Lavrentiev regularization for nonlinear equation with monotone operator

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Statement of problem

X is a Hilbert space
Nonlinear equation

$$F(x) = f, \tag{1}$$

where $F : D(F) \subset X \to X$ possesses a locally uniformly bounded Frechet-derivative F'
$R(F) \neq \overline{R(F)}$

Assumption 1.

Let exist such v, that the relation

$$x^0 - x^\dagger = \phi(F'(x^\dagger))v \tag{2}$$

fulfils, where index function ϕ, $\phi(0) = 0$ is continuous non-decreasing function defined on some interval $[0, \sigma]$ containing the spectrum of $F'(x^\dagger)$.

Statement of problem

Perturbation of the right-hand side:

$$f_\delta \in X : \quad \|f - f_\delta\| \leq \delta.$$

F is a monotone operator, i.e. for all $x_1, x_2 \in D(F)$

$$(F(x_1) - F(x_2), x_1 - x_2) \geq 0 \tag{3}$$

Lipschitz condition is fulfilled, i.e. there is a constant R, such that for all $x_1, x_2 \in D(F)$

$$||F(x_1) - F(x_2)|| \leq R||x_1 - x_2||. \tag{4}$$

Proposed method

Lavrentiev regulatization:

$$(F+\alpha I)x_{\alpha}^{\delta} = f_{\delta} + \alpha x^0, \quad (5)$$

where x^0 is a initial guess, α is a regularization parameter.

Fixed-point method (FPM):

$$x_{\alpha,k+1}^{\delta} = G_{\alpha}(x_{\alpha,k}^{\delta}), \quad (6)$$

where $G_{\alpha}(x) = (I - \gamma(F+\alpha I))(x) + \alpha\gamma x^0 + \gamma f_{\delta}, \gamma > 0.$

(5)+ (6) – $M_{\alpha,\gamma}$

1. A. Bakyshinskiy, A. Smirnova, A posteriori stopping rule for regularized fixed point iterations, Nonlinear Analysis, 64 (2006), pp. 1255–1261.

$$\|F(x_\alpha^\delta) - y_\delta\| \leq b_2\delta, \quad b_2 \text{ is some constant.}$$

2. U. Tautenhahn, On the method of Lavrentiev regularization for nonlinear ill-posed problems, Inverse Problems, 18 (2002), pp. 191–207.

$$\|\alpha(F'(x_\alpha^\delta) + \alpha I)^{-1}[F(x_\alpha^\delta) - y^\delta]\| = C\delta.$$

Balancing principle

S. Pereverzev, E. Schock, On the adaptive selection of the parameter in regularization of ill-posed problems, SIAM J. Numer. Anal., 43 (2005), pp. 2060–2076.

$$D_M = \{\alpha_i = \alpha_0 q^i, i = 0, 1, ..., M\}, \quad q > 1,$$

$$i_+ = \max\{i : \alpha_i \in D_M^+\}, \tag{7}$$

where

$$D_M^+ = \{\alpha_i \in D_M : \|x_i - x_j\| \leq \frac{4\delta(c_z + 1)}{\alpha_j}, j = 0, 1, ..., i-1\},$$

$x_i := x^\delta_{\alpha_i,k}$, where $x^\delta_{\alpha_i,k}$ is a approximate solution of Lavrentiev method (5) at α_i that obtained by FPM (6) after k iterations.

Goal of investigation

- to prove the optimality of constructed method $M_{\alpha,\gamma}$ with balancing principle;
- to find the quantity of necessary iterations by fixed-point method for given accuracy.

Assumptions

Let $\rho(\alpha)$ is qualification of Lavrentiev method

- If $\phi(\lambda) = \lambda^{p}$ then qualification $\rho(\alpha) := \alpha^{q}, \quad 0 < q \leq 1.$
  So if $0 < p \leq 1$ then the accuracy by the order is $O(\delta^{\frac{p}{p+1}})$.
  If $p > 1$, than of accuracy by the order is $\delta^{1/2}$.
- If $\phi(\lambda) = \ln^{-p}(\lambda)^{-1}$ then $\rho(\alpha) := \ln^{-q}(\alpha)^{-1}$, where $q \geq 1$.
  A rate of convergence is $O(\ln^{-p}(\delta^{-1}))$ for any $p > 1$.

Assumptions

Assumption 2.[1]

Let exist $æ > 0$ such that the relation

$$æ\frac{\rho(\alpha)}{\phi(\alpha)} \leq \inf_{\alpha\leq\lambda\leq\sigma}\frac{\rho(\lambda)}{\phi(\lambda)}, \quad 0 < \alpha \leq \sigma. \tag{8}$$

takes place.

