ITERATIVE PROCESSES FOR NONLINEAR EQUATIONS WITH QUASI-MONOTONE OPERATOR AND ITS APPLICATIONS TO INVERSE GEOPHYSICAL PROBLEMS

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Outline of talk

1. Statement of inverse problem.

2. Pseudo-contractive operators and its properties.

3. Weak convergence theorem.

4. A priori information and strong convergence of iterations.

5. Local condition for integral operators for inverse gravimetry and magnetometry problems.

6. Other approaches.

7. Applications to inverse geophysical problems.

1. Statement of inverse problem

It is very easy to make a cake from a recipe, but can we write down the recipe if we are given a cake ?
R. Feynman

Let $A: U \to F$ be a nonlinear Frechet differentiable operator acting on a pair of Hilbert spaces and having the discontinuous inverse operator A^{-1}, therefore, the problem

$$A(u) = f \Leftrightarrow \min\{1/2\|A(u) - f\|^2 : u \in U\} \Leftrightarrow S(u) \equiv A'(u)^*(A(u) - f) = 0 \qquad (1)$$

with the perturbed right-hand side f, $\|f - f_\delta\| \le \delta$, is an essentially ill-posed one. Let the operator A satisfy the following *local condition* in the neighborhood of a solution z of problem (1) (**Scherzer,1995; Vasin, 1998**):

$$\|A(u) - A(z)\|^2 \le \kappa < S(u) - S(z), u - z > \quad \forall u \in S_\rho(z), \qquad (2)$$

or $\|A(u) - A(z)\|^2 \le \kappa < F(u) - F(z), u - z >, \quad F(u) = (A'(u)^* A(u)' + \alpha I)^{-1} S(u);$ (2a)

from (2) *quasi-uniformly monotonicity* of the operator S (*or* F) follows:

$$\|S(u) - S(z)\|^2 \le \kappa N^2 < S(u) - S(z), u - z > \quad \forall u \in S_\rho(z), \ \kappa > 0, \ \|A'(u)\| \le N.$$

We investigate the iterative processes in the form

$$u^{k+1} = u^k - \gamma \beta_k [A'(u^k)^* (A(u^k) - f)] = T(u^k), \tag{3}$$

where, in particular:

$\gamma = 1,\ \beta_k = 1$ (**Landweber method** - **LM**);

$\gamma = 1,\ \beta_k = \left\| A(u^k) - f \right\|^2 / \left\| S(u^k) \right\|$ (**Minimal error method** - **MEM**);

$\gamma = 1,\ \beta_k = \left\| S(u^k) \right\|^2 / \left\| A'(u^k)\ S(u^k) \right\|^2$ (**Steepest descent method** - **SDM**);

$\gamma = 1,\ \beta_k = (A'(u^k)^* A'(u^k) + \alpha_k I)^{-1}$ **(Levenberg-Marquardt method).**

Remark. Introducing the parameter γ allow us to obtain *pseudo-contractivity* of the step operator T for these methods, to guarantee weak convergence of iterations.

2. Pseudo-contractive operators and its properties

Definition 1. A mapping $T:U\to U$ is called M - pseudo-contractive if $M=Fix(T)\neq\varnothing$ and there exists a constant $\upsilon>0$ such that

$$\|T(u)-z\|^2\le\|u-z\|^2-\upsilon\|u-T(u)\|^2\quad\forall u\in U,\ z\in M\ .$$

Theorem 1. Let T_i be M_i - pseudo-contractive and $\cap_{i=1}^m M_i=M$. Then

$$T=T_{i_1}T_{i_2}....T_{i_m}\ ,\quad T=\sum_{i=1}^m\alpha_i T_i,\ \ \alpha_i\in(0.1),\ \sum_{i=1}^m\alpha_i=1$$

are M - pseudo-contractive operators **(Vasin, 1988)**.

Corollary 1. If T_i is pseudo-contractive operator and $Fix(T_i)=M_i=M$, then

$$T=\sum_{i=1}^m\alpha_i T_{i_1}^{n_{i_1}}T_{i_2}^{n_{i_2}}....T_{i_m}^{n_{i_m}}\ ,\ \ \alpha_i\in(0.1),\ \sum_{i=1}^m\alpha_i=1$$

is pseudo-contractive one.

3. Weak convergence theorem

Theorem 3. Let at a neighborhood $S_\rho(z)$ of a solution z the following conditions hold:

$$\|A(u)-A(z)\|^2 \le \kappa < S(u)-S(z), u-z > \quad \forall u \in S_\rho(z) \ , \qquad (4)$$

$$u^k \to \bar{u} \ (weakly), \quad S(u^k) = A'(u^k)^*(Au - f) \to 0 \Rightarrow S(\bar{u}) = 0 \Rightarrow \bar{u} = z. \qquad (*)$$

Then for **LM** as $\beta = 1, \ \gamma < 2/\kappa N^2$, **MEM** and **SDM** as $\gamma < 2/\kappa$, the step operator is pseudo-contractive, and the following properties are valid:

1) $u^k \to z$ (weakly);

2) $\|u^{k+1} - z\| < \|u^k - z\|, \ k = 0,1,\dots;$

3) $\sum_{k=0}^{\infty} \|u^{k+1} - u^k\|^2 \le \|u^0 - z\|^2 / \nu(\kappa, \beta);$

4) $lim_{k\to\infty} \|A(u) - f\| = 0.$

Theorem 4. Let instead of (4) the following condition hold:

$$\|A(u)-A(z)\|^2 \le \kappa < F(u)-F(z), u-z >, \quad F(u) = (A'(u)^* A(u)' + \alpha I)^{-1} S(u). \qquad (4a)$$

Then for L-MM with $\gamma < 2\alpha^2/\kappa N^2$ properties 1)-4) are valid.

