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Trefftz method in solving the inverse problems

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Trefftz method has been known since the work of Trefftz in 1926*.

An approximate solution of a problem is a linear combination of functions that satisfy the governing differential linear equation or such one that is possible to be converted into such a form.

The unknown coefficients are determined from the conditions of approximate fulfilling the boundary and initial conditions, finally having a form of a system of algebraic equations.

* **E. Trefftz, *Ein Gegenstuek zum Ritz'schen Verfahren*. Proceedings of the 2nd International Congress of Applied Mechanics, 131–137, Zurich, 1926.**

Generally, Trefftz bases fall into two broad classes, F-Trefftz bases based on fundamental solutions and T-Trefftz bases, which are usually (but not always) obtained by separation of variables in polar and Cartesian coordinate systems.

In 2000 Ciałkowski presented two other methods, one of which, based on developing function in Taylor series, is particularly simple and effective. We will focus our attention on T-Trefftz bases.

In eighties and nineties T-complete functions have been used to find approximate solutions of BVP, also with the use of FEM. However, no IBVP were investigated. Researchers used to get rid of time variable, and as a result they considered the Helmholtz type equation. Therefore the first T-complete functions were found for Laplace equation, biharmonic one and for reduced wave equation (Helmholtz type eq.)

M.J. Ciałkowski, A. Frąckowiak, ***Heat functions and their application to solving heat conduction and mechanical problems.*** **Wyd. Politechniki Poznańskiej, Poznań, 2000. (in Polish)**

During the last 10 years Trefftz method has been applied to find approximate solutions of the inverse problems.

Up to now the following inverse problems have been considered:
- boundary value determination inverse problems,
- material properties determination inverse problems,
- sources determination inverse problems.

At first a "global" approach (i.e. looking for an approximate solution in the whole domain) was applied with good results for simple geometry and initial-boundary conditions. However, a great majority of more complex problems of mathematical modeling cannot be solved without division of the area Ω into subregions (elements). FEM with Trefftz functions as trial functions is then used.

Trefftz functions for some linear differential equations without time*

Laplace equation: $$\nabla^2 u = 0$$

2D in Ω $$\left\{1,\ \mathrm{Re}\left(z^n\right),\ \mathrm{Im}\left(z^n\right)\right\}$$

3D in Ω $$\left\{ r^n P_n^q\left(\cos\theta\right)e^{iq\phi};\ n = 0,1,2,\ldots,\ -n \le q \le n \right\}$$

$P_n^q\left(\cos\theta\right)$ **- associated Legendre functions**

Reduced wave equation: $$\nabla^2 u + u = 0$$

2D in Ω $$\left\{ J_0(r), J_n(r)\cos(n\theta), J_n(r)\sin(n\theta),\ n = 1,2,\ldots \right\}$$

3D in Ω $$\left\{ j_n(r)P_n^q\left(\cos\theta\right)e^{iq\phi};\ n = 1,2,\ldots, -n \le q \le n \right\}$$

$$j_n(r) = \sqrt{\frac{\pi}{2r}}J_{n+1/2}(r)$$

*A.P. Zieliński, *On trial functions applied in the generalized Trefftz method.* Adv. in Eng. Software, 24, 147-155, 1995.

Trefftz functions for some linear differential equations with time

Heat conduction equation: $\nabla^2 u = \dfrac{\partial u}{\partial t}$

1D in Ω $\left\{ v_n(x,t) = \sum_{k=0}^{[n/2]} \dfrac{x^{n-2k} t^k}{(n-2k)!k!}, \; n = 0,1,2\ldots \right\}$

2D in Ω $\left\{ V_m(x,y,t) = v_{n-k}(x,t) v_k(y,t), \; n = 0,1,\ldots, \; k = 0,\ldots,n, \quad m = \dfrac{n(n+1)}{2} + k \right\}$

Beam vibration equation * **:** $\dfrac{\partial^4 u}{\partial x^4} + \dfrac{\partial^2 u}{\partial t^2} = 0$

$S_0 = 1$

$S_1 = x$

$S_2 = t$

and for $n \geq 3$

$$S_n(x,t) = \sum_{j=0}^{\left[\frac{n-3\left[\frac{n-2}{4}\right]-3}{2} \right]} \frac{(-1)^j t^{n-3\left[\frac{n-2}{4}\right]-3-2j} x^{4\left[\frac{n-2}{4}\right]-n+5+4j}}{\left(n-3\left[\dfrac{n-2}{4}\right]-3-2j\right)!\left(4\left[\dfrac{n-2}{4}\right]-n+5+4j\right)!}$$

M.J. Al-Khatib, K. Grysa, A. Maciąg, *The method of solving polynomials in the beam vibration problems.* J. Theoret. Appl. Mech., 46, 2, 347-366, 2008.

Trefftz Methods

Trefftz Methods have not received a precise definition, although this terminology has had wide acceptance. Herrera's definition of what is meant by a Trefftz Method is:

Given a region of an Euclidean space or some partitions of that region, a „Trefftz Method" is any procedure for solving initial boundary value problems of partial differential equations or systems of such equations, on such region, using solutions of that differential equation or its adjoint, defined in its subregions.

When Trefftz Method is conceptualized in this manner, it includes many of the basic problems considered in numerical methods for partial differential equations and becomes a fundamental concept of that subject.

