On the impact of smoothness in regularization
– selected aspects –

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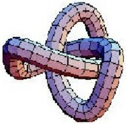

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Inverse Problems: developments in theory and applications

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Outline

1. Introduction
2. Variational inequalities and convergence rates
3. A case distinction for κ and the structure of nonlinearity
4. Extensions in nonlinearity and variational inequalities
5. Extended variational inequalities based on the benchmark source condition and on approximate source conditions

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Outline

The talk partially presents joint work with:

TORSTEN HEIN, RADU BOŢ (CHEMNITZ)

MASAHIRO YAMAMOTO (TOKYO)

OTMAR SCHERZER, CHRISTIANE PÖSCHL (INNSBRUCK)

BARBARA KALTENBACHER (GRAZ)

For another aspect see

JENS GEISSLER's talk on Friday

References

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- B. H.; B. Kaltenbacher; C. Pöschl; O. Scherzer: A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators. *Inverse Problems* **23** (2007), 987-1010.
- B. H.; P. Mathé; S.V. Pereverzev: Regularization by projection: Approximation theoretic aspects and distance functions. *J. Inv. Ill-Posed Problems* **15** (2007), 527-545.
- T. Hein; B. H.: Approximate source conditions for nonlinear ill-posed problems – chances and limitations. *Inverse Problems* **25**, 035003.
- B. H.; M. Yamamoto: On the interplay of source conditions and variational inequalities for nonlinear ill-posed problems. *Applicable Analysis* **89** (2010).
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Nonlinear inverse problems

Let U, V be infinite dimensional Banach spaces with strong convergence $\to$ in norms $\|\cdot\|_U, \|\cdot\|_V$, and well-defined weak convergences $\rightharpoonup$ based on weak topologies.

$F : D(F) \subseteq U \longrightarrow V$ **forward operator** with domain $D(F)$.

We consider the **ill-posed nonlinear** operator equation

$$F(u) = v \qquad (u \in D(F) \subseteq U,\ v \in V) \qquad (*)$$

with solution $u^* \in \mathcal{D}(F)$ and exact right-hand side $v^* = F(u^*)$.

For the stable approximate solution of $(*)$ we consider with

stabilizing functional $\Omega : \mathcal{D}(\Omega) \subseteq U :\to \mathbb{R}$

and for noisy data v^δ assuming a deterministic noise model

$$\|v^* - v^\delta\|_V \le \delta$$

variational regularization (Tikhonov type regularization)

$$T_\alpha^\delta(u) := \psi(\|F(u) - v^\delta\|_V) + \alpha\,\Omega(u) \to \min,$$

subject to $u \in \mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(\Omega),$ with $\alpha > 0$, minimizers u_α^δ,

and an **index function** ψ defined on $[0, \infty)$

(continuous, strictly increasing, $\psi(0) = 0$).

Standing assumptions

Assumption 1

- U, V are reflexive Banach spaces.
- F is weakly-weakly continuous and $\mathcal{D}(F)$ is weakly closed, hence F is weakly closed.
- Ω is convex and weakly lower semi-continuous.
- $\mathcal{D} = \mathcal{D}(F) \cap \mathcal{D}(\Omega) \neq \emptyset$.
- For every $\alpha > 0$ and $c \geq 0$ the sets

$$\mathcal{M}_\alpha(c) := \left\{ u \in \mathcal{D} : T_\alpha^0(u) \leq c \right\} ,$$

  are weakly sequentially pre-compact in the sense that every sequence $\{u_k\}$ in $\mathcal{M}_\alpha(c)$ has a subsequence, which is weakly convergent in U to some element from U.

We exploit for Ω with subdifferential $\partial\Omega$ the **Bregman distance** $D_\xi(\cdot, u)$ of Ω at $u \in U$ and $\xi \in \partial\Omega(u) \subseteq U^*$ defined as

$$D_\xi(\tilde{u}, u) := \Omega(\tilde{u}) - \Omega(u) - \langle \xi, \tilde{u} - u \rangle_{U^*,U} \quad (u, \tilde{u} \in \mathcal{D}(\Omega) \subseteq U) \,.$$

The set

$$\mathcal{D}_B(\Omega) := \{u \in \mathcal{D}(\Omega) : \partial\Omega(u) \neq \emptyset\}$$

is called Bregman domain. An element $u^* \in \mathcal{D}$ is called an Ω**-minimizing solution** if

$$\Omega(u^*) = \min \{\Omega(u) : F(u) = v^*, \ u \in \mathcal{D}\} < \infty \,.$$

Such Ω-minimizing solutions exist under Assumption 1 if $(*)$ has a solution $u \in \mathcal{D}$.

For results on **existence, stability and convergence** see
▷ H./KALTENBACHER/P./SCHERZER 2007, ▷ PÖSCHL 2008.

