An inverse boundary problem for a semi-linear parabolic equation

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Statement of the problem. The inverse boundary problem is to determine the function $z(t) = u(1,t) \in L_2[0,\infty)$ (i.e. to determine the boundary condition), where $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u) \quad (0 < x < 1; t > 0) \qquad (1)$$

$$u(x,0) = 0; \quad u(0,t) = 0; u_x(0,t) = \varphi(t).$$

Here $\varphi \in L_2[0,\infty)$ is the given function, $f : L_2[0,\infty) \to L_2[0,\infty)$ is a mapping under the Lipshcitz condition

$$\|f(u_1) - f(u_2)\|_{L_2[0,\infty)} \leq L\|u_1 - u_2\|_{L_2[0,\infty)}.$$

The problem (1) is ill-posed. Assume that (1) has an exact solution that belongs to the uniform regularization class:

$$M_r = \{z(t) : z(t), z'(t) \in L_2(0,\infty);\ \|z'(t)\|_{L_2[0,\infty)} \leq r\}$$

but only a $\delta-$ approximation φ_δ and the error level $\delta > 0$ such that $\|\varphi_\delta - \varphi\| < \delta$ are known instead of the exact data $\varphi(t)$. The task is, given the input data M_r, φ_δ, δ , to find a stable approximate solution to the problem (1) and to estimate its deviation from the exact solution.

The exact solution to the linear inverse problem. Before studying the nonlinear problem (1) consider the corresponding linear problem:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad (0 < x < 1; t > 0) \qquad (2)$$

$$u(x,0) = 0; \quad u(0,t) = 0; u_x(0,t) = \varphi(t);$$

$Z = u(1,t)$ is to be determined. The problem (2) is ill-posed in $L_2(0,\infty)$. Consider the following uniform regularization set for (2):

$$\mathbf{M_R = \{Z(t) : Z(t), Z'(t) \in L_2(0,\infty);\ \|Z'(t)\|_{L_2(0,\infty)} \leq R\}}$$

Lemma 1. Let the problem (1) have the solution that belongs to M_r. Then there exists $R > 0$ such that the problem (2) has the solution that belongs to $\mathbf{M_R}$.

Using a priori estimates for the solution to the corresponding forward problem we prove that the Fourier transform on the half-line $t \in (0,\infty)$ can be applied to the problem (2).

Applying the Fourier transform with respect to t we obtain the following problem for the ordinary linear equation:

$$\widetilde{u}_{xx}(x,\lambda) = i\lambda\widetilde{u}(x,\lambda); \tag{3}$$

$$\widetilde{u}(0,\lambda) = 0 \;; \widetilde{u}_x(0,\lambda) = \widetilde{\varphi}(\lambda).$$

Solving (3) we obtain the transform of the exact solution to the problem (2):

$$\widetilde{u_0}(\lambda) = \frac{\text{sh } \mu_0\sqrt{\lambda}}{\mu_0\sqrt{\lambda}}\widetilde{\varphi}.$$

Here $\mu_0 = \frac{1+i}{\sqrt{2}}$; $\widetilde{\varphi}(\lambda) = F\varphi = \frac{1}{\sqrt{\pi}}\int\limits_0^\infty \varphi(t)e^{-i\lambda t}dt$ is the Fourier transform of the function $\varphi(t)$ ($\lambda > 0$).

Consider the Hilbert space $X = L_2[0,\infty)$ of complex-valued square integrable functions . We can reformulate the problem (2) as the problem of solving the operator equation

$$AZ = \varphi,$$

where the linear operator $A : X \to X$ has an unbounded inverse operator.

An approximate solution to the problem (2). Instead of the ill-posed problem (2) consider the following problem for the hyperbolic equation with a small parameter :

$$\varepsilon \frac{\partial^2 u(x,t,)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < 1; t > 0) \quad (4)$$

$$u(x,0) = u_t(x,0) = 0; \quad u(0,t) = 0; u_x(0,t) = \varphi_\delta(t).$$

Here the constant $\varepsilon > 0$ has a simple physical interpretation. That is we consider the function

$$Z_\delta^\varepsilon(t) = u_\delta^\varepsilon(1,t)$$

with $u_\delta^\varepsilon(x,t)$ satisfying (4) as an approximate solution to the problem (2).

Lemma 2. The problem (4) is well-posed in $L_2[0,\infty)$.

Using the a priori estimate for the solution to the first boundary problem for the hyperbolic equation we show that the Fourier transform can be applied to the problem (4). Applying the Fourier transform to (4) we obtain the following problem for the ordinary linear equation :

$$\widetilde{u}_{xx}(x,\lambda) = i\lambda\widetilde{u}(x,\lambda) - \varepsilon\lambda^2\widetilde{u}(x,\lambda);$$

$$\widetilde{u}(0,\lambda) = 0; \widetilde{u}_x(0,\lambda) = \widetilde{\varphi}(\lambda).$$

So, we consider the function

$$Z_\delta^\varepsilon = R_\varepsilon\varphi_\delta$$

with the transform

$$\widetilde{Z}_\delta^\varepsilon(\lambda) = \frac{\operatorname{sh}\sqrt{i\lambda - \varepsilon\lambda^2}}{\sqrt{i\lambda - \varepsilon\lambda^2}}\widetilde{\varphi}_\delta \tag{4'}$$

as an approximate solution to (2). Here $R_\varepsilon : X \to X$ is the regularizing operator for (2).