I. Herrera, *Trefftz method: A general theory.* Numer. Meth. Partial Diff. Eq. 16, 561-580, 2000.

Indirect Trefftz Methods

Consider a linear diff eq Lu=0 in Ω with values of $u=u_1$ prescribed on Γ_1 and $\partial u/\partial n=q_2$ prescribed on Γ_2 . Γ_1 and Γ_2 are parts of $\partial\Omega$ or are included in Ω .

$$u(P)\cong\tilde{u}(P)=\sum_{n=1}^{N}a_n u_n^* = \mathbf{a}^{\mathrm{T}}\mathbf{u}^*(P)$$

$$q(P)\cong\tilde{q}(P)=\frac{\partial\tilde{u}}{\partial n}(P)=\mathbf{a}^{\mathrm{T}}\mathbf{q}^*(P)$$

$\left\{u_n^*\right\}$ **- T-complete functions**

Residuals: $R_1\equiv\mathbf{a}^{\mathrm{T}}\mathbf{u}^*(P)-u_1(P)\neq 0$ for $P\in\Gamma_1$

$R_2\equiv\mathbf{a}^{\mathrm{T}}\mathbf{q}^*(P)-q_2(P)\neq 0$ for $P\in\Gamma_2$

1. Collocation method: residuals at the points P_i placed on Γ_1 and Γ_2 are forced to vanish $\Rightarrow$ Ka=f $\Rightarrow$ a $\Rightarrow$ $\tilde{u}(P)$

2. Least-square-method: $\Rightarrow F(\mathbf{a})=\int_{\Gamma_1}R_1^2 d\Gamma+\alpha\int_{\Gamma_2}R_2^2 d\Gamma\rightarrow\min\Rightarrow$ **Ka=f $\Rightarrow$**

$\Rightarrow$ a $\Rightarrow$ $\tilde{u}(P)$

α **- weighting parameter**

Indirect Trefftz Methods

Consider a linear diff eq Lu=0 in Ω with values of $u=u_1$ prescribed on Γ_1 and $\partial u/\partial n=q_2$ prescribed on Γ_2 . Γ_1 and Γ_2 are parts of $\partial\Omega$ or are included in Ω .

3. Galerkin method formulation: $F(\mathbf{a}) = \int\limits_{\Gamma_1} \tilde{q} R_1 d\Gamma - \int\limits_{\Gamma_2} \tilde{u} R_2 d\Gamma = 0$

$\tilde{q}$ **and** $-\tilde{u}$ **- weighting parameter**

$\Rightarrow$ **Ka=f** $\Rightarrow$ $\tilde{u}(P)$

4. Modified Trefftz formulation (T-Trefftz approach)

In this case the T-complete functions are built using fundamental solution for the eq Lu=0. The f.s. is a function of $r(P,Q_i)$ with P being an integral point of Ω and Q_i standing for external source placed on the imaginary boundary surrounding the (real) boundary. Then

$$u(P) \cong \tilde{u}(P) = \sum_{n=1}^{N} a_n u^*(P, Q_i) = \mathbf{a}^{\mathrm{T}} \mathbf{u}^*(P) \quad \textbf{and} \quad q(P) \cong \tilde{q}(P) = \frac{\partial \tilde{u}}{\partial n}(P) = \mathbf{a}^{\mathrm{T}} \mathbf{q}^*(P)$$

Direct Trefftz Methods

The direct Trefftz formulation is based on the boundary integral equation. Details for the Laplace eq: in the article
E. Kita, N. Kamiya, *Trefftz method: an overview.* **Adv. in Eng. Software, 24, 3-12, 1995.**

„Global" approach, BVP

Consider a stationary heat conduction problem in a hollow cylinder:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0, \quad r \in (a,b) \subset R,\ z \in (0,H) \subset R$$

$$T(a,z) = C \sin \frac{z\pi}{H}$$

$$T(b,z) = T_b$$

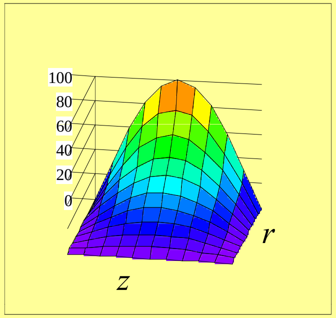
$$T(r,0) = T_b$$

$$T(r,H) = T_b$$

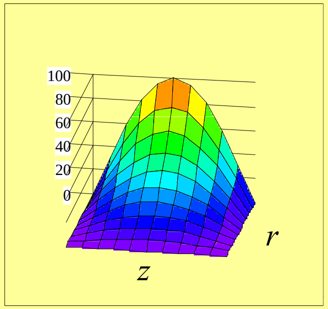
$$T(r,z) \cong \theta(r,z) = \sum_{n=0}^{N} c_n h_n(r,z) + \sum_{m=0}^{M} d_m w_m(r,z)$$

Exact solution:

$$T(r,z) = C \sin \frac{z\pi}{H} \frac{F_0\left(\frac{r\pi}{H}; \frac{b\pi}{H}\right)}{F_0\left(\frac{a\pi}{H}; \frac{b\pi}{H}\right)} \qquad F_0(x;y) = I_0(x)K_0(y) - K_0(x)I_0(y)$$