Example: Standard situation in Hilbert spaces

$$U, V \text{ **Hilbert spaces**}, \qquad \psi(t) = t^2\,,$$

$\Omega(u) := \|u - \bar{u}\|_U^2, \qquad u^*$ is called $\bar{u}$-minimum norm solution

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^2 + \alpha\, \|u - \bar{u}\|_U^2$$

$\mathcal{D}(\Omega) = \mathcal{D}_B(\Omega) = U$, since $\partial\Omega(u)$ is singleton

$$\xi := \Omega'(u^*) = 2(u^* - \bar{u})$$

$$D_\xi(\tilde{u}, u) = \|\tilde{u} - u\|_U^2$$

Example: Regularization with differential operators

$$U, V \ \textbf{Hilbert spaces}, \qquad \psi(t) = t^2\,,$$

$$\Omega(u) := \|Bu\|_U^2 \ \text{ with unbounded s.a. operator } B : \mathcal{D}(B) \subset U \to U$$

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^2 + \alpha\, \|Bu\|_U^2$$

$$\mathcal{D}(\Omega) = \widetilde{U} \quad \text{Hilbert space with stronger norm } \ \|u\|_{\widetilde{U}} := \|Bu\|_U$$

$$\xi := \Omega'(u^*) = 2B^2 u^*$$

$$D_\xi(\tilde{u}, u) = \|B(\tilde{u} - u)\|_U^2 \quad \text{with} \quad \mathcal{D}_B(\Omega) = \mathcal{D}(B^2)$$

Example: Power type penalties in Banach spaces

U, V **Banach spaces**, $\psi(t) = t^p \; (p > 1)$, $\Omega(u) := \frac{\|u\|_U^q}{q} \; (q > 1)$,

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^p + \alpha \left(\frac{1}{q} \|u\|_U^q \right) \qquad (p, q > 1)$$

$\mathcal{D}(\Omega) = \mathcal{D}_B(\Omega) = U$, since $\Omega(u)$ is differentiable with

$\xi := \Omega'(u^*) = J_q(u^*)$ with $J_q : U \to U^*$ duality mapping

$$D_\xi(\tilde{u}, u) = \frac{1}{q}\|\tilde{u}\|_U^q - \frac{1}{q}\|u\|_U^q - \langle J_q(u), \tilde{u} - u \rangle_{U^*, U}$$

Assumption 2

Let F, Ω, U, V and $\mathcal{D}$ satisfy Assumption 1.

- There exists an Ω-minimizing solution u^* which is an element of the Bregman domain $\mathcal{D}_B(\Omega)$.
- $\mathcal{D}$ is starlike with respect to u^*, that is, for every $u \in \mathcal{D}$ there exists t_0 such that

$$u^* + t\,(u - u^*) \in \mathcal{D} \qquad (0 \le t \le t_0).$$

- There is a bounded linear operator $F'(u^*) : U \to V$ such that we have for the one-sided directional derivative at u^* and for every $u \in \mathcal{D}$ the equality

$$\lim_{t \to 0+} \frac{1}{t}\,(F(u^* + t(u - u^*)) - F(u^*)) = F'(u^*)(u - u^*)\,.$$

The operator $F'(u^*)$ has Gâteaux derivative like properties, and there is an adjoint operator $F'(u^*)^* : V^* \to U^*$

Outline

Proposition 1 – weak convergence

Consider an a priori choice $\alpha = \alpha(\delta) \le \bar{\alpha}$ with

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\psi(\delta)}{\alpha(\delta)} \to 0 \quad \text{as} \quad \delta \to 0.$$

Then every sequence $\{u_n\}_{n=1}^{\infty} := \{u_{\alpha(\delta_n)}^{\delta_n}\}_{n=1}^{\infty}$ of regularized solutions corresponding to a data sequence $\{v^{\delta_n}\}_{n=1}^{\infty}$ of data with $\lim_{n\to\infty} \delta_n = 0$ has a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ which is weakly convergent in U to some u^*. This limit element is always an Ω-minimizing solution of $(*)$ with $\Omega(u^*) = \lim_{k\to\infty} \Omega(u_{n_k})$.

