A Preconditioned GMRES Method for Solving the Sideways Parabolic Equation in Two Space Dimensions

Zohreh Ranjbar

Department of Mathematics, Linköping University

Linköping, Sweden.

Joint work with Lars Eldén

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Outline

- 2D Sideways Parabolic Equation with variable coefficients
- Generalized Minimum Residual method (GMRES)
- Preconditioner
- Implementation
- Numerical Experiments
- Summary

Two Dimensional Sideways Parabolic Equation

We want to solve the following problem:

$$
\begin{array}{ll}
u_t = (a(x,y)u_x)_x + (b(x,y)u_y)_y, & 0 < x < 1, \quad 0 < y < 1, \quad 0 \leq t \leq 1 \\
u(x,y,0) = 0, & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\
u(1,y,t) = g(t,y), & 0 \leq y \leq 1, \quad 0 \leq t \leq 1 \\
u_x(1,y,t) = 0, & 0 \leq y \leq 1, \quad 0 \leq t \leq 1 \\
u_y(x,0,t) = 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq 1 \\
u(x,1,t) = 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq 1
\end{array}
$$

where $u(0,y,t) = f(y,t)$ is sought from the data at the right boundary.

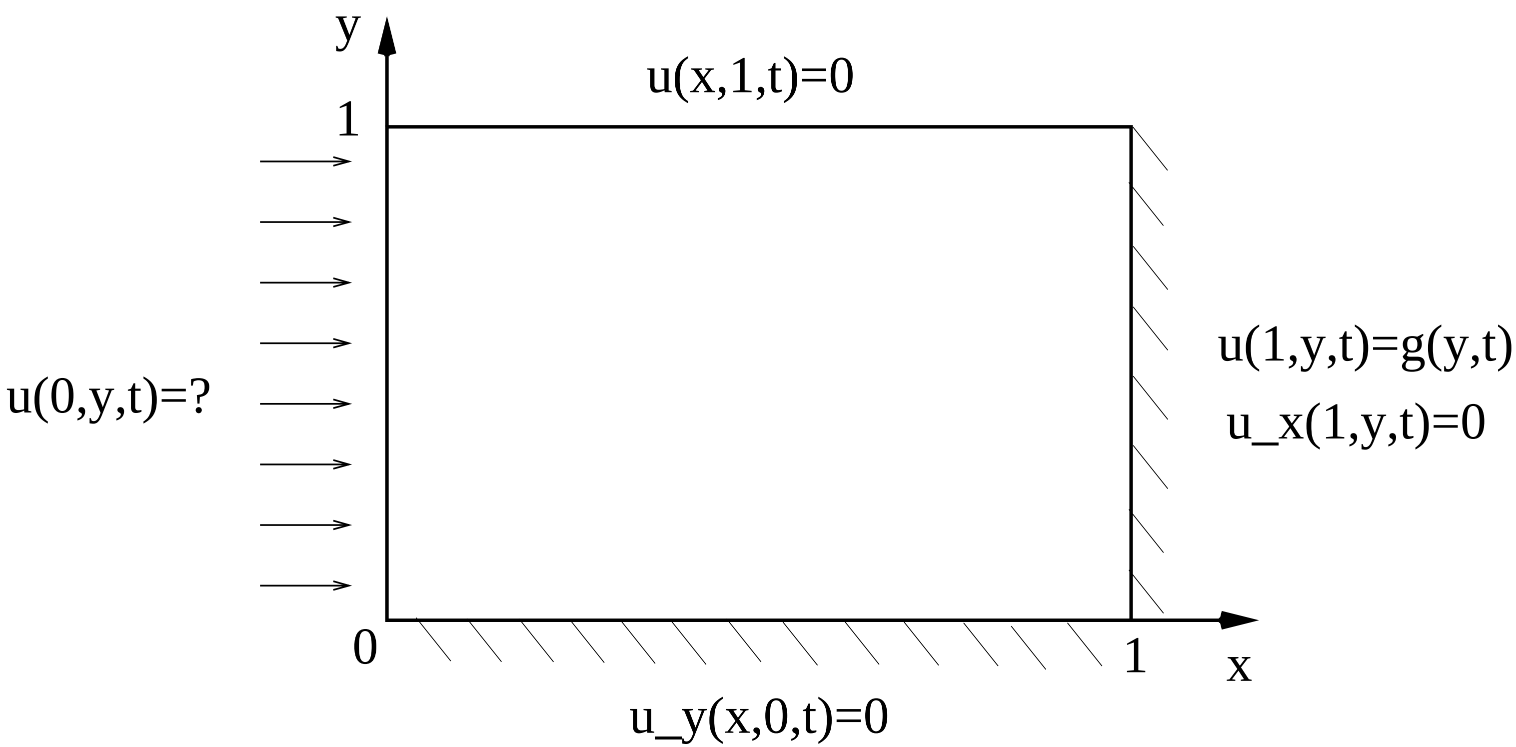
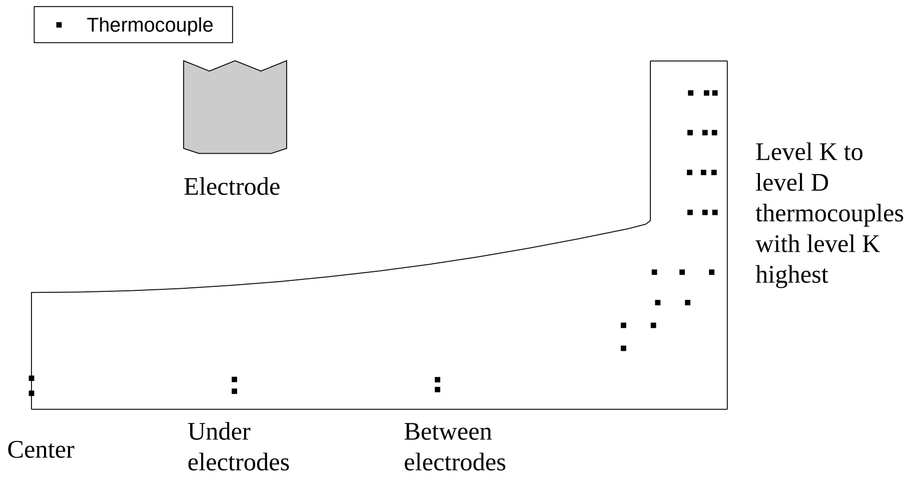


Figure 1: Two dimensional sideways heat problem.

Application



Skaar, I. Monitoring the Lining of a Melting Furnace Norwegian University of Science and Technology, Trondheim, Department of Mathematics, 2001

We want to solve

$$Kf = g$$

where f is the solution and K is the operator that maps f to the data g.