Exact



Approx

a = 0.1 m, b =0.2 m , H = 0.2 m

Approximate solution has been obtained with 8 1st kind h_n and 10 2nd kind w_n T-functions

$$h_n(r,z) = \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(k!)^2 (n-2k)!} \left(\frac{r}{2}\right)^{2k} z^{n-2k}$$

$$w_n(r,z) = h_n(r,z) \ln r - \sum_{k=0}^{[n/2]} \frac{(-1)^k a_k}{(k!)^2 (n-2k)!} \left(\frac{r}{2}\right)^{2k} z^{n-2k}$$

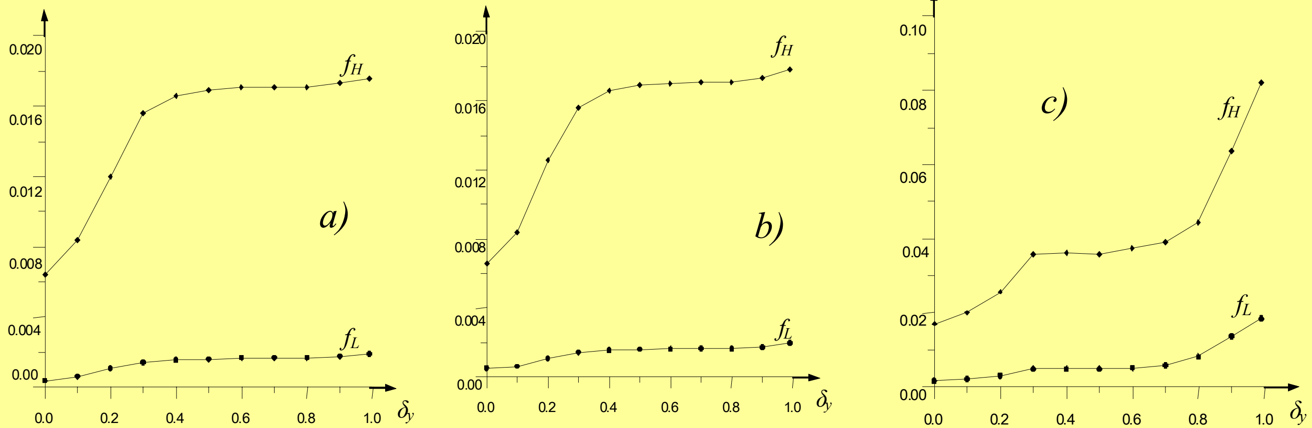
A square, 4 elements, inverse BVP

On three sides of the square the Dirichlet boundary conditions are prescribed in accordance with the accurate solution*:

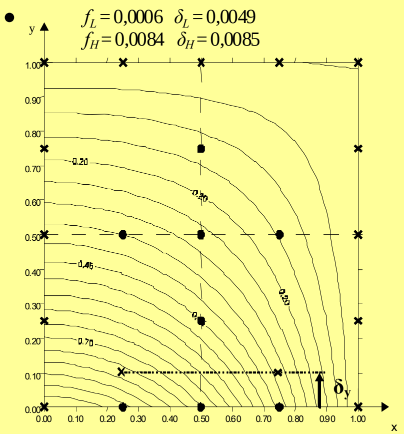
$$T(x,y)=\frac{\cos\left(\frac{\pi x}{2}\right)\sinh\left(\frac{\pi(1-y)}{2}\right)}{\sinh\left(\frac{\pi}{2}\right)}$$

Using norms L² and H¹ $f_H=\left\|T-\tilde{T}\right\|_{H^1}$ $f_L=\left\|T-\tilde{T}\right\|_{L^2}$ $\delta_L=\frac{\left\|T-\tilde{T}\right\|_{L^2}}{\left\|T\right\|_{L^2}}$ $\delta_H=\frac{\left\|T-\tilde{T}\right\|_{H^1}}{\left\|T\right\|_{H^1}}$

the approximate $\tilde{T}$ **and exact** T **solutions are compared.**



Distance from the boundary with unknown condition δ_y versus norms f_H and f_L for a) 3 b) 2 and c) 1 points with measured temperature



Approximate solution for 2 points with measured temperature

* M.J. Ciałkowski, A. Frąckowiak, *Heat functions and their application to solving heat conduction and mechanical problems.* Wyd. Politechniki Poznańskiej, Poznań, 2000. (in Polish)

FEMT for the IHCP:

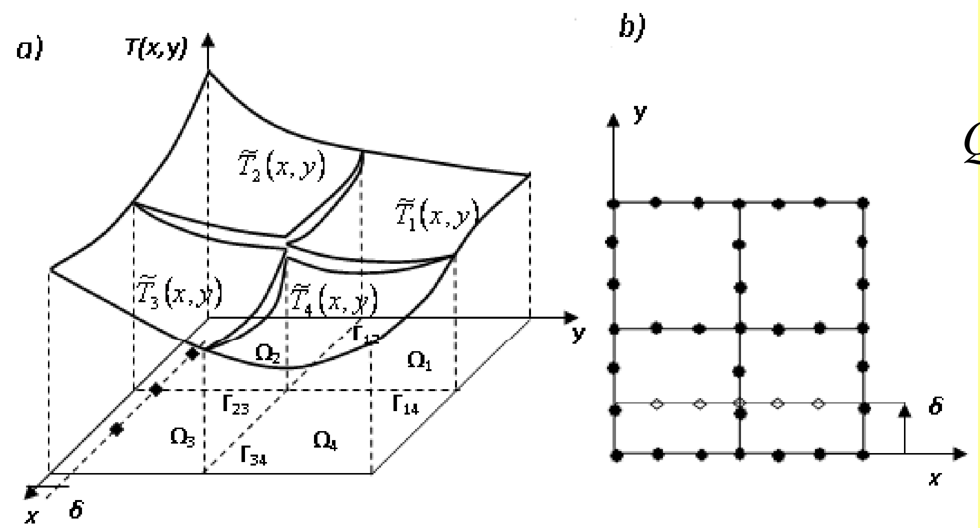
1° FEMT with condition of continuity of temperature in the common nodes of elements.

