Some remarks on general similarity functionals in variational regularization

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Outline

- Introduction
- Example
- Existence, stabilty, convergence
- Convergence rates

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- (X, τ_X) and (Y, τ_Y) topological spaces, $F : D(F) \subseteq X \to Y$
- for $y^0 \in Y$ we want to solve the ill-posed equation

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- $S : Y \times Y \to [0, \infty]$ (non-metric) fitting functional
- $\Omega : X \to (-\infty, \infty]$ stabilizing functional
- assumptions on F, S, Ω that guarantee existence, stability, and convergence?
- convergence rates?
- based on: PhD thesis C. Pöschl (Innsbruck, Austria), Diploma thesis J. Geissler

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    $$\|F(x) - y^\delta\|^p \quad \rightsquigarrow \quad S(F(x), y^\delta)^{(p)},$$
  - bounding noisy data
    $$\|y^0 - y^\delta\| \le \delta \quad \rightsquigarrow \quad S(y^0, y^\delta) \le \delta,\ S(y^\delta, y^0) \le \delta,$$
  - sufficient condition for convergence rates
    $$\|F(x) - y^0\|^\kappa \quad \rightsquigarrow \quad S(F(x), y^0)^\kappa$$

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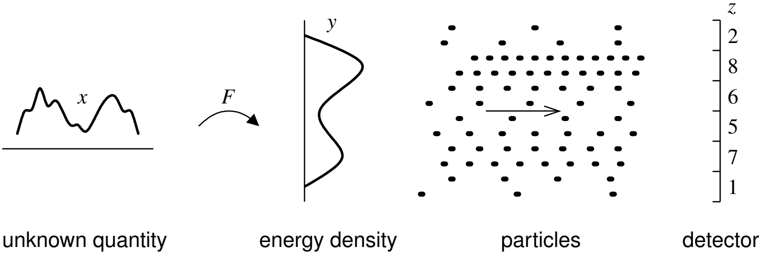
- $y^\delta \in Y$ in practice? (often discrete data)

Example: modeling



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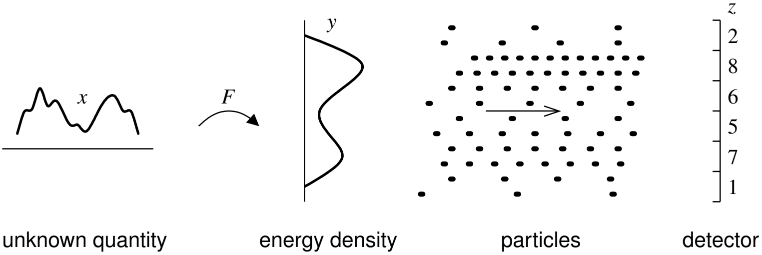

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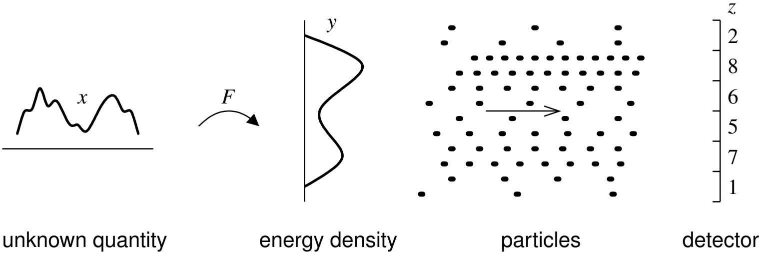

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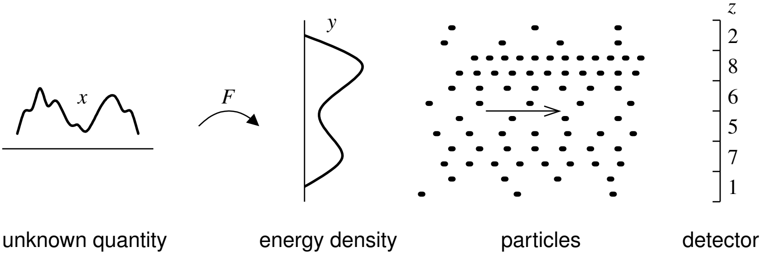

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- $T_1, \ldots, T_n \subseteq T$ pairwise disjoint: detector cells (pixels)
- MAP estimation: z Poisson distribution (n-dimensional)
- leads to

$$\sum_{i=1}^{n} \left(z_i \ln \frac{z_i}{\int_{T_i} F(x)\,\mathrm{d}\mu} + \int_{T_i} F(x)\,\mathrm{d}\mu - z_i \right) + \alpha\Omega(x) \to \min_{x \in D(F)}$$

Example: definition of the fitting functional

- auxiliary function $g : [0,\infty) \times [0,\infty) \to (-\infty,\infty]$,

$$g(u,v) := \begin{cases} v \ln \frac{v}{u} + u - v, & u, v \in (0,\infty), \\ u, & u \in (0,\infty),\ v = 0, \\ \infty, & u = 0,\ v \in (0,\infty), \\ 0, & u = v = 0 \end{cases}$$