4. A priori information and strong convergence of iterations

For nonlinear equations the condition

$$u^k \to \bar{u}\ (weakly), \quad S(u^k) = A'(u^k)^*(Au - f) \to 0 \Rightarrow S(\bar{u}) = 0$$

(which can be changed by weak closeness of operator A) is difficult checked. To overcome this problem, we suppose additionally that a solution of equation (1) belongs to a boundedly compact set which can be presented in the form **(Ivanov et al, 1978)** $Q = K + U_n$, where K is a absolutely convex compact set and U_n is a finite-dimensional space, in particular; one of these sets can be trivial.

Let us pass from the previous gradient type methods to the modified processes

$$u^{k+1} = P_Q T(u^k) \equiv V(u^k), \qquad u^0 \in S_\rho(z), \qquad (5)$$

where P_Q the metric projection onto the set Q.

Theorem 5. Let the operator A meet the local condition (4). Then the modified gradient type methods (5): **MLM** as $\beta = 1,\ \gamma < 2/\kappa N^2$, **MMEM** , **MSDM** as $\gamma < 2/\kappa$, converge strongly to a solution z.

If local condition (4a) is fulfilled, then **ML-MM** converges strongly to z.

Now we consider the modified iterative process for an approximate given right-hand side of equation (1): $\| f - f_\delta \| \le \delta$:

$$u_\delta^{k+1} = P_Q\{u_\delta^k - \gamma \beta_k [A'(u_\delta^k)^* (A(u_\delta^k) - f_\delta)]\} \qquad (6)$$

where β_k determines one of three gradient methods.

Theorem 6. Let Q be a compact set. Then iterations (6) with the stopping rule on the base of the discrepancy principle (existence of such $k(\delta)$ is assumed)

$$k(\delta): \left\| A(u_\delta^{k(\delta)}) - f_\delta \right\| \le \tau\delta < \left\| A(u_\delta^k) - f_\delta \right\|, \quad k = 0,1,\ldots,k(\delta)-1,$$

generate the regularized family of approximate solutions for problem (1), i.e.,

$$\lim_{\delta \to 0} \left\| u_\delta^{k(\delta)} - z \right\| = 0.$$

5. Local condition for integral operators

Theorem 7. Let A be n-dimensional nonlinear integral operator

$$[A(u)](x) = \int_{\Pi} K(x,x',u(x'))dx' = f(x),$$

$$\left|K'_u(x,x,z)\right| \geq K_1 > 0, \quad \left|K'_u(x,x',u) - K'_u(x,x',z)\right| \leq L\left|u-z\right| \qquad (**)$$

for $x, x' \in \Pi \subset R^n$, $u, z \in M \in U$.

Then for the operator A the local condition is fulfilled for $u \in M \cap S_\rho(z)$, $\rho < K_1 / L$.

Remark 1. This property follows from the presentation $A'(u) = R_u A'(z)$, $\left\|R_u - I\right\| \leq C\left\|u-z\right\|$ that for special one-dimensional integral operators was earlier considered **(Hanke, Neubauer, Scherzer, 1995; Gilyazov, Gol'dman, 2000).**

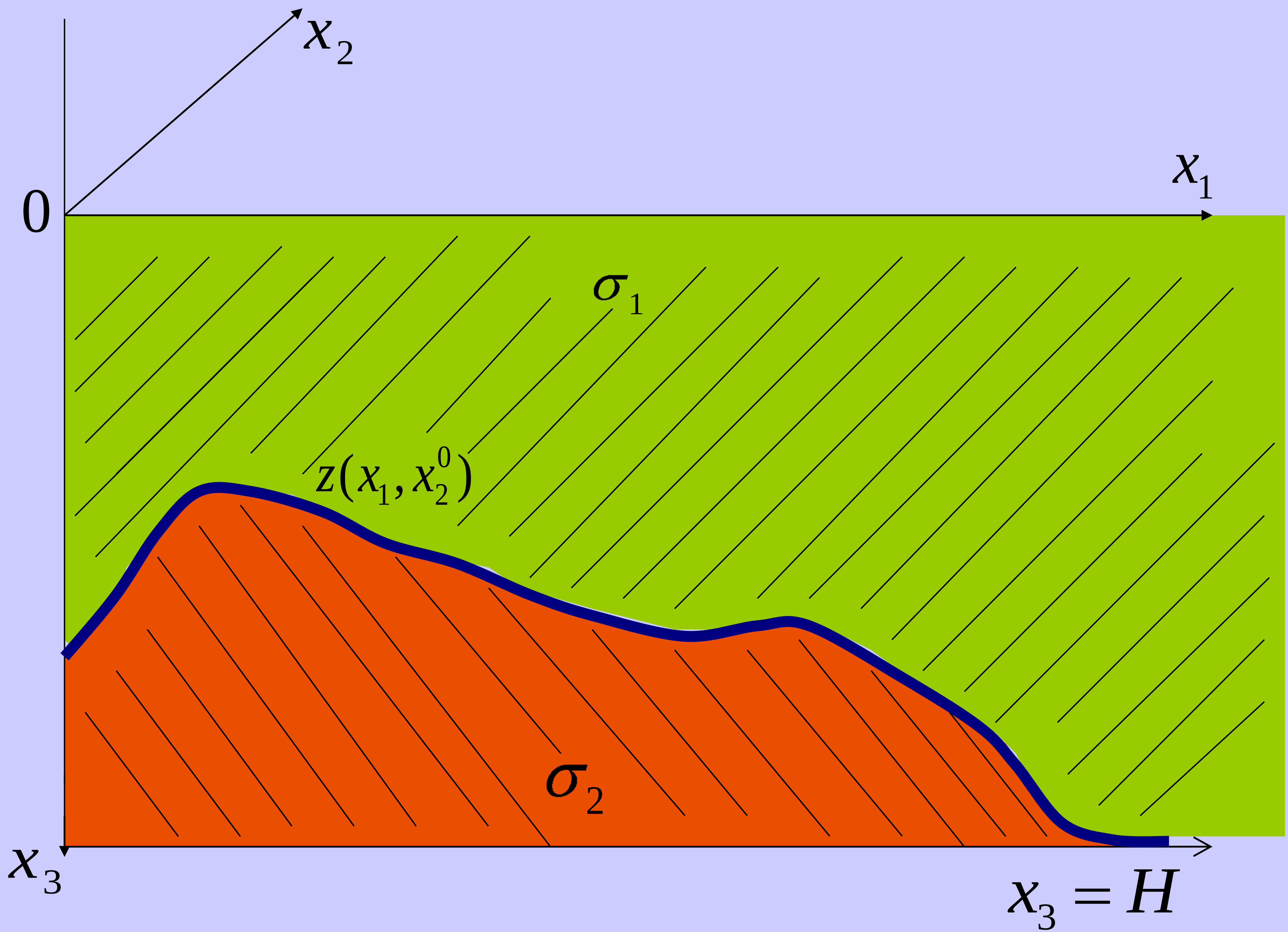
$$[A(u)](x) \equiv \gamma \Delta\sigma \{ \int_{\Pi} \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + H^2]^{1/2}} dx' - \\ - \int_{\Pi} \frac{1}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + u(x')^2]^{1/2}} dx' \} = f(x) \qquad (7)$$