2° No temperature continuity at any point between elements.

3° Nodeless FEMT. Instead, in each finite element the temperature is approximated with the linear combination of the Trefftz functions. The unknown coefficients of the combination are calculated from the condition of minimizing the functional that describes the mean-square fitting of the approximated temperature field in an element to the boundary and initial conditions.

Moreover, the energetic regularisation is used to improve the approximate solution, i.e. one minimizes defect of energy dissipation between elements or numerical entropy production, (being a result of discontinuity of heat flux between elements).

A square, 4 elements, temperature discontinuous between elements



Objective functional:

$$Q(T)=\sum_{ij}\left[\int_{\Gamma_{ij}}(q_{n+}-q_{n-})^2\,d\Gamma_{ij}+\alpha\int_{\Gamma_{ij}}(T_+-T_-)^2\,d\Gamma_{ij}\right]$$

Accuracy of the approximate solution $\tilde{T}$

$$\delta\tilde{T}=\left(\frac{\int_\Omega\left[\left(\tilde{T}-T_e\right)^2+\left(\frac{\partial\tilde{T}}{\partial x}-\frac{\partial T_e}{\partial x}\right)^2+\left(\frac{\partial\tilde{T}}{\partial y}-\frac{\partial T_e}{\partial y}\right)^2\right]d\Omega}{\int_\Omega\left[T_e^2+\left(\frac{\partial T_e}{\partial x}\right)^2+\left(\frac{\partial T_e}{\partial y}\right)^2\right]d\Omega}\right)^{1/2}\cdot 100[\%]$$

$$T(r,z)\cong\tilde{T}(r,z)=\sum_{n=0}^{N}a_nV_n(x,y)$$

δ	FEMT continuous			FEMT discontinuous		
	α=0.001	α=1	α=1000	α=0.001	α=1	α=1000
0.00	0.0042	0.0042	0.0042	0.0207	0.0141	0.0161
0.10	0.0046	0.0046	0.0047	0.0852	0.0153	0.0168
0.30	0.0184	0.0184	0.0178	0.1805	0.0376	0.0333
0.50	0.3471	0.3471	0.3471	0.1856	0.0383	0.0523
0.70	0.1655	0.1656	0.3076	0.3954	0.1711	0.3990
0.90	0.3426	0.3425	0.3153	0.2029	0.0753	0.3492
0.99	0.2097	0.2101	0.0958	0.3669	0.1173	0.7800

Norm $\delta\tilde{T}$ as a function of distance δ for 12 trial functions. Input data are accurate.

The energetic regularisation for IHCP*

In order to find an approximate solution one minimizes *a heat flux jump* or *defect of energy dissipation* or *defect of numerical entropy production* between elements. The terms in the functional are minimized to find coefficients in the formula describing $\tilde{T}_i$. They read:

the heat flux jump

$$\sum_{i,j}\int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\tilde{\dot{q}}_{ni} - \tilde{\dot{q}}_{nj}\right)^2 d\Gamma \qquad \tilde{\dot{q}} = \frac{\partial \tilde{T}}{\partial n}$$

the defect of energy dissipation

$$\sum_{i,j}\int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{\tilde{\dot{q}}_{ni}}{\tilde{T}_i} - \frac{\tilde{\dot{q}}_{nj}}{\tilde{T}_j}\right)^2 d\Gamma$$

the defect of numerical entropy production

$$\sum_{i,j}\int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\tilde{\dot{q}}_{ni} \ln \tilde{T}_i - \tilde{\dot{q}}_{nj} \ln \tilde{T}_j\right)^2 d\Gamma$$

*M.J.Ciałkowski, A. Frąckowiak, K. Grysa, *Solution of a stationary inverse heat conduction problem by means of Trefftz non-continuous mthod.* Int. J. Heat Mass Transfer, 50, 2170-2181, 2007.