GMRES

The algorithm GMRES

1. Compute $r_0 = g - Kx_0$, $\beta = \|r_0\|_2$, and $v_1 = r_0/\beta$
2. **for** $j = 1, \ldots, m$, (m is a regularization parameters)

   Compute $w := Kv_j$

   for $i = 1, \ldots, j$ **do** (Gram-Schmidt)

   $h_{i,j} := w^T v_i$

   $w := w - h_{i,j} v_i$

   end

   Compute $h_{j+1,j} = \|w\|_2$ and $v_{j+1} = w/h_{j+1,j}$

   end
3. Define $V_m := [v_1, \ldots, v_m]$, $\bar{H}_m = \{h_{i,j}\}_{1 \le i \le j+1, 1 \le j \le m}$
4. Compute $y_m = \mathrm{argmin}_y \|\beta e_1 - \bar{H}_m y\|_2$ and $x_m = x_0 + V_m y_m$

The multiplication by the operator,

$$w = Kv_j,$$

corresponds solving the following well-posed parabolic problem in two space dimensions.

$$\begin{array}{ll}
u_t = (a(x,y)u_x)_x + (b(x,y)u_y)_y, & 0 < x < 1, \quad 0 < y < 1, \quad 0 \le t \le 1, \\
u(x,y,0) = 0, & 0 \le x \le 1, \quad 0 \le y \le 1, \\
u(0,y,t) = v_j(y,t), & 0 \le y \le 1, \quad 0 \le t \le 1, \\
u_x(1,y,t) = 0, & 0 \le y \le 1, \quad 0 \le t \le 1, \\
u_y(x,0,t) = 0, \quad u(x,1,t) = 0, & 0 \le x \le 1, \quad 0 \le t \le 1,
\end{array} \tag{1}$$

and evaluating the solution at $x = 1$ to get w. This can be done efficiently using any standard method for the solution of a 2D parabolic equation.

GMRES dose not work without preconditioner

Preconditioner

Let M represent a discretization of the 2D sideways heat equation with constant coefficients, where the constants are chosen as the mean values of the respective variable coefficients in the original problem 2D SHE. If the variations of the coefficients $a(x, y)$ and $b(x, y)$ are moderate then M is close to the operator K, and it is appropriate to use it as a preconditioner.

$$KM^{-1}\psi = g, \qquad \text{where} \qquad \psi = Mf.$$

Instead of M^{-1} we use the pseudo-inverse $M^{\dagger}$.

2D SHE with constant Coefficients

Consider the following equations,

$$
\begin{array}{lll}
u_t = a u_{xx} + b u_{yy}, & & 0 < x < 1, \quad 0 < y < 1, \quad 0 \le t \le 1, \\
u(x,y,0) = 0, & & 0 \le x \le 1, \quad 0 \le y \le 1, \\
u(1,y,t) = g(y,t), & & 0 \le y \le 1, \quad 0 \le t \le 1 \\
u_x(1,y,t) = 0, & & 0 \le y \le 1, \quad 0 \le t \le 1 \\
u_y(x,0,t) = 0, & u(x,1,t) = 0, & 0 \le x \le 1, \quad 0 \le t \le 1.
\end{array}
$$

Lemma. *[Reinhardt:91]The explicit representation of the solution of 2D SHE in terms of the unknown function $f(y,t)$ is*

$$u(x,y,t) = \int_0^t 2a \sum_{n,j=0}^{\infty} (-1)^n \nu_n \exp(-(a\nu_n^2 + b\mu_j^2)(t-s)) \Psi_{nj}(x,y) f_j(s) ds,$$

where

$$\nu_n = (2n+1)\frac{\pi}{2}, \qquad \mu_j = (2j+1)\frac{\pi}{2},$$
$$\Psi_{nj}(x,y) = \cos(\nu_n(1-x)) \cos(\mu_j y),$$

and

$$f_j(s) = 2\int_0^1 f(y,s) \cos(\mu_j y) dy, \quad j = 0, 1, \cdots$$

From Lemma we have for $x = 1$,

$$\begin{aligned} g(y,t) &= u(1,y,t) \\ &= \int_0^t 2a \sum_{n,j=0}^{\infty} (-1)^n \nu_n \exp(-(a\nu_n^2 + b\mu_j^2)(t-s)) \cos(\mu_j y) f_j(s) ds, \end{aligned}$$

Expanding $g(y,t)$ in the same cosine series

$$g(y,t) = \sum_{j=0}^{\infty} g_j(t)\cos(\mu_j y),$$

leads to

$$g_j(t) = \int_0^t 2a \sum_{n,j=0}^{\infty} (-1)^n \nu_n \exp(-(a\nu_n^2 + b\mu_j^2)(t-s)) f_j(s) ds.$$

1. Compute the **cosine transform** of the data function:

$$g_j(t) = 2\int_0^1 g(y,t)\cos(\mu_j y)dy, \qquad j = 0, 1, \ldots$$

2. Solve the **Volterra integral equations**

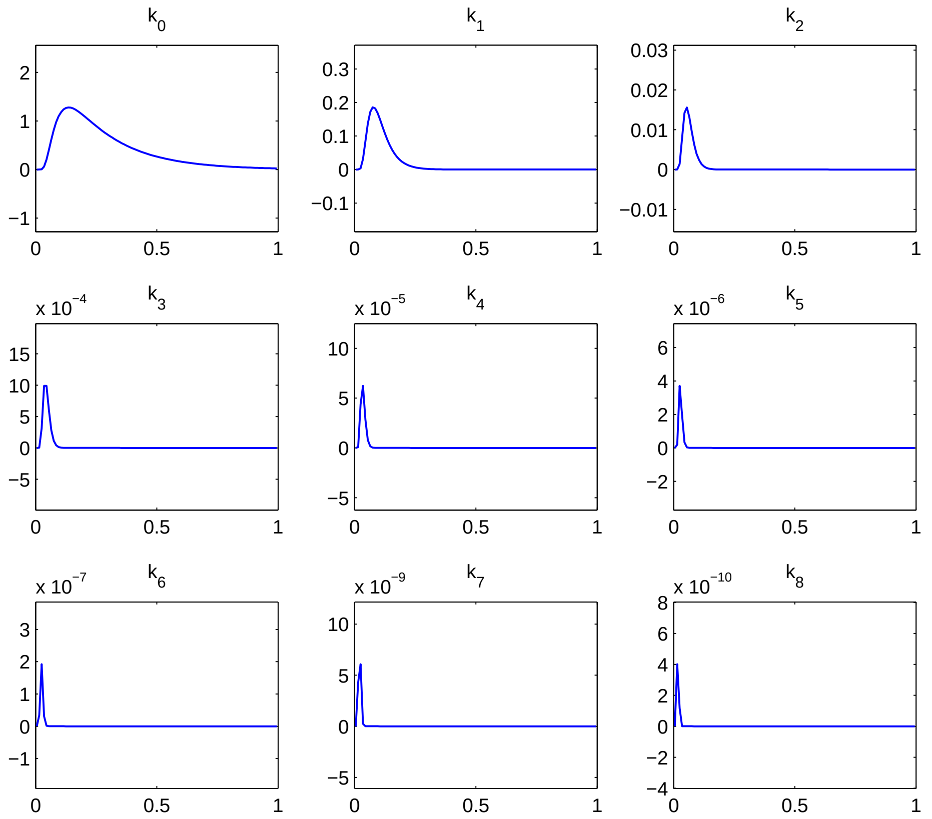
$$g_j(t) = \int_0^t k_j(t-s) f_j(s) ds, \qquad j = 0, 1, \ldots$$