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- fitting functional $S : Y \times Z \to [0,\infty]$,

$$S(y,z) := \begin{cases} \sum_{i=1}^n g\left(\int_{T_i} y \,\mathrm{d}\mu, z_i\right), & y \in Y_a^b, \\ \infty, & \text{else} \end{cases}$$

Basic assumptions

- (X, τ_X), (Y, τ_Y), (Z, τ_Z) topological spaces; consider

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- assumptions on $\Omega : X \to (-\infty, \infty]$:
  - Ω sequentially lower semi-continuous w.r.t. τ_X
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- assumptions on $S : Y \times Z \to [0, \infty]$:
  - S sequentially lower semi-continuous w.r.t. $\tau_Y \times \tau_Z$
  - if $S(y, z_k) \to 0$, then there is some $z \in Z$ with $z_k \rightharpoonup z$
  - if $z_k \rightharpoonup z$ and $S(y, z) < \infty$, then $S(y, z_k) \to S(y, z)$

Remarks, definitions, propositions

- in case $Y = Z$: $S(y_1, y_2) = 0 \Leftrightarrow y_1 = y_2$
- now $Y \neq Z$:

  $y_1 \in Y$ and $y_2 \in Y$ are called S-equivalent, if there is some $z \in Z$ with $S(y_1, z) = 0 = S(y_2, z)$.

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- S from example with $a > 0$ satifies all assumptions

Existence and stability

- For all $z \in Z$ and all $\alpha > 0$ the minimization problem

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- Let $z \in Z$ and $\alpha > 0$ be fixed and let $(z_k)_{k\in\mathbb{N}}$ be a sequence in Z satisfying $z_k \rightharpoonup z$. Further, assume that there is some $\bar{x} \in D(F)$ with $S(F(\bar{x}), z) < \infty$ and $\Omega(\bar{x}) < \infty$.

  Then each sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in \operatorname{argmin}_{x\in D(F)} T_\alpha^{z_k}(x)$ has a τ_X-convergent subsequence and each limit $\tilde{x}$ of such a subsequence minimizes T_α^z.

Convergence

- Let $y \in Y$, let $(z_k)_{k\in\mathbb{N}}$ be a sequence in Z satifying $S(y, z_k) \to 0$, let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence in $(0, \infty)$ with $\alpha_k \to 0$ and $\frac{S(y,z_k)}{\alpha_k} \leq c$ for some $c > 0$ and sufficiently large k, and assume that there exists some $\bar{x} \in D(F)$ with $F(\bar{x}) = y$ and $\Omega(\bar{x}) < \infty$.

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- Then each sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in \operatorname{argmin}_{x\in D(F)} T_{\alpha_k}^{z_k}(x)$ has a τ_X-convergent subsequence and each limit $\tilde{x}$ of such a convergent subsequence $(x_{k_l})_{l\in\mathbb{N}}$ is S-equivalent to a solution of $F(x) = y$ (i.e. $F(\tilde{x})$ is S-equivalent to y).

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- Then each sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in \operatorname{argmin}_{x\in D(F)} T^{z_k}_{\alpha_k}(x)$ has a τ_X-convergent subsequence and each limit $\tilde{x}$ of such a convergent subsequence $(x_{k_l})_{l\in\mathbb{N}}$ is S-equivalent to a solution of $F(x) = y$ (i.e. $F(\tilde{x})$ is S-equivalent to y).
- If we have $\frac{S(y,z_k)}{\alpha_k} \to 0$, then $\Omega(\tilde{x}) \leq \Omega(x^*)$ for each limit $\tilde{x}$ and all solutions x^* of $F(x) = y$.

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- If $\tilde{x}$ is a solution of $F(x) = y$, then $\tilde{x}$ is an Ω-minimizing solution and $\Omega(x_{k_l}) \to \Omega(\tilde{x})$.

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  - examples: Bregman distance, norm

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- There exists some $\beta > 0$, a (sufficiently large) set $M \subseteq D(F)$ and a function $\varphi : [0, \infty) \to [0, \infty)$, such that

$$E_{x^\dagger}(x) \leq \beta(\Omega(x) - \Omega(x^\dagger)) + \varphi\big(S_{y^0}(F(x))\big)$$

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- this form of a variational inequality is equivalent to the form known from the Banach space setting with $E_{x^\dagger} =$ Bregman

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- applies to the S-example with $E_{x^\dagger}$ = Bregman and $S_{y^0} = \|y^0 - \bullet\|_{L^1(T,\mu)}^2$ for the distance in Y

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- complete removal of the odds and ends of the norm case
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- practically relevant example for non-metric similarity measures, which fits completely into the theoretic results

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