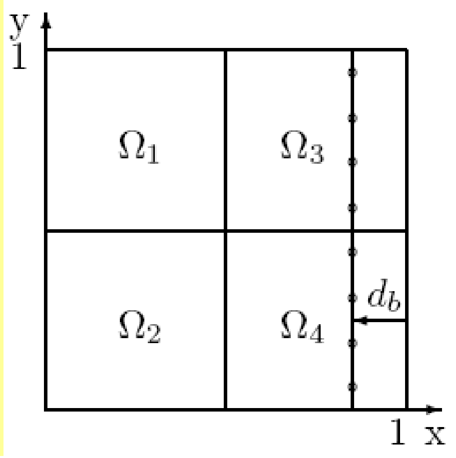
A square, three FEMT approaches, the energetic regularisation

An IHCP in a square is considered*. Conditions:

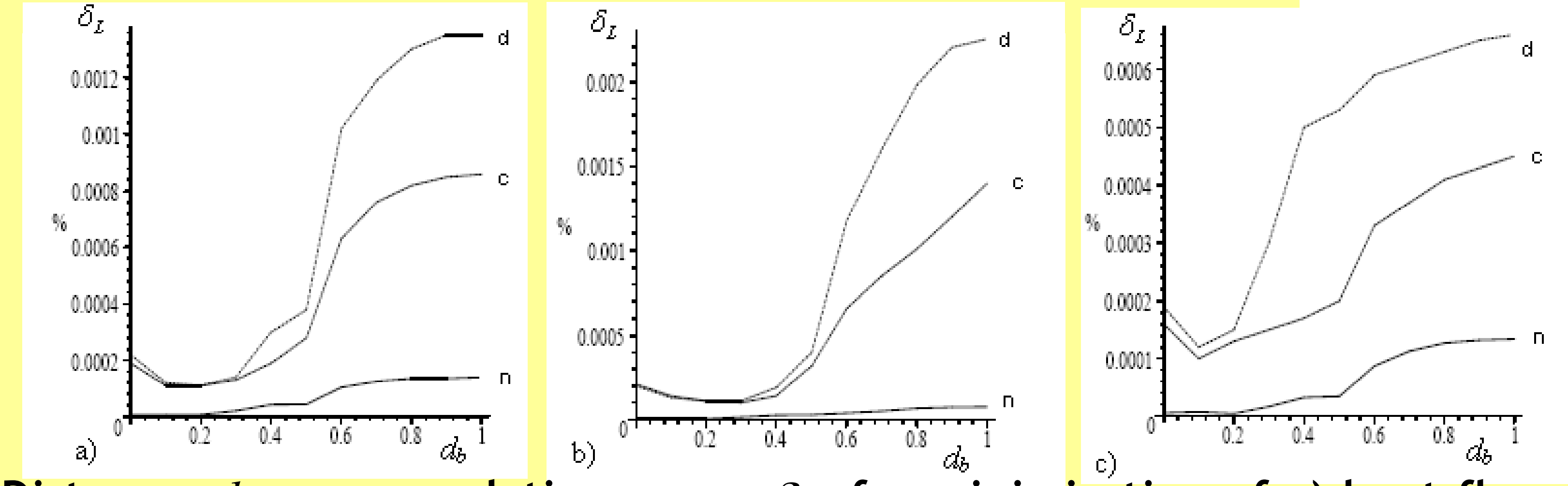
$$\left.\frac{\partial T}{\partial x}\right|_{(0,y)} = h_1(y) = e^y + e^{-y} \qquad \left.\frac{\partial T}{\partial y}\right|_{(x,1)} = h_3(x) = (\cos x + \sin x)(e - e^{-1})$$

$$\left.\frac{\partial T}{\partial y}\right|_{(x,0)} = h_2(x) = 0 \qquad T(1-d_b, y_i) = T_i \quad i = 1,\dots,8$$

Accurate internal temperatures



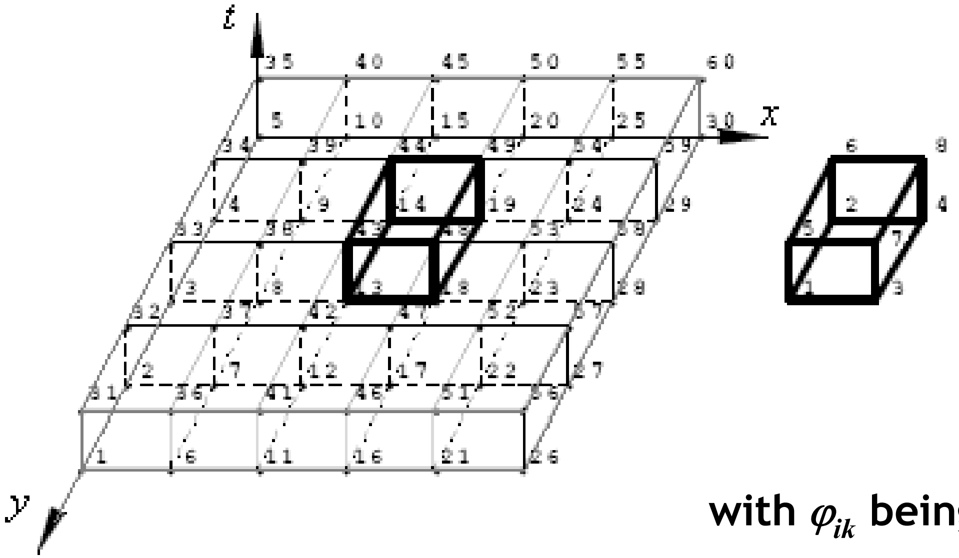
Accurate solution: $T(x,y) = (\cos x + \sin x)(e^y - e^{-y})$



Distance d_b versus relative error δ_L for minimisation of a) heat flux, b) entropy production, c) energy dissipation for c(ontinuous), d(iscontinuous), n(odeless) FEM

A. Maciąg, *Trefftz functions for some direct and inverse problems of mechanics.* Politechnika Świętokrzyska, Kielce, 2009 (in Polish).