   where the kernel k_j is given by

$$k_j(r) = 2a \sum_{n=0}^{\infty} (-1)^n \nu_n \exp(-(a\nu_n^2 + b\mu_j^2)r)$$

3. Evaluate the solution $f(y,t)$ by computing the **inverse cosine transform**:

$$f(y,t) = \sum_{j=0}^{\infty} f_j(t)\cos(\mu_j y).$$



Kernel functions $k_j(r)$, $\quad j = 0, 1, 2, \ldots, 8.$

PGMRES

The algorithm GMRES with Right Preconditioning

1. Compute $r_0 = g - Kx_0$, $\beta = \|r_0\|_2$, and $v_1 = r_0/\beta$
2. **for** $j = 1, \ldots, m$,

   Compute $w := KM^{\dagger}v_j$

   for $i = 1, \ldots, j$ **do**

   $h_{i,j} := w^T v_i$

   $w := w - h_{i,j}v_i$

   end

   Compute $h_{j+1,j} = \|w\|_2$ and $v_{j+1} = w/h_{j+1,j}$

   end
3. Define $V_m := [v_1, \ldots, v_m]$, $\bar{H}_m = \{h_{i,j}\}_{1\leq i\leq j+1, 1\leq j\leq m}$
4. Compute $y_m = \text{argmin}_y \|\beta e_1 - \bar{H}_m y\|_2$ and $x_m = x_0 + M^{\dagger}V_m y_m$

Implementation

Implementation of the preconditioner $M^{\dagger}v$:

1. $\hat{v} = \mathtt{DFT}(v), \quad \hat{v} = \begin{pmatrix} \hat{v}_1^T & \cdots & \hat{v}_n^T \end{pmatrix}^T$
2. **for** $j = 1, p$

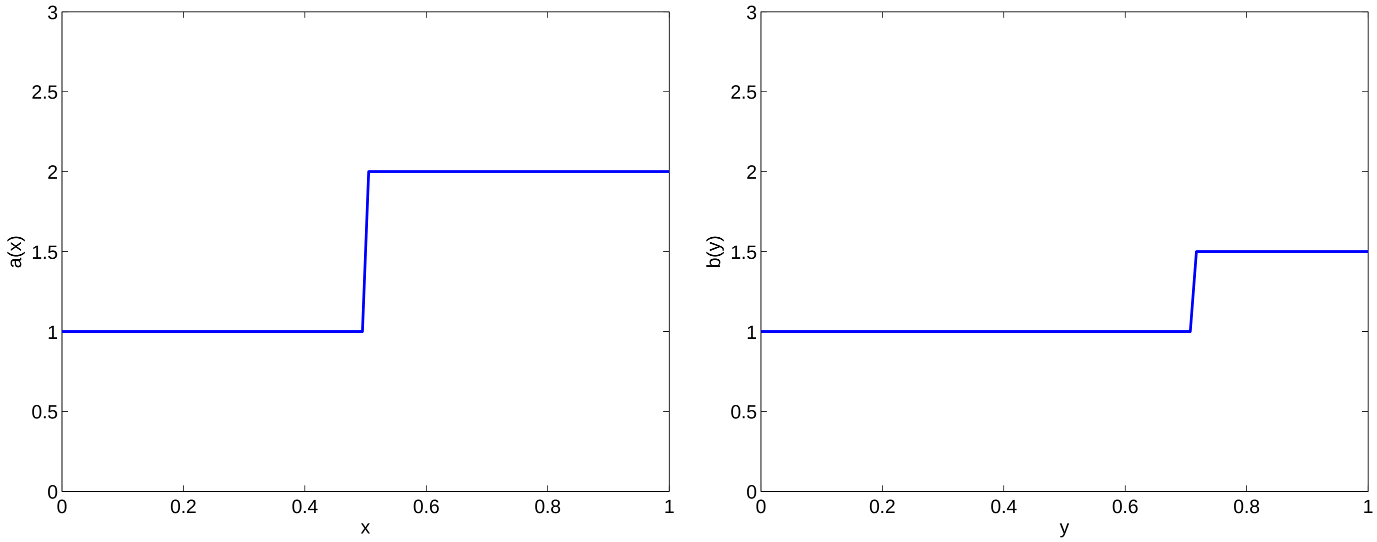
   $\hat{v}_j = \hat{v}(j, :)$,

   Solve $M_j \hat{u}_j = \hat{v}_j$ by Tikhonov regularization

   $\hat{u}(j, :) = \hat{u}_j$
3. **end**
4. $\hat{u} = (\hat{u}_1^T, \cdots, \hat{u}_p^T, 0, \cdots, 0)^T$
5. $u = \mathtt{IDFT}(\hat{u})$