$$[A(u)](x) \equiv \Delta J \{ \int_{\Pi} \frac{H}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + H^2]^{3/2}} dx' - \\ - \int_{\Pi} \frac{u(x')}{[(x_1 - x_1')^2 + (x_2 - x_2')^2 + u(x')^2]^{3/2}} dx' \} = f(x), \qquad (8)$$

where $x \in \Pi \subset R^2$, $\Delta\sigma = \sigma_1 - \sigma_2$ is a jump of density between layers of media, ΔJ is a jump of magnetization vector x_3-component.

Corollary 1. Let z be a unique solution of gravimetry or magnetometry problems (7), (8) in the set $M = \{u:\ 0 < k_0 \le u \le k_1\}$.

Then for the integral operator A from (7) or (8) in $S_\rho(z) \cap M$ $(\rho < K_1 / 2L)$ the local condition holds.



x_2
0
x_1
σ_1
$z(x_1, x_2^0)$
σ_2
x_3
$x_3 = H$

Theorem 8. Let z be a unique solution of gravimetry (6) or magnetometry problem (7) in $S_\rho(z)\cap Q\cap M$, where $M=\{u:\ 0<m_0\le u(x)\le m_1\}\ (x\in\Pi\subset R^2)$ and Q is a boundedly compact set in $L_2(\Pi)$.

Then the iterative process

$$u^{k+1}=P_Q T P_M(u^k),\qquad u^0\in S_\rho(z)\cap M\cap Q \tag{9}$$

where T is the step operator of the gradient type method and P_M, P_Q are metric projection , generates the sequence $\{u^k\}$ and $\lim_{k\to\infty}\left\|u^k-z\right\|=0$.

Remark 1. If $z\in Q=\{u:\ \|u\|_{W_2^1(\Pi)}\le r\}$, then Q is a compact set in $L_2(\Pi)$.

Remark 2. If iterative process (9) is realized after finite-dimensional approximation of integral equations, then it is not obligatory to use the projections P_Q, P_M. We can construct iteration for all methods in the form

$$u_n^{k+1}=T(u_n^{k}),\qquad u_n^{0}\in S_\rho(z_n)\cap M_n\ ,$$

because in real calculation for gravimetry and magnetometry problems the condition $u^k\in M_n=\{u_{i,j}: 0<m_0\le u_{i,j}\le m_1,\ i,j=1,2,\dots,n\}$ is usually fulfilled.

6. Other approaches

6.1. Additional Tikhonov regularization. In structural inverse magnetometry problem, outside oscillations of solution on boundary of the range Π can arise. In this case using the Tikhonov regularization allow us to obtain more smooth solution. Let us consider the Tikhonov method in the form

$$\min\{\|A(u)-f_\delta\|^2+\alpha\|u-u^0\|:\ u\in M\cap W_2^1(\Pi)\} \qquad (10)$$

where $A: W_2^1(\Pi)\to L_2(\Pi)$ is the integral operator gravimetry or magnetometry problem, $M=\{u(x_1,x_2):\ 0<m_0\le u(x_1,x_2)\le m_1\ a.e.\}$.

Theorem 9. Let z be a unique solution of equation (7) or (8) in $M\cap W_2^1(\Pi)$. Then the minimization problem (10) has a solution u^α and the strong convergence holds:

$$\lim_{\delta\to 0}\left\|u^{\alpha(\delta)}-z\right\|_{L_2(\Pi)}=0,$$

where $\alpha(\delta)$ is such a dependence that $\alpha(\delta)\to 0,\ \delta^2/\alpha(\delta)\to 0$ as $\delta\to 0$.

