, \ldots,$$

and $\{Y_{k,i}\}$ is a system of spherical harmonics.

Satellite gravity gradiometry (SGG)

$A = A_{SGG}\colon L^2(\Omega_R) \rightarrow L^2(\Omega_r)$

$$
\begin{aligned}
A_{SGG}F(t) &:= \frac{1}{4\pi R}\int_{\Omega_R} \frac{\partial^2}{\partial r^2}\left[\frac{r^2-R^2}{(r^2+R^2-2t\cdot\tau)^{3/2}}\right] F(\tau)\,d\Omega_R(\tau), \quad t\in\Omega_r. \\
a_j &= \left(\frac{R}{r}\right)^k \frac{(k+1)(k+2)}{r^2}, \quad j=i+k^2, \quad i=1,2,\ldots,2k+1.
\end{aligned}
$$

Ill-posed problems and regularization

$$
\begin{aligned}
y_\varepsilon &= Ax^\dagger + \varepsilon, \\
\Phi(\alpha;x) &= \|Ax-y_\varepsilon\|^2 + \alpha\|Bx\|^2, \\
\Phi(\alpha_1,\alpha_2,\ldots,\alpha_l;x) &= \|Ax-y_\varepsilon\|^2 + \sum_{i=1}^{l}\alpha_i\|B_ix\|^2.
\end{aligned}
$$

An appearance of multiparameter regularization in the geodetic context

$$B_i = \begin{pmatrix} 0 & & & & \\ & \dots & & & \\ & & I_i & & \\ & & & \dots & \\ & & & & 0 \end{pmatrix},$$

where I_i is $(2i+1)\times(2i+1)$-identity matrix.

Kaula's rule of thumb: heuristic *a priori* choice of α_i;

Xu, Rummel, Fukuda, Liu (1992-2006):

heuristic *a posteriori* parameter choice strategy.

Remark: In our tests for Xu et.al. approach α_i will be chosen in "the best possible way".

Regularization in Hilbert scale

- $A \in \mathscr{L}(X, Y)$ with non-closed range $\mathscr{R}(A)$
- X, Y - Hilbert spaces
- B - strictly positive self-adjoint (unbounded) operator in X

Assumption A1 (Link condition between A and B^{-1})

$$m\|B^{-a}x\| \leq \|Ax\| \quad \text{for some} \quad a > 0, m > 0.$$

Assumption A2 (Solution smoothness)

$$x^{\dagger} \in M^{p}_{B,\rho} = \{x \in X :, \|B^{p}x\| \leq \rho\} \quad \text{for some } p > 0.$$

Spherical Sobolev spaces

Define an unbounded self-adjoint strictly positive definite multiplicator operator in $L^2(\Omega_R)$

$$Df(t) := \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \left(k + \frac{1}{2}\right) Y_{k,i}^R(t) \langle Y_{k,i}^R, f\rangle_{L^2(\Omega_R)},$$

where $Y_{k,i}^R(t) = \frac{1}{R} Y_{k,i}(\frac{t}{R})$ are $L^2(\Omega_R)$-orthonormal spherical harmonics.

Spherical Sobolev spaces

We introduce the space

$$\mathscr{H}_s(\Omega_R) = \{f : \|D^s f\|^2 = \sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} \left(k + \frac{1}{2}\right)^{2s} \langle Y_{k,i}^R, f\rangle_{L^2(\Omega_R)}^2 < \infty\}$$

with the scalar product $\langle f, g\rangle_s := \langle D^s f, D^s g\rangle_{L^2(\Omega_R)}$ and the associated norm $\|f\|_s = \langle f, f\rangle_s^{1/2}$.

Spherical Sobolev spaces

The known (unitless) leading coefficients $\langle x^\dagger, Y_{k,i}^R \rangle_{L^2(\Omega_R)}$ of the Earth's anomalous potential allow the estimates

$$\|x^\dagger\|_{3/2} := \left(\sum_{k=0}^{\infty} \sum_{i=1}^{2k+1} (k+\tfrac{1}{2})^3 \langle x^\dagger, Y_{k,i}^R \rangle^2 \right)^{1/2} \approx 1.934 \times 10^{-3},$$

$$\|x^\dagger - P_{31}x^\dagger\|_{3/2} := \left(\sum_{k=32}^{\infty} \sum_{i=1}^{2k+1} (k+\tfrac{1}{2})^3 \langle x^\dagger, Y_{k,i}^R \rangle^2 \right)^{1/2} \approx 4.3 \times 10^{-8},$$

$x^\dagger = F(t)$ is of class $\mathscr{H}_s(\Omega_R)$ with $s = \frac{3}{2}$.