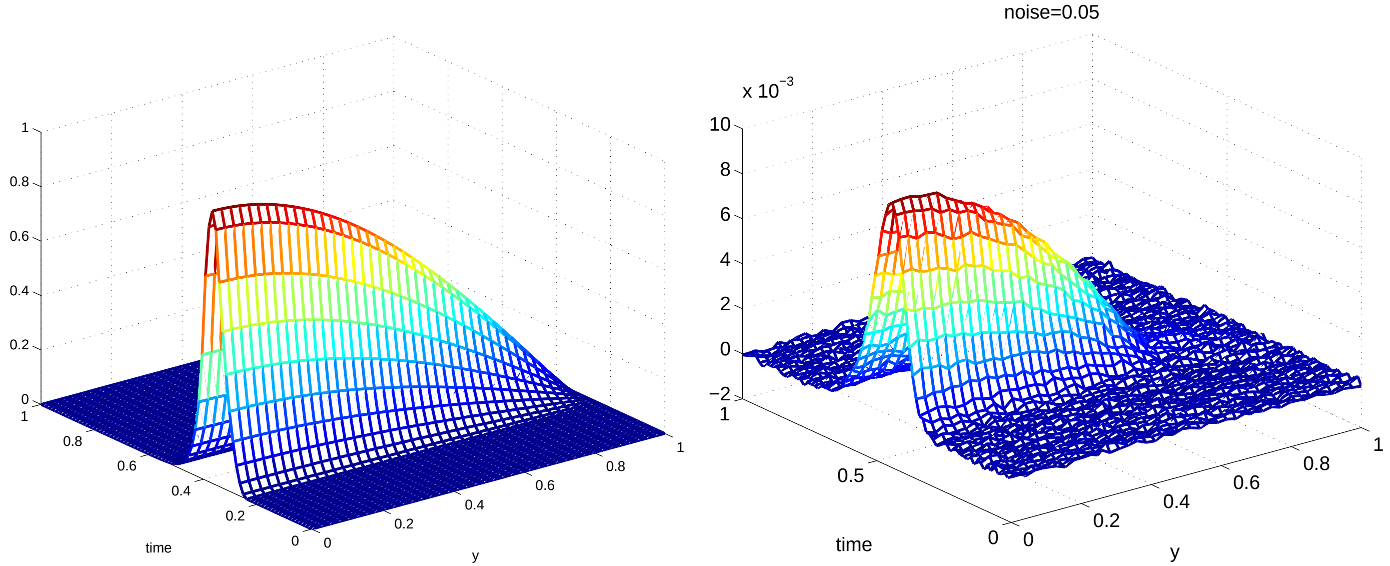
Here p is the number of terms in the cosine expansion.

Numerical Experiments

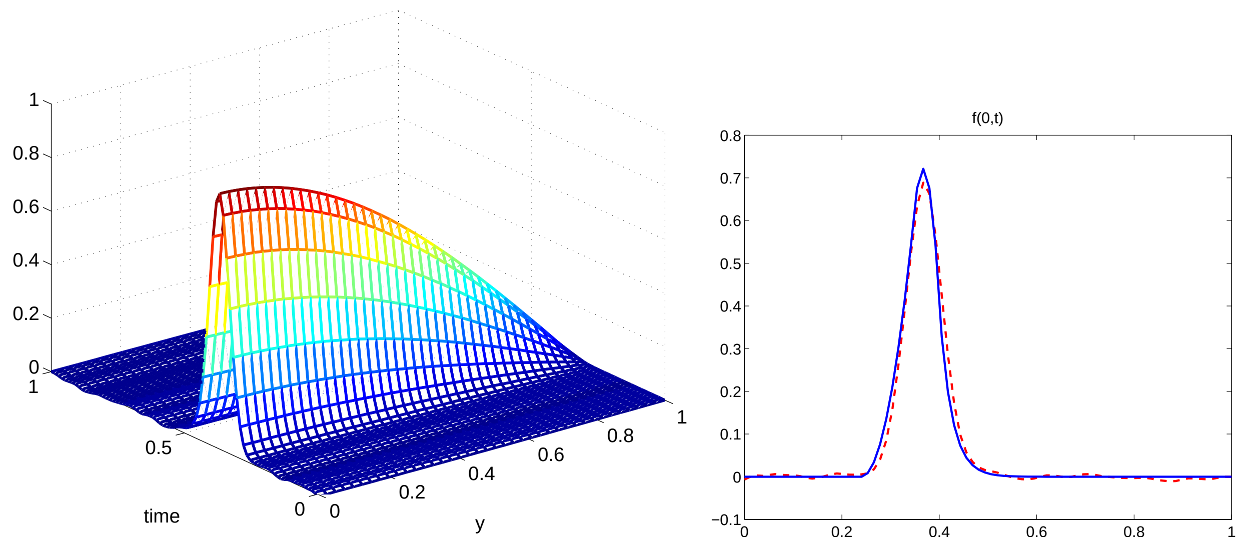


Plot of variable coefficients $a(x)$ (left) and $b(y)$ (right).

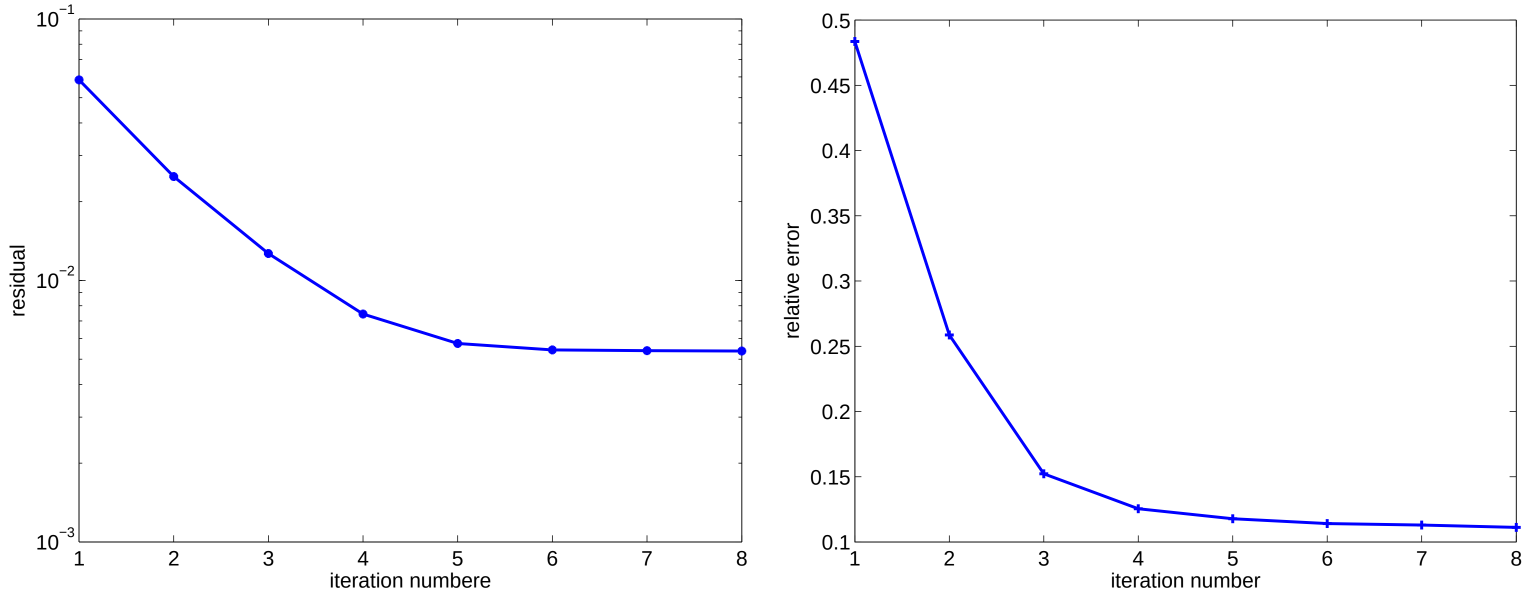
Test example 1:



Example 1. Exact solution $f(y,t)$ (left) and data function with 5% noise(right)

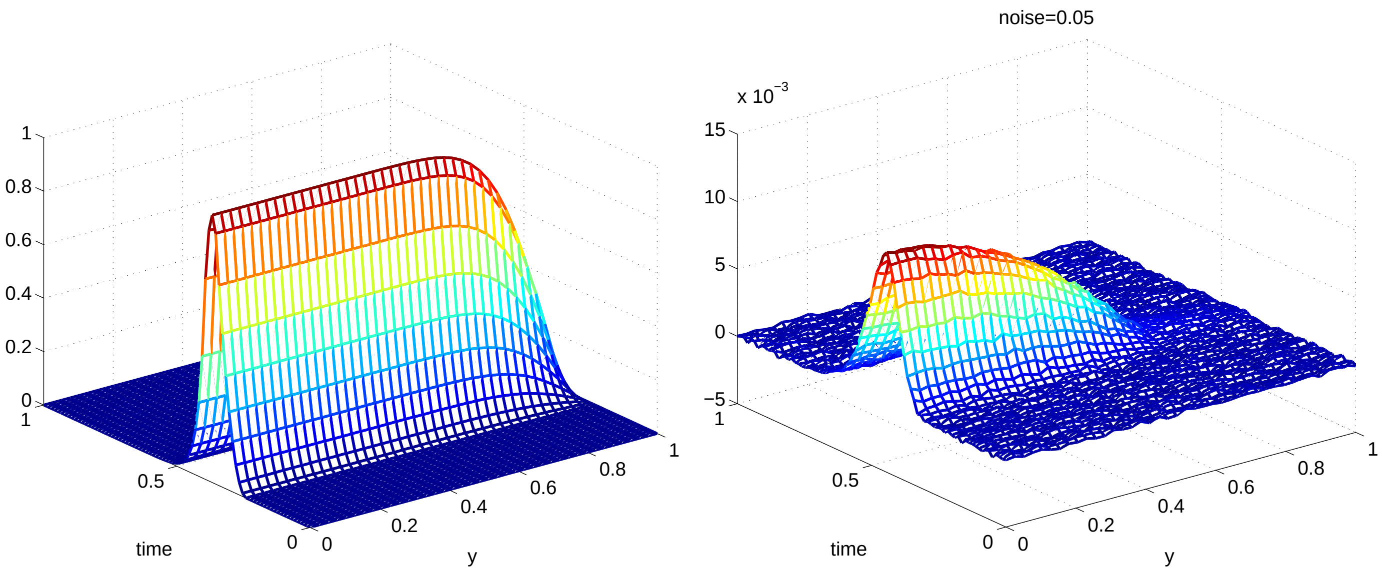


Example 1. Approximate solution after 6 iterations of PGMRES for $0 \leq y \leq 1$ (left) and plot of f and the approximated solution at $y = 1/2$(right). $\lambda \in (0.05, 0.5)$ and $p = 1$

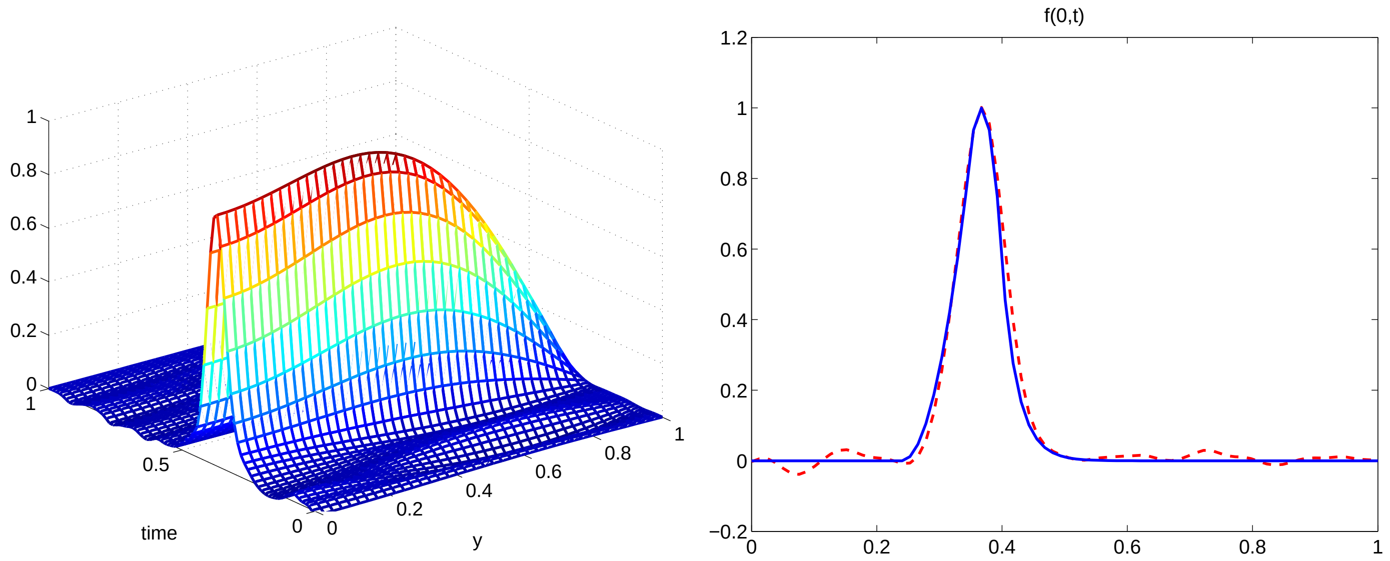


Example 1. Residuals (left) and relative errors (right) of PGMRES as functions of iteration index for 5% perturbation in the data.