Remark 1. For solving (10) gradient method like the Landweber process or Fletcher-Reevs type method can be applied. Besides, for regularized problem (10) the quadrature method of approximation can be justified (**Vasin, Ageev, 2005**).

7. Applications to geophysical problems

Not everything that can be counted count, and not everything that count can be counted.

A. Einstein

For a domain D of **the Middle Urals** three-layer model of the lower half-space with two interfaces S_1, S_2 and asymptotic planes $x_3 = 2,\ x_3 = 10$ was investigated. The domain D has sizes

$$D = \{0 \le x_1 \le 55\,km,\ \ 0 \le x_2 \le 68\,km\}.$$

The gravity field was measured on the mesh with the steps

$$\Delta x_1 = 1\,km,\ \ \Delta x_2 = 2\,km.$$

After finding the anomalous gravity field for every contact surface S_1, S_2 the gravity equation for the density jumps $\Delta_1\sigma = 0.48\ g/cm^3,\ \Delta_2\sigma = 0.23\,g/cm^3$ was solved.

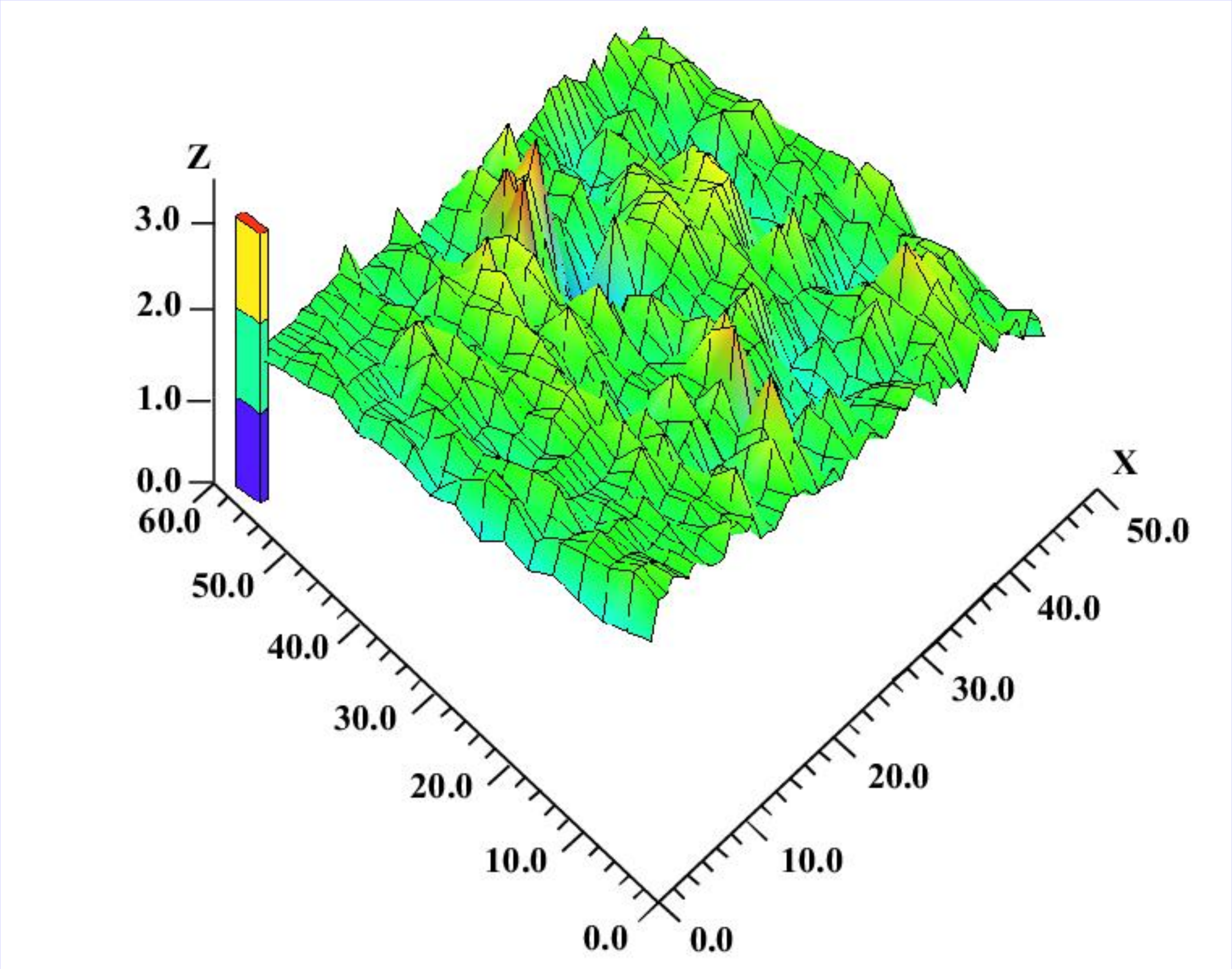


Fig.1. Interface between media S_1 (Middle Urals, H=2 km, $\Delta\sigma = 0.48$)