Time-space finite elements



$v_n(x_j, y_j, t_j)$ the *n-th* trial Trefftz function

$$\tilde{T}_i(x_j, y_j, t_j) \equiv T^{ij} = \sum_{n=1}^{N} a_{in} v_n(x_j, y_j, t_j)$$

$$j = 1, \dots, N$$

hence the temperature in the *i-th* element:

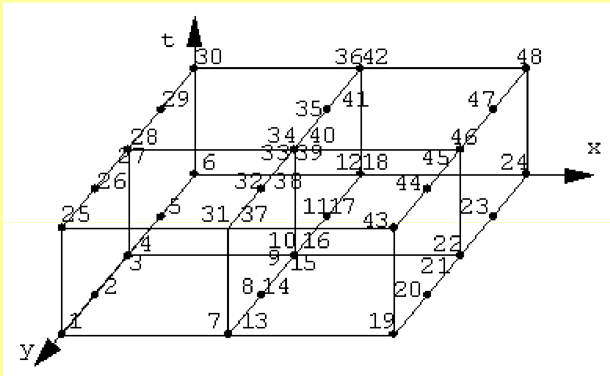
$$\tilde{T}_i(x, y, t) = \sum_{k=1}^{N} \varphi_{ik}(x, y, t) T^{ik}$$

with φ_{ik} being a linear combination of the Trefftz function.

Time-space elements continuous in nodes

$$\delta L_2 = \left[\frac{\int_D \left(\tilde{T}(x, y, t_e) - T(x, y, t_e) \right)^2 dD}{\int_D T(x, y, t_e)^2 dD} \right]^{1/2}$$

Relative error



Time-space elements discontinuous in nodes

A square, non-stationary IHCP, FEMT*

Heat transfer eq. $\Delta T = \partial T / \partial t \qquad (x, y) \in (0,1) \times (0,1) \qquad t \in (0, t_e)$

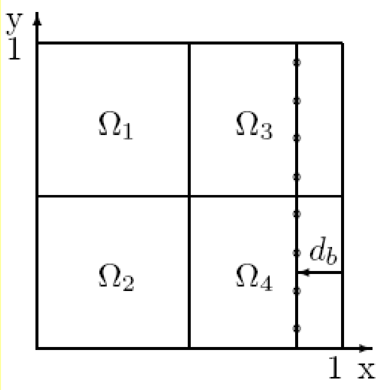
Conditions:

$$T(x,y,t)\big|_{t=0} = T_0(x,y)$$

$$T(x,y,t)\big|_{x=0} = h_1(y,t) = e^{y+2t}$$

$$\frac{\partial T}{\partial y}(x,y,t)\bigg|_{y=1} = h_2(x,t) = e^{x+1+2t}$$

$$\frac{\partial T}{\partial y}(x,y,t)\bigg|_{y=0} = h_3(x,t) = e^{x+2t}$$



Internal temperatures

$$T(1-\delta_b, y_i, t_k) = T_{ik}$$

$$y_i \in \{\, 0.1, 0.2, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9 \,\}$$

$$t_k \in \{\, 0.0025, 0.0050, 0.0075, 0.01 \,\}$$

The exact solution

$$T(x,y,t) = e^{x+y+2t}$$

*K. Grysa, R. Leśniewska, *Different finite element approaches for inverse heat conduction problems.* Inv. Probl. in Science and Eng., 18: 1, 3 — 17, 2010,

Objective functional

$$J = \sum_i \int_{D_i} \left(\tilde{T}_i(x,y,0) - f(x,y) \right)^2 dD + \sum_i \int_0^{t_e} dt \int_{\Gamma_i} \left(\tilde{T}_i(0,y,t) - h_1(y,t) \right)^2 d\Gamma +$$

$$+ \sum_i \int_0^{t_e} dt \int_{\Gamma_i} \left(\frac{\partial \tilde{T}_i}{\partial y}(x,1,t) - h_2(x,t) \right)^2 d\Gamma + \sum_i \int_0^{t_e} dt \int_{\Gamma_i} \left(\frac{\partial \tilde{T}_i}{\partial y}(x,0,t) - h_3(x,t) \right)^2 d\Gamma +$$

$$+ \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{i,j}} \left(\tilde{T}_i - \tilde{T}_j \right)^2 d\Gamma + \sum_i \sum_{k=1}^{I_{ITR}} \left(\tilde{T}_i(x_k, y_k, t_k) - T_{ik} \right)^2 \Big|_{x=1-\delta_b} +$$

$$+ \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{\partial \tilde{T}_i}{\partial x} - \frac{\partial \tilde{T}_j}{\partial x} \right)^2 d\Gamma + \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{\partial \tilde{T}_i}{\partial y} - \frac{\partial \tilde{T}_j}{\partial y} \right)^2 d\Gamma$$

heat flux jump, J_S

$$+ \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{1}{\tilde{T}_i} \frac{\partial \tilde{T}_i}{\partial x} - \frac{1}{\tilde{T}_j} \frac{\partial \tilde{T}_j}{\partial x} \right)^2 d\Gamma + \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{1}{\tilde{T}_i} \frac{\partial \tilde{T}_i}{\partial y} - \frac{1}{\tilde{T}_j} \frac{\partial \tilde{T}_j}{\partial y} \right)^2 d\Gamma$$