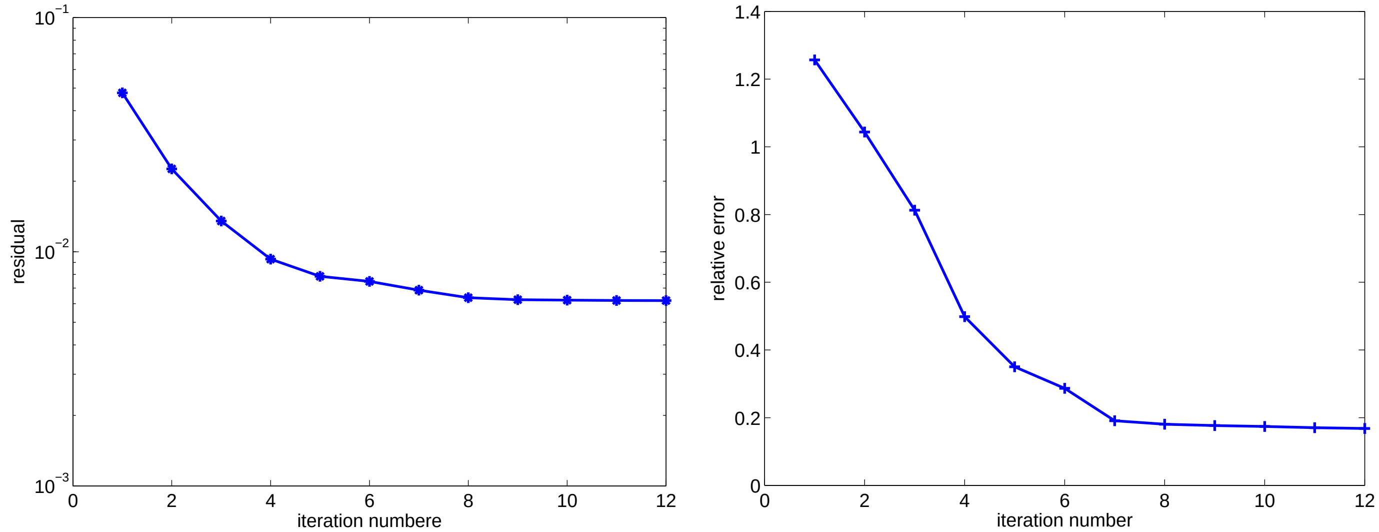
Test exapmle 2:



Example 2. Exact solution $f(y,t)$ and data function $g_\delta(y,t)$ with 5% noise.



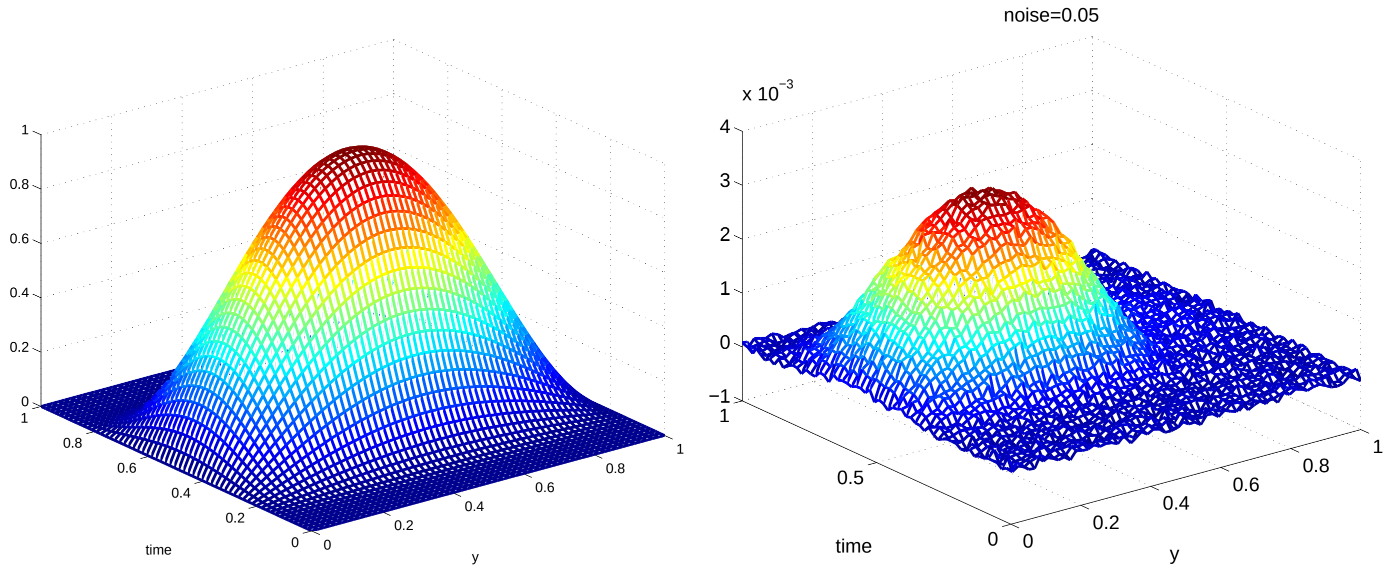
Example 2. Approximate solution (left), and the exact solution(solid) and the approximate solution (dashed) at $y = 1/2$ after 12 iterations of PGMRES with 5% perturbation in the data, and $\lambda = 0.06$ and $p = 2$.



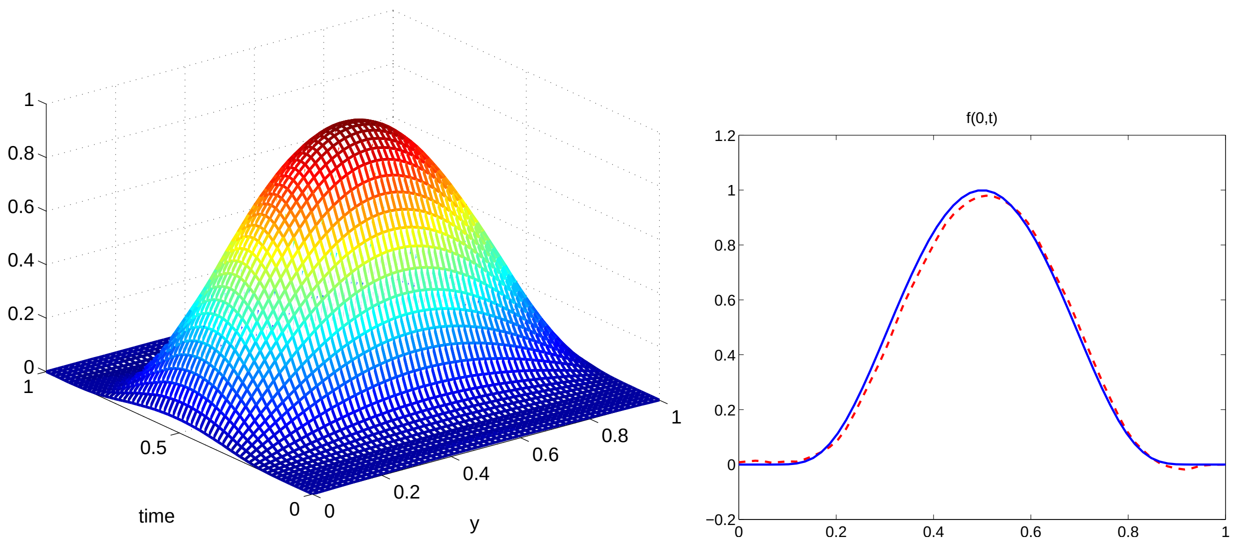
Example 2. Residuals (left) and relative errors (right) of PGMRES as functions of iteration index.