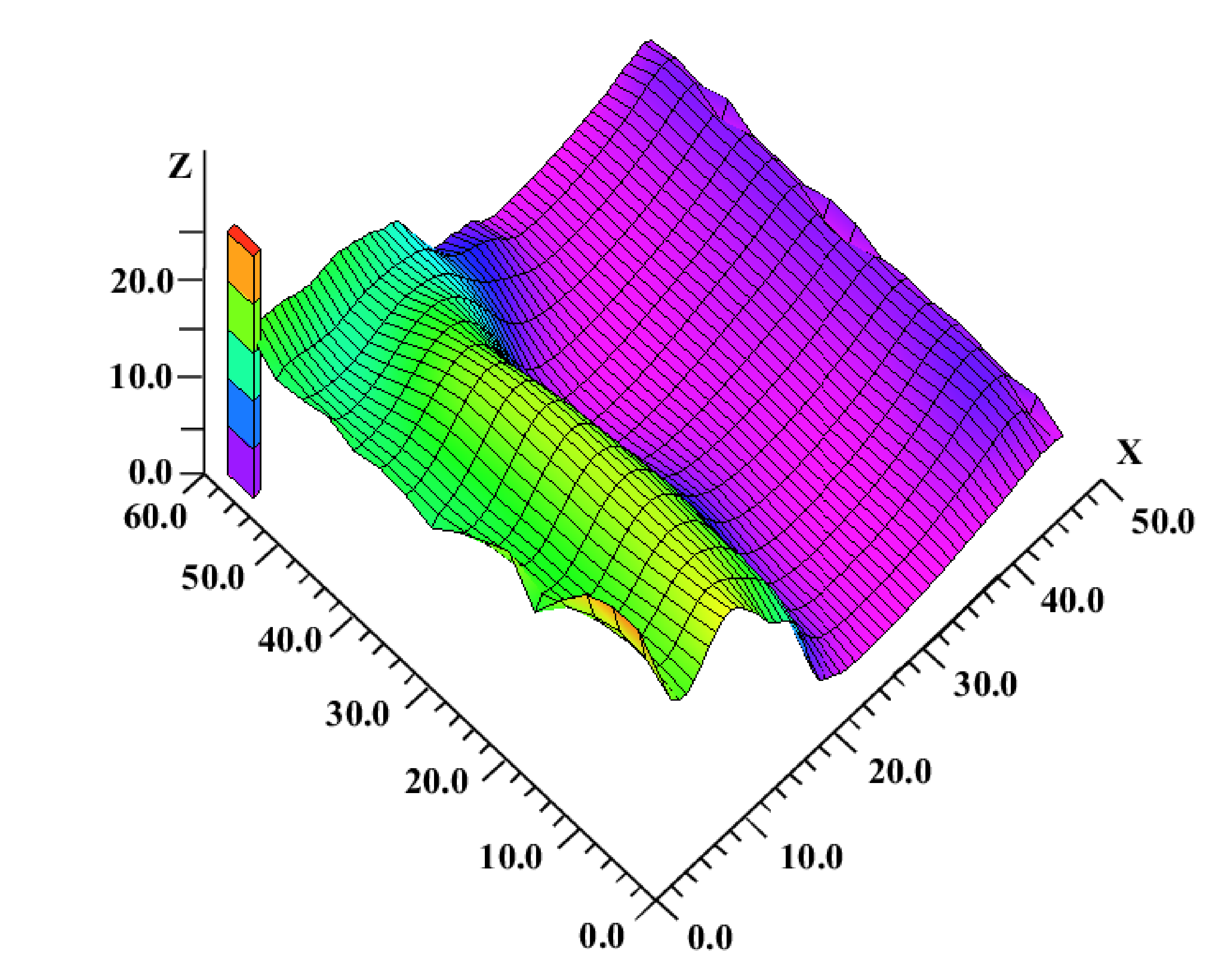


Fig. 2. Interface between media S_2 (Middle Urals, H=10 km, $\Delta\sigma = 0.23$)

Fig. 3. Interface reconstructed from real gravity data (Bashkiriya region)