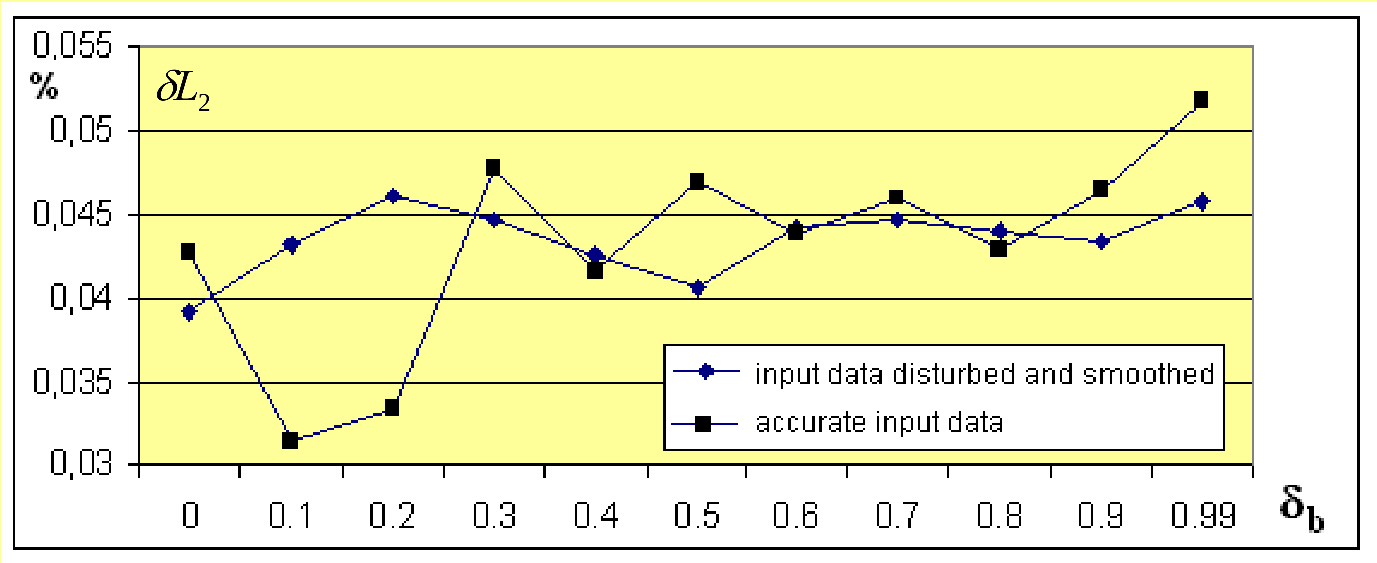
entropy prod. jump, J_E

$$+ \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{\partial \tilde{T}_i}{\partial x} \ln \tilde{T}_i - \frac{\partial \tilde{T}_j}{\partial x} \ln \tilde{T}_j \right)^2 d\Gamma + \sum_{i,j} \int_0^{t_e} dt \int_{\Gamma_{ij}} \left(\frac{\partial \tilde{T}_i}{\partial y} \ln \tilde{T}_i - \frac{\partial \tilde{T}_j}{\partial y} \ln \tilde{T}_j \right)^2 d\Gamma$$

energy diss. jump, J_{RE}

Inaccurate input data

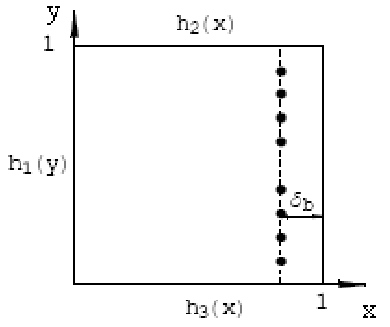
The time-spatial domain is divided into 4 elements. The measurements are simulated from the exact solution and disturbed with a noise with normal distribution not greater than 5% of the exact value. In order to obtain good results the input data have been smoothed with the use of 18 Trefftz functions.



The relative error δL_2 versus the distance δ_b for accurate and inaccurate smoothed input data for nodeless FEMT without energetic regularisation

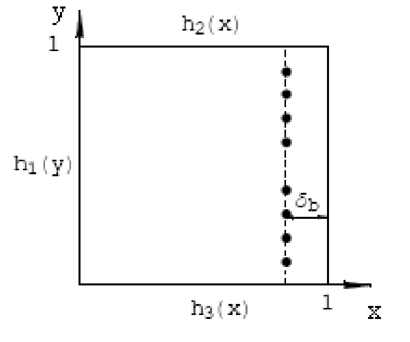
The approximate temperature accuracy in the nodeless FEMT for 12 Trefftz functions