Assumption A2 (Solution smoothness)

$$x^\dagger \in M_{B,\rho}^p = \{x \in X :, \|B^p x\| \le \rho\} \quad \text{for some } p > 0.$$

is satisfied with any $B = D^q$, $p = \frac{3}{2q}$, $q > 0$ and $\rho \approx 1.934 \times 10^{-3}$.

Surrogate operator

$$A_{SGG}F(t) := \frac{1}{4\pi R}\int_{\Omega_R}\frac{\partial^2}{\partial r^2}\left[\frac{r^2-R^2}{(r^2+R^2-2t\cdot\tau)^{3/2}}\right]F(\tau)d\Omega_R(\tau), \quad t\in\Omega_r.$$

$$a_j = \left(\frac{R}{r}\right)^k\frac{(k+1)(k+2)}{r^2}, \quad j=i+k^2, \quad i=1,2,\dots,2k+1.$$

$B=D^q$, the singular values b_j of B^{-a} have the form $b_j=(k+1/2)^{-aq}$, $j=k^2+i$, $i=1,2,\dots,2k+1$, $k=0,1,\dots$

GOCE-mission (Gravity Field and Steady-State Ocean Circulation): provides a high-accuracy model of the Earth's gravitational field based on potential coefficients $\langle x^\dagger, Y_{k,i}^R\rangle_{L^2(\Omega_R)}$ up to degree $k=300$. Then the singular values $a_j=a_{k,i}=(R/r)^k(k+1)(k+2)/r^2$, $j=k^2+i$, behave like $(k+1/2)^{-s}$ with $s=5.5$.

Surrogates operator

A straightforward calculation shows that assuming a mean Earth's radius $R = 6378 \times 10^3 [m]$ and an altitude of GOCE-satellite of about $r - R = 250 \times 10^3 [m]$ we obtain, in particular

$$0.2(k+1/2)^{-\frac{11}{2}} \leq a_{k,i} \leq 3(k+1/2)^{-\frac{11}{2}}, \quad k = 100, 101, \ldots, 300.$$

Surrogate operator

$$\tilde{A}_{SGG} := \sum_{k=0}^{300} \left(\frac{R}{r}\right)^k \frac{(k+1)(k+2)}{r^2} \sum_{i=1}^{2k+1} Y^r_{k,i} \langle Y^R_{k,i}, \cdot \rangle_{L^2(\Omega_R)} + \sum_{k=301}^{\infty} (k+1/2)^{-\frac{11}{2}} \sum_{i=1}^{2k+1} Y^r_{k,i} \langle Y^R_{k,i}, \cdot \rangle_{L^2(\Omega_R)},$$

where $Y^r_{k,i}(t) = \frac{1}{r} Y_{k,i}(\frac{t}{r})$.

Surrogate operator

Then for any $x \in L^2(\Omega_R)$

$$m\|D^{-\frac{11}{2}}x\|_{L^2(\Omega_R)} \leq \|\tilde{A}_{SGG}x\|_{L^2(\Omega_r)} \leq M\|D^{-\frac{11}{2}}x\|_{L^2(\Omega_R)}.$$

Assumption A1 (Link condition between A and B^{-1})

$$m\|B^{-a}x\| \leq \|Ax\| \quad \text{for some} \quad a > 0, m > 0.$$

The approximate design operator $\tilde{A}_{SGG}$ satisfies the assumption for any $B = D^q$ and $a = \frac{11}{2q}$, $q > 0$.