Δx_1=0.66 km, Δx_2=1.1 km, $\Delta\sigma$=0.2 g/cm^3

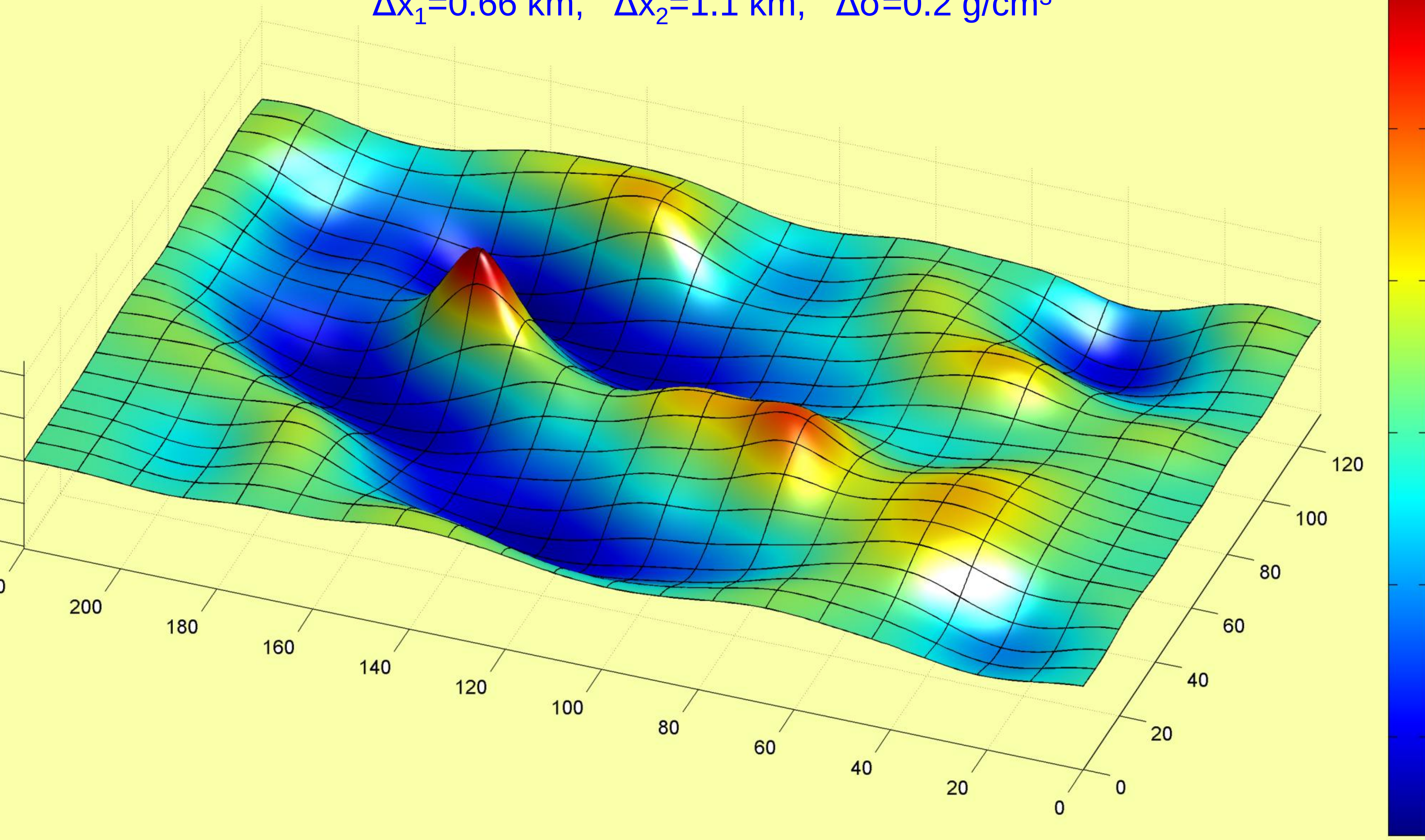


Fig. 4. Interface reconstructed from real magnetic data (North Urals region)