δ_b	J_S			J_E			J_{RE}		
	δ	δL_2 [%]	δH^1 [%]	δ	δL_2 [%]	δH^1 [%]	δ	δL_2 [%]	δH^1 [%]
0	0,0095	0,0428	0,3197	0,0070	0,0308	0,2488	0,0128	0,0487	0,3499
0,1	0,0079	0,0315	0,2510	0,0060	0,0236	0,2089	0,0116	0,0384	0,2860
0,2	0,0100	0,0334	0,2623	0,0079	0,0285	0,2368	0,0109	0,0380	0,2846
0,3	0,0124	0,0478	0,3388	0,0153	0,0450	0,3290	0,0114	0,0489	0,3471
0,4	0,0109	0,0416	0,3105	0,0143	0,0394	0,3003	0,0117	0,0468	0,3381
0,5	0,0130	0,0469	0,3436	0,0154	0,0465	0,3380	0,0130	0,0530	0,3780
0,6	0,0120	0,0438	0,3236	0,0142	0,0393	0,2999	0,0121	0,0486	0,3451
0,7	0,0122	0,0459	0,3331	0,0143	0,0411	0,3090	0,0114	0,0500	0,3520
0,8	0,0120	0,0429	0,3171	0,0149	0,0407	0,3080	0,0123	0,0449	0,3250
0,9	0,0131	0,0464	0,3382	0,0154	0,0416	0,3130	0,0139	0,0480	0,3460
0.99	0,0141	0,0517	0,3713	0,0157	0,0446	0,3308	0,0149	0,0550	0,3850



The approximate temperature accuracy in the nodeless FEMT for 15 Trefftz functions

δ_b	J_S			J_E			J_{RE}		
	δ	δL_2 [%]	δH^1 [%]	δ	δL_2 [%]	δH^1 [%]	δ	δL_2 [%]	δH^1 [%]
0	0,00111	0,00419	0,03820	0,00082	0,00361	0,03470	0,00137	0,00475	0,04230
0,1	0,00100	0,00334	0,03210	0,00081	0,00288	0,02930	0,00137	0,00390	0,03570
0,2	0,00121	0,00397	0,03750	0,00118	0,00364	0,03510	0,00138	0,00412	0,03810
0,3	0,00142	0,00447	0,04100	0,00168	0,00509	0,04360	0,00153	0,00484	0,04240
0,4	0,00142	0,00445	0,03910	0,00164	0,00512	0,04110	0,00150	0,00483	0,04030
0,5	0,00178	0,00537	0,04540	0,00190	0,00553	0,04520	0,00195	0,00591	0,04820
0,6	0,00139	0,00431	0,03880	0,00168	0,00450	0,03870	0,00142	0,00466	0,04090
0,7	0,00137	0,00453	0,04110	0,00171	0,00472	0,04070	0,00154	0,00474	0,04210
0,8	0,00125	0,00401	0,03750	0,00171	0,00479	0,04100	0,00126	0,00443	0,04010
0,9	0,00130	0,00374	0,03530	0,00165	0,00442	0,03840	0,00111	0,00419	0,03780
0.99	0,00131	0,00380	0,03560	0,00161	0,00435	0,03810	0,00108	0,00440	0,03860



Remarks

The test problems with energetic regularization lead to very good results for the all three methods.

The best to apply seem to be the nodeless FEMT.

The relative error does not exceed 1% even in norm δH^1 for inaccurate and smoothed input data and 12 trial functions.

The greater number of T-functions the better results one obtains.

Smoothing the inaccurate data with the use of Trefftz functions leads to results comparable with those obtained with accurate input data.

In the case of a direct problem all three methods lead to good results.

Open problems:

1. A type of time-space element depends on the type of governing equation, because the „length” of the time side of the element should probably depend on the signal propagation velocity.
2. For the HC problems a „velocity” of temperature propagation should be related to the temperature measurement accuracy.
3. Generally in FEMT big finite elements can be used. However, their size depends on the number of trial functions.
4. In the places of accumulation of the investigated phenomenon the elements should be concentrated in space and time. Far from such places the time-space elements can be greater in time and space.
5. An approximate solution of a (direct, inverse) problem seems to be of better quality if the conditions and input data are formulated in the same subspace of the space generated by the T-complete functions.

Open problems:

6. The nodeless FEMT seems to be an interesting method to examine in terms of solution discontinuities at the edges between the elements versus the size of the elements and dimensions of the subspace generated by T-functions.

7. In our study on T-functions application, small number of points with input data (internal responses) led to good results. Also the incomplete data (eg lack of initial condition or the boundary conditions known only on a part of the boundary) lead to good (comparable with accurate) results.

8. Regularisation with the use of normal derivative jump on the borders between time-space elements seems to be interesting to investigate.

9. In the case of nonhomogeneous diff equation an idea of approximating the right side of the eq. with the T-functions seems to simplify the investigation.

Mathematicians and engineers

In the article of Z.C. Li, T.T. Lu, H.T. Huang, A.H.-D. Cheng, *Trefftz, Collocation, and Other Boundary Methods – A Comparison.* Numer. Meth. Par.Diff. Eq, 23,93-144, 2007 I have found the following remark:

It is of interest to point out that some of the TMs are developed in the engineering community [E. Kita, N. Kamiya, *Trefftz method: an overview.* Adv. in Eng. Software, 24, 3-12, 1995.], while others in the mathematical community [Z.C. Li, *Combined methods for elliptic equations with singularities, interfaces and infinities.* Kluwer Academic Publisher, Dordrecht, 1998]. As commented in an article on the history of boundary methods [A.H.-D. Cheng, D.T. Cheng, *Heritage and early developmentof boundary element.* Eng. Anal. Bound. Elem. 29, 268-302, 2005], "*the developments in the two communities, the applied mathematics and the engineering, seems to ran parallel to each other, almost devoid of any cross citations…although many techniques have much in common, and cross-fertilization is needed*."

It would be useful to unify the studies of the two communities, and to stimulate cross-fertilization and citation.

Thank you for your attention

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