Assumption 3.[2]

Let exist a constant $k_0 \geq 0$ such that for all $x \in D(F)$ and $\omega \in X$ there exists some element $k(x, x^\dagger, \omega) \in X$ with property

$$[F'(x) - F'(x^\dagger)]\omega = F'(x)k(x, x^\dagger, \omega) \quad \text{and} \quad \|k(x, x^\dagger, \omega)\| \leq k_0\|\omega\|.$$

[1] P. Mathe, S. Pereverzev, Geometry of linear ill-posed problems in variable Hilbert scales, Inverse Problems, 19 (2003)

[2] U. Tautenhahn, On the method of Lavrentiev regularization for nonlinear ill-posed problems, Inverse Problems, 18 (2002)

Bound of error

$$\|x_i - x^\dagger\| \leq \|x^\dagger - x_\alpha\| + \|x_\alpha - x_\alpha^\delta\| + \|x_\alpha^\delta - x_i\| \tag{9}$$

The stability bound on Lavrentiev method

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha} \tag{10}$$

The stopping rule for FPM

$$\|x_{\alpha_i}^\delta - x_i\| \leq \frac{c_z \delta}{\alpha_i} \tag{11}$$

Proposition 1

Let assumptions A1, A2, A3 fulfil. Then for all $\alpha > 0$ the approximation error for Lavrentiev method is

$$\|x_\alpha - x^\dagger\| \leq (1 + k_0)æ^{-1}\|v\|\phi(\alpha). \tag{12}$$

Theorem about optimality

Theorem 1.

Let operator F of equation (1) is monotone and condition (4) takes place. Assume that, solution $x^\dagger$ is such that condition (2) is true. Then at choosing index i_+ according to (7) the method $M_{\alpha,\gamma}$ is optimal by the order, i.e. the following bound fulfils

$$\|x^\dagger - x_{i_+}\| \leq \frac{c\delta}{\alpha_{opt}} = c\phi(\alpha_{opt}), \tag{13}$$

where $\alpha_{opt} = ((c_z+1)(1+k_0)æ^{-1}\phi(\delta)\delta)^{-1}$, and constant c is independent from δ.

Fixed Point Method

Theorem 2.

Let operator $F+\alpha I$ is strong monotone and Lipschitz condition takes place. Besides assume that $\gamma\alpha<1$. Then operator G_α is contractive with constant $\beta=\sqrt{(1-\alpha\gamma)^2+\gamma^2R^2}$ at $\gamma<\frac{2\alpha}{\alpha^2+R^2}$ and the following bound takes place

$$\|x_{\alpha_i}^{\delta}-x_i\|\leq\frac{\beta^k||x_{\alpha_i,1}^{\delta}-x_{\alpha_i,0}^{\delta}||}{1-\beta}. \tag{14}$$

The best value of bound (14) is achieved at $\gamma=\frac{\alpha}{\alpha^2+R^2}$ and $\beta=\frac{R}{\sqrt{\alpha^2+R^2}}$.

Corollary

The quantity of iterations :

$$N_{iter} = \log_{\beta} \frac{\varepsilon(1-\beta)}{\|x^{\delta}_{\alpha_i,1} - x^{\delta}_{\alpha_i,0}\|}, \tag{15}$$

The quantity of remaining iterations that should be checked after each step:

$$N^{n}_{iter} = \log_{\beta} \frac{\varepsilon(1-\beta)}{\|x^{\delta}_{\alpha_i,n} - x^{\delta}_{\alpha_i,n-1}\|}, \tag{16}$$

Algorithm of computation

To apply Balancing Principle two strategies are possible:

- Test the condition

$$\|x_i - x_j\| \le \frac{4\delta(c_z+1)}{\alpha_j}, j = 0, 1, \ldots, i-1, \tag{17}$$

  starting from small α_0 and going forward to $\alpha_k = \alpha_0 q^k$;
- Test the condition (17) starting from large value α_M and continuing to smaller regularization parameter $\alpha_{M-k} = \alpha_M q^{-k}$.

$$\alpha_0 = C\sqrt{\delta}, \tag{18}$$

where C is some constant.

$$M = \log_q \frac{\alpha_M}{C\sqrt{\delta}}. \tag{19}$$

[2] S. Lu, S. Pereverzev, and R. Ramlau, An analysis of Tikhonov Regularization for nonlinear ill-posed problems under a general smoothness assumption, Inverse Problems, 23 (2007)

Algorithm of computation

1. Choose sufficiently big α_M and parameter $q > 1$, $C > 0$. It allows to define the grid

$$D_M = \{\alpha_i = \alpha_0 q^i, i = 0, 1, ..., M\}.$$

2. Compute the approximation x_i

$$x^{\delta}_{\alpha,k+1} = G_{\alpha}(x^{\delta}_{\alpha,k}), \tag{20}$$

where $G_{\alpha}(x) = (I - \gamma(F + \alpha I))(x) + \alpha\gamma x^0 + \gamma f_{\delta}, \gamma > 0$. with accuracy $c_z\delta/\alpha_i$ for $i = M-1, ..., 0$.

3. For $i = M, M-1, ..., 1$ check the condition (17). As soon as for some $i = i_+$ this condition carries out, the test is stopped. As approximate solution of equation (20) take the element x_{i_+}.

Conclusion

- We have considered a nonlinear equation with monotone operator; for its solving the combination of Lavrentiev method with FPM has been proposed. The regularization parameter has been chosen according to BP;
- The optimality of this method on the set of smooth solutions has been proved;
- The FPM has been investigated for the problem under consideration; bound for the accuracy has been obtained; the necessary quantity of iterations has been established.