Δx_1=2.08 km, Δx_2=1.38 km, ΔJ= - 0.1 A/m

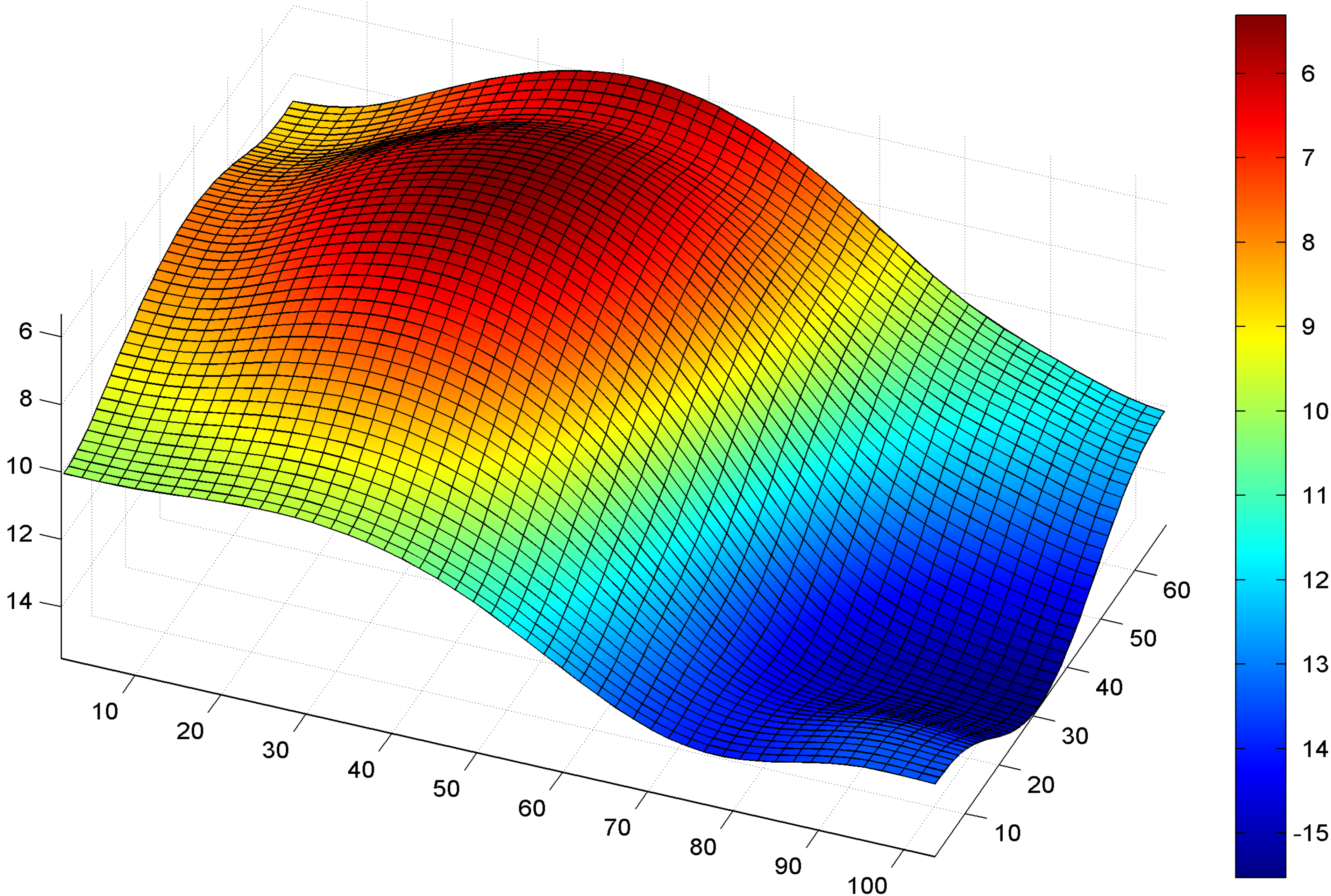


Fig. 5. Interface reconstructed from model gravity data

Δx_1=0.5 km, Δx_2=2 km, $\Delta\sigma$=0.5 g/cm^3, rel. error = 2.7%

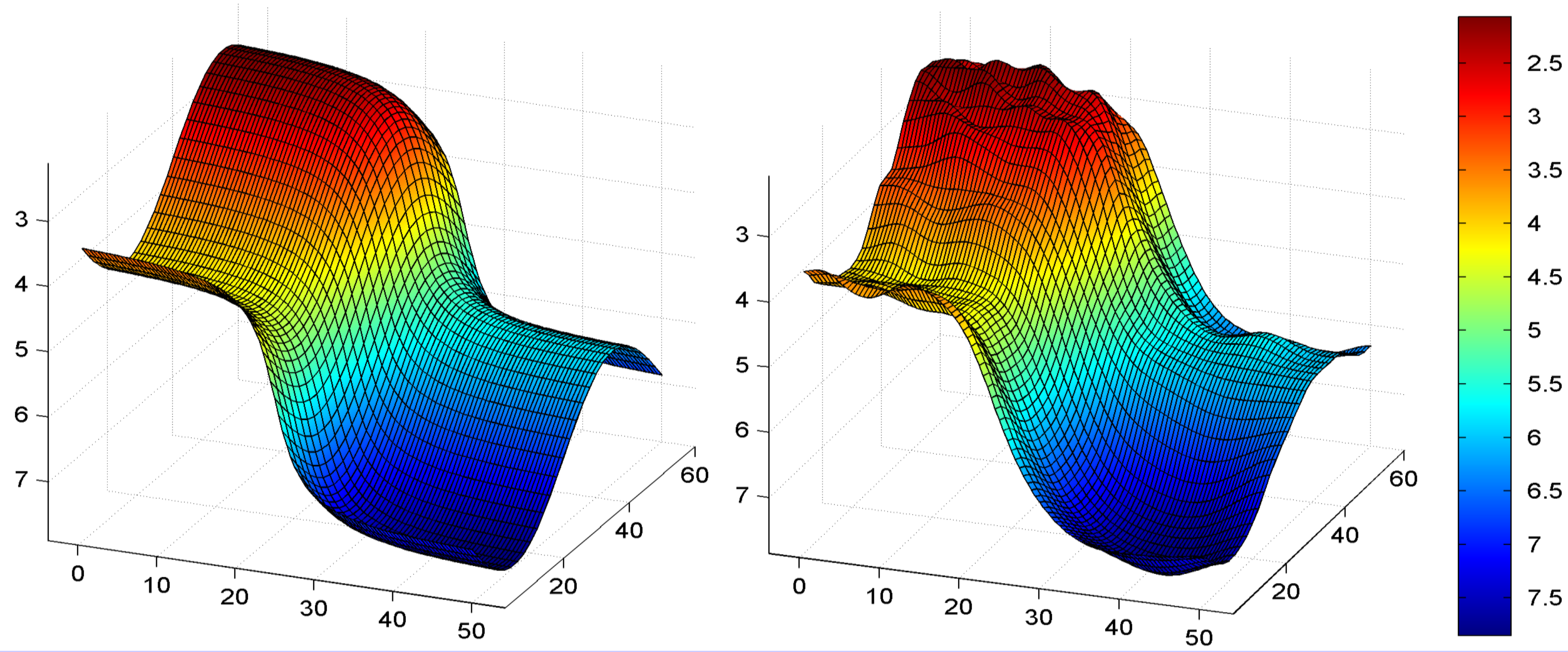
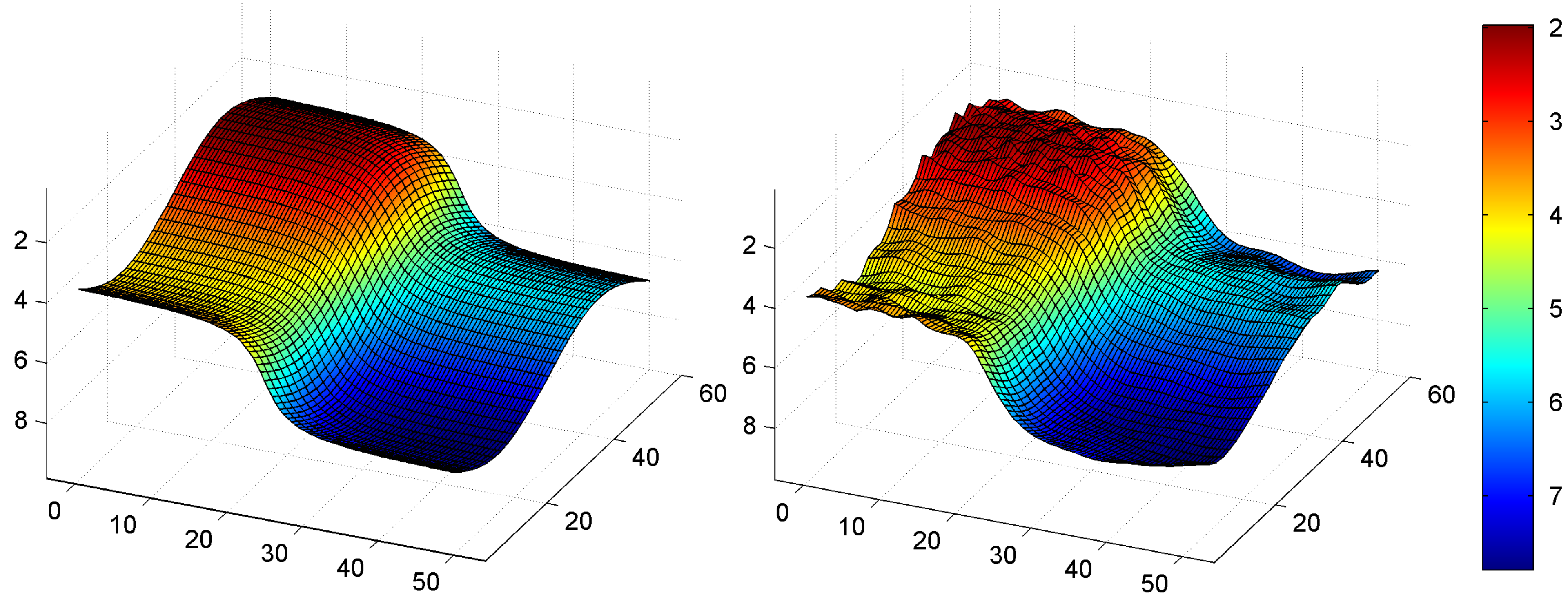