The error estimation for the approximate solution to (2). We obtain first the sharp error estimate for the approximate solution to the problem (2) defined by (4') on the regularization class M_R. Denote $Z^\varepsilon(\lambda) = R_\varepsilon \varphi$. We shall use the value

$$\Delta(\varepsilon,\delta) = \sup\{\|Z^\varepsilon_\delta(t) - Z(t)\| : Z \in M_R; \|\varphi - \varphi_\delta\| \le \delta\}$$

to characterize the accuracy of the approximate solution under consideration.

We apply M.M.Lavrentyev's scheme to choose the regularization parameter. We use the evident inequality

$$\Delta(\varepsilon,\delta) \le \Delta_1(\varepsilon,\delta) + \Delta_2(\varepsilon),$$

where

$$\Delta_2(\varepsilon,\delta) = \sup_{\|\varphi-\varphi_\delta\|\le\delta} \|Z^\varepsilon_\delta - Z^\varepsilon\|; \;\; \Delta_1(\delta) = \sup_{Z\in M_R} \|Z^\varepsilon - Z\|.$$

The sharp estimates of Δ_2 and Δ_1 are following:

$$\Delta_2(\varepsilon,\delta) \le \delta \sup_{\lambda>0} \left| \frac{\operatorname{sh}\sqrt{i\lambda - \varepsilon\lambda^2}}{\sqrt{i\lambda - \varepsilon\lambda^2}} \right| \le \delta e^{\frac{1}{\sqrt{2\varepsilon}}};$$

$$\Delta_1(\varepsilon) \le r \sup_{\lambda\ge 0} \frac{1}{\sqrt{1+\lambda^2}} \left| \frac{\operatorname{sh}\mu_0\sqrt{\lambda}\sqrt{1+i\varepsilon\lambda}}{\sqrt{1+i\varepsilon\lambda}\operatorname{sh}\mu_0\sqrt{\lambda}} - 1 \right| \le r\varepsilon.$$

Choosing the relation $\varepsilon = \varepsilon(\delta)$ such that

$$\varepsilon \simeq \frac{1}{\ln^2 \delta}$$

(quasi-optimal choice of the regularization parameter) we obtain the error estimate of the method under consideration on the set M_R:

$$\Delta(\varepsilon(\delta), \delta) \simeq \frac{1}{\ln^2 \delta}$$

as $\delta \to 0$.

Taking into account the error estimates of an optimal method for (4) on the set M_R obtained in [2] and reminding that the operator $F : X \to X$ is isometric we prove the following theorem.

Theorem 1. The method for approximate solution of (2) defined by (4) is order-optimal on the uniform regularization set M_R.

An approximate solution to the problem (1). Instead of the nonlinear ill-posed problem (1) consider the problem

$$\varepsilon\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+f(u) \qquad (7)$$

$u(x,0)=u_t(x,0)=0;\ u(0,t)=0;\ u_x(0,t)=\varphi_\delta(t).$

Lemma 2. The problem (7) is well-posed in $L_2[0,\infty)$.

Therefore, we consider the function $z_\delta^\varepsilon(t)=u_\delta^\varepsilon(1,t)$ with $u_\delta^\varepsilon(x,t)$ satisfying (7) as an approximate solution of (1).

The error estimation for the problem (1). We use the value

$$\Delta(\varepsilon,\delta) = \text{sup}\{\|z_\delta^\varepsilon(t) - z(t)\| : z \in M_r; \|\varphi - \varphi_\delta\| \le \delta\}$$

to characterize the accuracy of the approximate solution under consideration. We use the evident estimate

$$\Delta(\varepsilon,\delta) \le \Delta_1(\varepsilon) + \Delta_2(\varepsilon,\delta),$$

where

$$\Delta_2(\varepsilon,\delta) = \sup_{\|\varphi-\varphi_\delta\|\le\delta} \|z_\delta^\varepsilon(t) - z^\varepsilon(t)\|$$

$$\Delta_1(\varepsilon) = \sup_{z\in M_r} \|z^\varepsilon(t) - z(t)\|$$

Here $z^\varepsilon(t)$ is the solution of the quasi-inversion problem (7) with the exact data $\varphi(t)$. Taking into account Lemma 1 and the above error estimate for the linear problem (2), we obtain the sharp estimates:

$$\Delta_2(\varepsilon,\delta) \le C_1 e^{\frac{1}{\sqrt{2\varepsilon}}}\delta;$$

$$\Delta_1(\varepsilon) \le C_2\varepsilon.$$

We choose the relation $\varepsilon = \varepsilon(\delta)$ to make the order of the above estimation minimal. We obtain

$$\varepsilon(\delta) \simeq \frac{1}{\ln^2 \delta}$$

as $\delta \to 0$. Consequently, the following theorem is true.

Theorem 2. There exist the constants δ_0, C_3, C_4 such that the error estimate for the solution of the problem (1) defined by (7) satisfies inequalities

$$C_3 \frac{1}{\ln^2 \delta} \leq \Delta(\varepsilon, \delta) \leq C_4 \frac{1}{\ln^2 \delta}$$

as $\delta < \delta_0$.

References

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