Analysis of two operators

Two models for the same observations

$$A_{SGG}x^{\dagger} = y_0,$$
$$\tilde{A}_{SGG}\tilde{x}^{\dagger} = y_0.$$

By using interpolation inequality

$$\begin{aligned}
\|x^{\dagger} - \tilde{x}^{\dagger}\|_{L^2(\Omega_R)} &\leq \|x^{\dagger} - \tilde{x}^{\dagger}\|_{-\frac{11}{2}}^{\frac{3}{14}} \|x^{\dagger} - \tilde{x}^{\dagger}\|_{\frac{3}{2}}^{\frac{11}{4}} \\
&\leq m^{-\frac{3}{14}} \|\tilde{A}_{SGG}(x^{\dagger} - \tilde{x}^{\dagger})\|_{L^2(\Omega_r)} \|x^{\dagger} - \tilde{x}^{\dagger}\|_{\frac{3}{2}}^{\frac{11}{4}}.
\end{aligned}$$

Analysis of two operators

We have

$$\langle Y^R_{k,i}, \tilde{x}^\dagger \rangle_{L^2(\Omega_R)} = \langle Y^R_{k,i}, x^\dagger \rangle_{L^2(\Omega_R)} = a^{-1}_{k,i} \langle Y^r_{k,i}, y_0 \rangle_{L^2(\Omega_r)}$$

up to the degree $k = 300$, while for $k = 301,\ 302,\ \ldots,\ a_{k,i} < (k + \frac{1}{2})^{-\frac{11}{2}}$ and

$$\begin{aligned} |\langle Y^R_{k,i}, \tilde{x}^\dagger \rangle_{L^2(\Omega_R)}| &= (k + \frac{1}{2})^{\frac{11}{2}} |\langle Y^r_{k,i}, y_0 \rangle_{L^2_{\Omega_r}}| \\ &< a^{-1}_{k,i} |\langle Y^r_{k,i}, y_0 \rangle_{L^2(\Omega_r)}| = |\langle Y^R_{k,i}, x^\dagger \rangle_{L^2(\Omega_r)}|. \end{aligned}$$

It means that

$$\|x^\dagger - \tilde{x}^\dagger\|_{\frac{3}{2}} \leq \|x^\dagger - P_{300} x^\dagger\|_{\frac{3}{2}} \leq \|x^\dagger - P_{31} x^\dagger\|_{\frac{3}{2}}.$$

Analysis of two operators

On the other hand, it is easy to see that

$$\begin{aligned} \|\tilde{A}_{SGG}(x^\dagger - \tilde{x}^\dagger)\|_{L^2(\Omega_r)} &= \|(A_{SGG} - \tilde{A}_{SGG})x^\dagger\|_{L^2(\Omega_r)} \\ &\le (300 + \frac{1}{2})^{-\frac{11}{2}} \|(I - P_{300})x^\dagger\|_{L^2(\Omega_R)} \\ &\le (300 + \frac{1}{2})^{-7} \|(I - P_{300})x^\dagger\|_{\frac{3}{2}} \\ &\le 4.5 \cdot 10^{-18} \|x^\dagger - P_{31}x^\dagger\|_{\frac{3}{2}}. \end{aligned}$$

Summing up, we obtain

$$\|x^\dagger - \tilde{x}^\dagger\|_{L^2(\Omega_R)} \le 2 \cdot 10^{-4} \|x^\dagger - P_{31}x^\dagger\|_{\frac{3}{2}} \approx 10^{-11}.$$

Remark: The expected accuracy for a satellite mission with the same altitude as GOCE is 10^{-11} [Freeden and Pereverzev 2001]. Thus, the surrogate operator $\tilde{A}_{SGG}$ allows keeping an error within desired range.

Noise in the operator

Assume that the surrogate design operator $\tilde{A}_{SGG}$ is spoiled by an operator noise hE

$$A = \tilde{A}_{SGG} + hE,$$

Total least squares

$$\|\hat{A} - A\|^2 + \|\hat{y} - y_\varepsilon\|^2 \to \min \quad \text{subject to} \quad \hat{A}\hat{x} = \hat{y}.$$

Regularized total least squares (RTLS)

$$\|\hat{A} - A\|^2 + \|\hat{y} - y_\varepsilon\|^2 \to \min \quad \text{subject to} \quad \hat{A}\hat{x} = \hat{y}, \quad \|B\hat{x}\| \leq \rho.$$

Regularized total least squares (RTLS)

RTLS constraint

$$\|x^\dagger - P_1 x^\dagger\|_{\frac{3}{2}} := \left(\sum_{k=2}^{\infty} \sum_{i=1}^{2k+1} \left(k+\frac{1}{2}\right)^3 \langle x^\dagger, Y_{k,i}^R \rangle^2 \right)^{1/2} \approx 3.033 \times 10^{-7}.$$