Fig. 5. Interface reconstructed from model magnetic data

Δx_1=0.5 km, Δx_2=2 km,   ΔJ=-2.5 A/m, rel. error = 2.2%



Conclusion

- Convergence for iterative methods of gradient type was investigated for nonlinear inverse gravimetry and magnetometry problems on retrieval interfaces between two media with different constant characteristics (density, magnetization vector component).

- Efficiency of these iterative methods was demonstrated for synthetic and real gravity and magnetic data.

- Numerical results for real gravity and magnetic data are in satisfactory concordance with *a priori* geophysical information getting, in particular, by seismic sounding.

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From the previous results and the work of **M.Hanke** **(Inv. Problems 13 (1997) 79-95)** the following theorem follows.

Theorem 4. Let in the Levenberg-Marquardt method (**L-MM**)

$$u^{k+1} = u^k - (A'(u^k)^* A'(u^k) + \alpha_k I)^{-1}(A'(u^k)^*(A(u^k) - f))$$

the parameter α_k be chosen by the discrepancy principle

$$\alpha_k : \left\| f - A(u^k) - F'(u^k)(u^{k+1}(\alpha_k) - u^k) \right\| = q \left\| f - F(u^k) \right\|, \quad q < 1,$$

the local condition (4), and the property

$$u^k \to \bar{u}\ (weakly),\ \ S(u^k) \to 0 \Rightarrow S(\bar{u}) = 0 \qquad (*)$$

be fulfilled. Then $u^k \to z$ (weakly) and the properties 2)-4) hold.

Remark 1. In finite-dimensional case the property (*) follows from the local condition.

6.2. Stabilizing methods of gradient type. In the approach suggested we use *a priori* information in the form $z \in Q = K + U_n$ where K is compactum and U_n is a finite-dimensional subspace. There is an another method of taking into account of *a priori* information **(Bakushinsky et al, 2007).** It is supposed that for a solution z the following estimate is known:

$$\|z - P_Q z\| \le \Delta ,$$

where Q is a finite-dimensional subspace and P_Q the operator of orthogonal projection. Here, for the same conditions

$$\|A'(u)\| \le N, \ \|A'(u) - A'(v)\| \le \|u - v\|, \ Ker\, A'(z) \cap Q = \varnothing$$

iterative process

$$u^{k+1} = P_Q\{u^k - \beta[A'(u^k)^*(A(u^k) - f_\delta)]\} \qquad (11)$$

is considered.

Theorem 10 (Bakushinsky et al, 2007). Iterative process (11) generates sequence $\{u^k\}$, for which the following relation hold:

$$u^k \to \bar{z} = \arg\min\{\|A(u) - f_\delta\| : u \in Q\}$$

$$\|z - \bar{z}\| \le C(\Delta + \delta), \quad \|f - f_\delta\| \le \delta.$$

Thus, the iterations converge to $\bar{z}$ from a neighborhood $S_r(z)$ of the solution z with radius r that is proportional to errors of approximation and input data. It should be noted that in this case a stopping rule is not necessary.

Remark. If instead estimate $\|z - P_Q z\| \le \Delta$ we have $z \in Q$, where Q is finite-dimensional subspace, then we arrive to the situation considered above; therefore, in this case gradient method (11) converges monotonically to a solution z (Theorem 3).

Theorem 2. If the local condition (2) or (2a) is fulfilled , then for appropriate the parameter γ the step operators for the gradient methods or the Levenberg-Marquardt method are pseudo-contractive (in detail, see Theorem 3).

Remark 1. The properties presented in Theorem1 and 2 provide an opportunity to implement a natural decomposition of a problem and to decompose an algorithm into some simple procedures, using parallelization technique.

Remark 2. As $M_i = M$ these properties mean that using several methods of such type, we can construct new and new hybrid iterative processes for solving equation (1).

Remark 3. In particular, due to this property and closeness of the operator S ($or\ A$), we have weak convergence of iterations (Theorem 3) and, using *a priori* information, we can modify the iterative process to obtain the strong convergence .

Discretization of problems

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