Test example 3: 2D SHE with zero boundary values in y-direction, i.e,

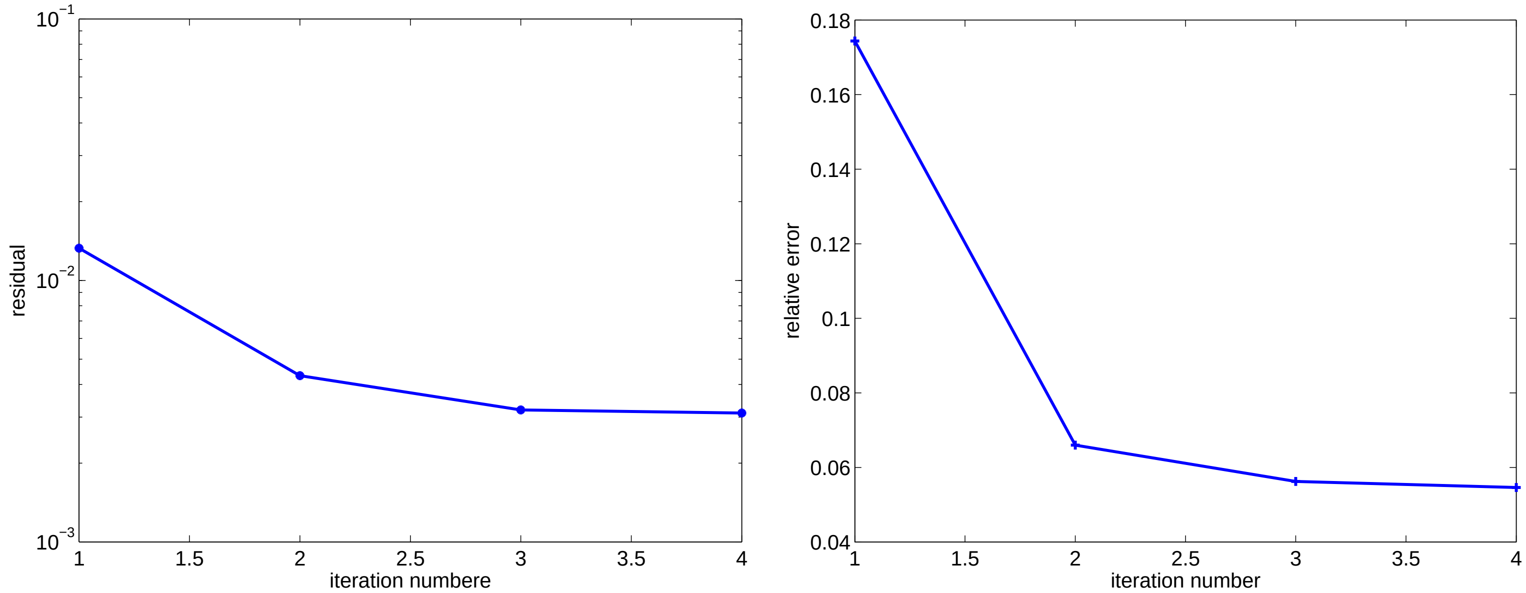
$$u(x,0,t) = u(x,1,t) = 0$$



Exact solution $f(y,t)$ (left) and data function with 5% noise(right)

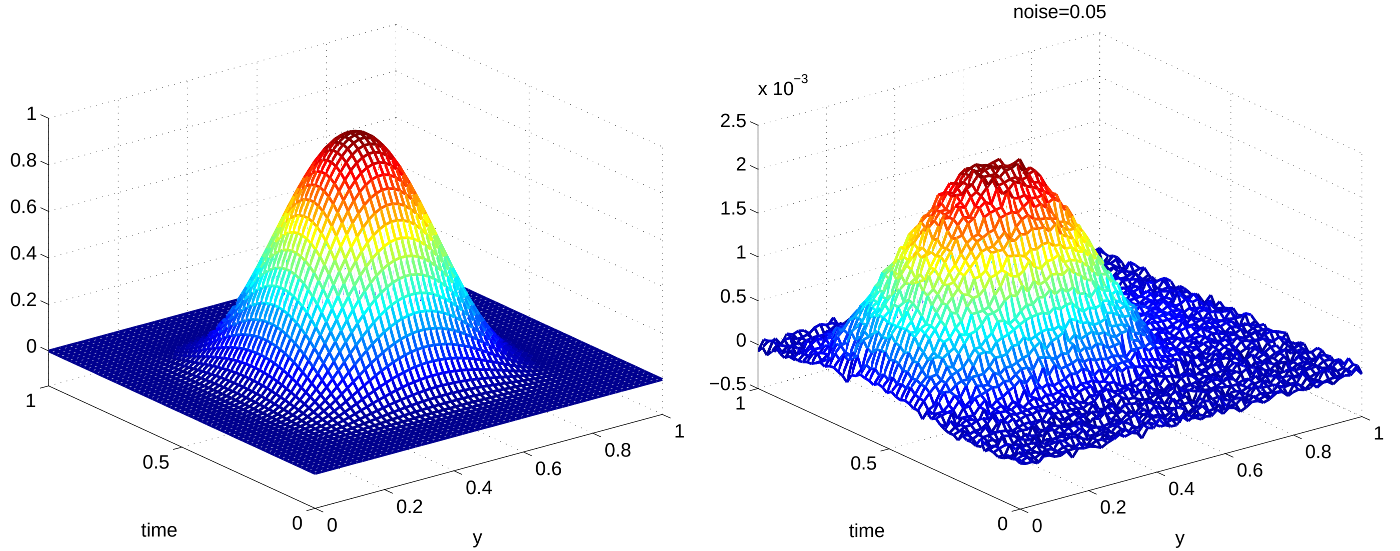


Approximate solution after 3 iterations of PGMRES for $0 \leq y \leq 1$ (left) and plot of f and the approximated solution at $y = 1/2$(right). $\lambda \in (0.05, 0.5)$ and $p = 1$