Within the framework of the RTLS-method it is reasonable to take

$$B = B_\eta = \eta \sum_{k=0}^{1} \sum_{i=1}^{2k+1} \langle Y_{k,i}^R, \cdot \rangle + \sum_{k=2}^{\infty} \sum_{i=1}^{2k+1} \left(k+\frac{1}{2}\right)^{\frac{3}{2}} \langle Y_{k,i}^R, \cdot \rangle, \quad \rho = 3.033 \times 10^{-7}$$

RTLS theorem

If in the RTLS problem the constraint $\|B\hat{x}\| \le \rho$ is active then the RTLS-solution $\hat{x} = x_\varepsilon^{\alpha,\beta}$ satisfies the equations

$$(A^T A + \alpha B^2 + \beta I) x_\varepsilon^{\alpha,\beta} = A^T y_\varepsilon$$

and

$$\|B x_\varepsilon^{\alpha,\beta}\| = \rho,$$

where the parameters α, β satisfy

$$\alpha = \frac{1}{\rho^2}\left(\beta + \|y_\varepsilon\|^2 - \langle y_\varepsilon, A x_\varepsilon^{\alpha,\beta}\rangle\right), \quad \beta = -\frac{\|A x_\varepsilon^{\alpha,\beta} - y_\varepsilon\|^2}{1 + \|x_\varepsilon^{\alpha,\beta}\|^2}.$$

Moreover, the RTLS-solution $\hat{x} = x_\varepsilon^{\alpha,\beta}$ is also the solution of the constrained minimization problem

$$\frac{\|Ax - y_\varepsilon\|^2}{1 + \|x\|^2} \to \min \quad \text{subject to} \quad \|Bx\| \le \rho.$$

Dual regularized total least squares (DRTLS)

$$\|Bx\| \to \min \quad \text{subject to } \hat{A}\hat{x} = \hat{y}, \quad \|\hat{y} - y_\varepsilon\| \le \delta, \quad \|\hat{A} - A\| \le h,$$

DRTLS theorem [Lu, Pereverzev and Tautenhahn 2009]

If in the DRTLS problem the two inequality constraints are active, then the DRTLS-solution $\hat{x} = x_\varepsilon^{\alpha,\beta}$ satisfies the equation

$$(A^T A + \alpha B^2 + \beta I) x_\varepsilon^{\alpha,\beta} = A^T y_\varepsilon,$$

where the regularization parameters α, β solve the following system of nonlinear equations

$$\begin{aligned} \|A x_\varepsilon^{\alpha,\beta} - y_\varepsilon\| &= \delta + h\|x_\varepsilon^{\alpha,\beta}\|, \\ \beta &= -\frac{h(\delta + h\|x_\varepsilon^{\alpha,\beta}\|)}{\|x_\varepsilon^{\alpha,\beta}\|}. \end{aligned}$$

DRTLS Convergence rate theorem [Lu, Pereverzev and Tautenhahn 2009]

Assume the assumptions hold with $1 \le p \le a+2$. Let $\hat{x} = x_\varepsilon^{\delta,\beta}$ be the DRTLS-solution. If the operator and right-hand side noise inequality constraints are active then

$$\|x^\dagger - \hat{x}\| \le 2E^{\frac{a}{a+p}} \left(\frac{\delta + h\|x^\dagger\|}{m} \right)^{\frac{p}{p+a}} = O\left((\delta + h)^{\frac{p}{p+a}} \right).$$

In addition let $\hat{x} = x_\varepsilon^{\alpha,\beta}$ be the RTLS-solution. If the exact solution $x^\dagger$ satisfies the side condition $\|Bx^\dagger\| = \rho$ then

$$\|x^\dagger - \hat{x}\| \le (2E)^{\frac{a}{a+p}} \left(\frac{\max\{1, \|\hat{x}\|\}(\sqrt{2}+1)}{m} \right)^{\frac{p}{p+a}} = O\left((\delta + h)^{\frac{p}{p+a}} \right).$$

Remark: The error bounds similar to one for DRTLS can be obtained for a version of the method of extending compacts ([Dombrovskaja and Ivanov 1965], [Dorofeev and Yagola 1998,2004]), and for Tikhonov regularization with a single regularization parameter chosen by the discrepancy principle ([Lu, Pereverzev, Shao and Tautenhahn 2010]). **Both methods require the value δ, h.**

RTLS Convergence rate theorem [Lu, Pereverzev and Tautenhahn 2009]

Let $\hat{x} = x_{\varepsilon}^{\alpha,\beta}$ be the RTLS-solution. If the exact solution $x^{\dagger}$ satisfies the side condition $\|Bx^{\dagger}\| = \rho$ then under the assumption of the previous theorem

$$\|x^{\dagger} - \hat{x}\| \leq (2E)^{\frac{a}{a+p}} \left(\frac{\max\{1, \|\hat{x}\|\}(\sqrt{2}+1)}{m} \right)^{\frac{p}{p+a}} = O\left((\delta + h)^{\frac{p}{p+a}} \right).$$

Numerical illustrations

Singular values:

$$a_j = (1.06)^{-k}, \quad j = i + k^2, \quad i = 1, 2, \ldots, 2k+1, \quad k = 0, 1, \ldots$$

correspond to an orbit height of about 400km.

Operators:

$$\begin{aligned} A_0 &= \operatorname{diag}(1, (1.06)^{-1}, (1.06)^{-1}, (1.06)^{-1}, \ldots, (1.06)^{-10}); \\ A &= A_0 + \|U\|_F^{-1} U. \end{aligned}$$

U is formed by random numbers uniformly distributed on $[0, 1]$

Numerical illustrations

The exact solution

$$x^\dagger = (x_{0,1}^\dagger, x_{1,1}^\dagger, x_{1,2}^\dagger, x_{1,3}^\dagger, \kappa x_{2,1}^\dagger, \ldots, \kappa x_{k,i}^\dagger, \ldots),$$

where $x_{k,i}^\dagger$ are uniformly distributed on [0,1], and

$$\begin{aligned} \kappa &= 3.033 \times 10^{-7} \times \left(\sum_{k=2}^{10} \sum_{i=1}^{2k+1} \left(k + \frac{1}{2}\right)^2 (x_{k,i}^\dagger)^2 \right)^{-1/2}, \\ y_\varepsilon &= A_0 x^\dagger + \delta \|e\|^{-1} e, \end{aligned}$$

e is formed by random numbers uniformly distributed on $[0,1]$.

Noise free operator

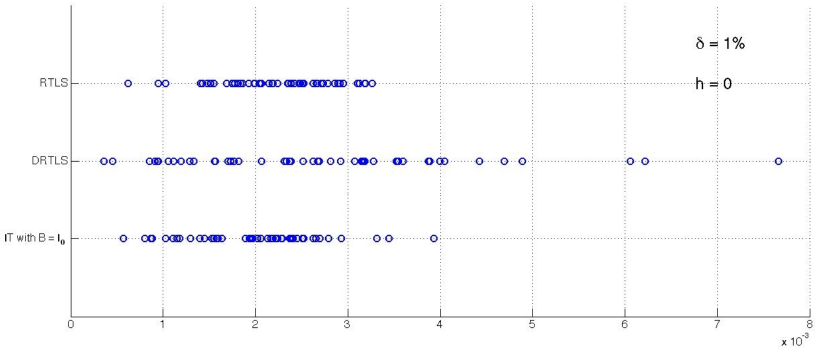


Figure: Comparison of the performances of considered multiparameter regularizations in the case of no modeling error (50 independent simulations of y_ε) .

Noise propagated operator

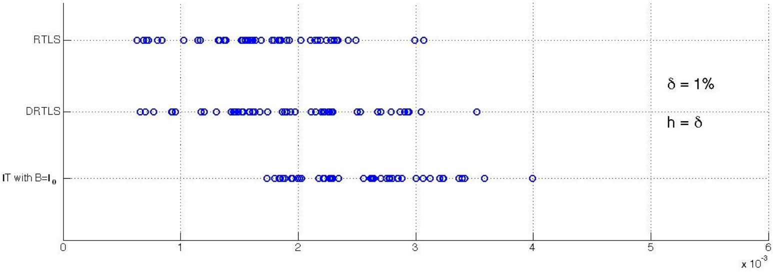


Figure: Comparison of the performances of considered multiparameter regularizations in the presence of modeling error, $h = \delta$ (50 independent simulations of A and y_ε) .

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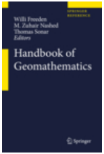

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