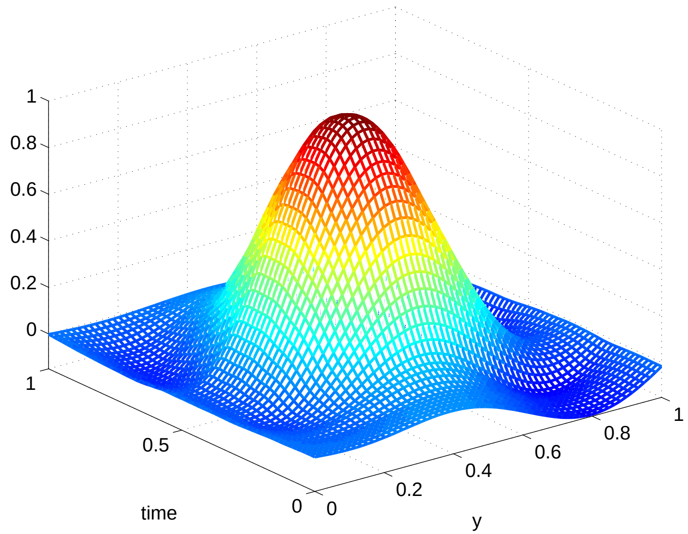


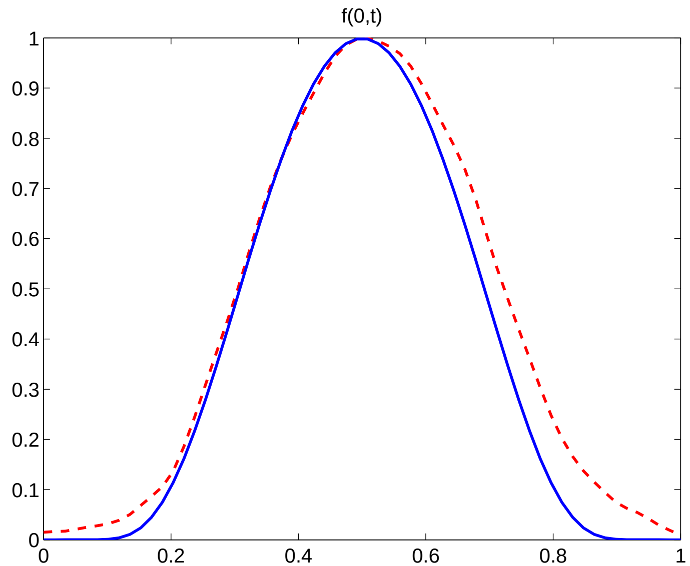
Residuals (left) and relative errors (right) of PGMRES as functions of iteration index for 5% perturbation in the data.

Test example 4:

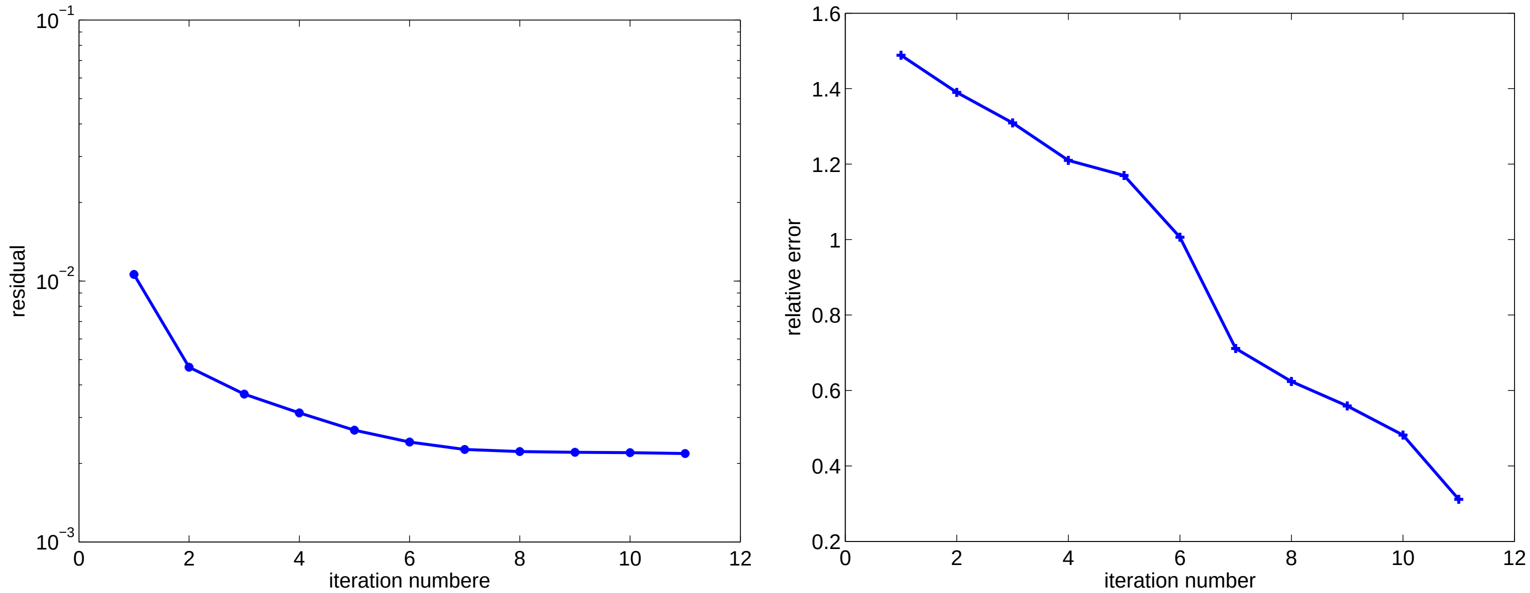


Example 4. Exact solution $f(y,t)$ and data function $g_\delta(y,t)$ with 5% noise.





Example 4. Approximated solutions(left) and the exact solution(solid) and the approximate solution at $y = 1/2$ after 11 iterations of PGMRES with 5% perturbation in the data $\lambda = 0.09$ and $p = 3$.



Example 4. Residuals (left) and relative errors (right) of PGMRES as functions of iteration index.

This problem is difficult because, due to the closeness to zero in the vicinity of 0 and 1, it is not possible to approximate the solution in terms of a small number of sine functions.

Summary and future works

- PGMRES: A new method for solving 2D sideways parabolic problem with variable coefficients which seems to be very efficient.

- Very few iterations

- Preconditioner is close to the original operator: just differ in coefficients

- Regularization is incorporated in the preconditioner

- Algorithms are computational feasible since we use Discrete Fourier Transform.

- Numerical analysis is more on experimental side and need to more study on theoretical parts

- Extension to more general form of multidimensional problems