Proposition 2 – regularized solutions stay inside level sets

Let $\alpha = \alpha(\delta)$ be a parameter choice from Proposition 1,
For given $\bar{\alpha} > 0$ and Ω-minimizing solution u^* set:

$$\rho := \bar{\alpha}\,(1 + \Omega(u^*))\,.$$

Then $u^* \in \mathcal{M}_{\bar{\alpha}}(\rho)$ and there exists some $\bar{\delta} > 0$ such that

$$u^\delta_{\alpha(\delta)} \in \mathcal{M}_{\bar{\alpha}}(\rho) \qquad \text{for all} \qquad 0 \le \delta \le \bar{\delta}\,.$$

In recent publications the distinguished role of

variational inequalities

$$\langle \xi, u^* - u \rangle_{U^*,U} \le \beta_1 D_\xi(u, u^*) + \beta_2 \|F(u) - F(u^*)\|_V^\kappa \quad (**)$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$ with some $\xi \in \partial\Omega(u^*)$,

two multipliers $0 \le \beta_1 < 1, \quad \beta_2 \ge 0$,

and an exponent $\kappa > 0$ was elaborated.

This talk outlines the chances of such variational inequalities and their **extensions** for ensuring **convergence rates** in Tikhonov type regularization.

Classical theory of convergence rates in Tikhonov regularization for nonlinear ill-posed equations in Hilbert spaces due to

▷ ENGL/KUNISCH/NEUBAUER *Inverse Problems* 1989

for the standard minimization problem

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^2 + \alpha\,\|u - \bar{u}\|_U^2 \to \min$$

separates the following both components

1. Smoothing properties and nonlinearity of the forward operator

$$\|F(u) - F(u^*) - F'(u^*)(u - u^*)\|_V \;\le\; \frac{L}{2}\,\|u - u^*\|_U^2\,.$$

2. Solution smoothness

$$u^* - \bar{u} \;=\; F'(u^*)^*\,w, \qquad L\|w\|_V < 1\,.$$

Both ingredients are **united in variational inequalities.**

This allows handling of **non-smooth** situations for u^* and F!

Theorem 1 – convergence rates & variational inequalities

Under the standing assumptions and assuming the existence of an Ω-minimizing solution from the Bregman domain $u^* \in \mathcal{D}_B(\Omega)$ let there exist an element $\xi \in \partial\Omega(u^*)$ and constants $0 \le \beta_1 < 1,\ \beta_2 \ge 0$, and $0 < \kappa \le 1$ such that the variational inequality $(**)$ holds for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

Then for $\psi(t) = t^p$ $(p > 1)$ we have the convergence rate

$$D_\xi(u^\delta_{\alpha(\delta)}, u^*) \;=\; \mathcal{O}\left(\delta^\kappa\right) \quad \text{as} \quad \delta \to 0$$

for an a priori parameter choice $\alpha(\delta) \asymp \delta^{p-\kappa}$.

Sketch of a proof:

As typical for **low rate world** using $T_\alpha^\delta(u_\alpha^\delta) \le T_\alpha^\delta(u^*)$ we obtain

$$\left\| F(u_\alpha^\delta) - v^\delta \right\|_V^p + \alpha D_\xi(u_\alpha^\delta, u^*) \le \delta^p + \alpha \left(\Omega(u^*) - \Omega(u_\alpha^\delta) + D_\xi(u_\alpha^\delta, u^*) \right).$$

Moreover, by exploiting the inequality $(a+b)^\kappa \le a^\kappa + b^\kappa$ $(a, b > 0,\ 0 < \kappa \le 1)$ from $(**)$ it follows

$$\Omega(u^*) - \Omega(u_\alpha^\delta) + D_\xi(u_\alpha^\delta, u^*) = -\left\langle \xi, u_\alpha^\delta - u^* \right\rangle_{U^*, U}$$

$$\le \beta_1 D_\xi(u_\alpha^\delta, u^*) + \beta_2 \left\| F(u_\alpha^\delta) - F(u^*) \right\|_V^\kappa$$

$$\le \beta_1 D_\xi(u_\alpha^\delta, u^*) + \beta_2 \left(\left\| F(u_\alpha^\delta) - v^\delta \right\|_V^\kappa + \delta^\kappa \right)$$

and hence

$$\left\| F(u_\alpha^\delta) - v^\delta \right\|_V^p + \alpha D_\xi(u_\alpha^\delta, u^*)$$

$$\le \delta^p + \alpha \left(\beta_1 D_\xi(u_\alpha^\delta, u^*) + \beta_2 \left(\left\| F(u_\alpha^\delta) - v^\delta \right\|_V^\kappa + \delta^\kappa \right) \right).$$

Using the variant

$$a\,b \le \varepsilon\, a^{p_1} + \frac{b^{p_2}}{(\varepsilon\, p_1)^{p_2/p_1}\, p_2} \qquad (a, b \ge 0, \;\; \varepsilon > 0)$$

of Young's inequality twice with $p_1, p_2 > 1,\; \frac{1}{p_1} + \frac{1}{p_2} = 1$ we get

$$\alpha D_\xi(u_\alpha^\delta, u^*) \le 2\delta^p + \alpha\beta_1 D_\xi(u_\alpha^\delta, u^*) + \frac{2(p-\kappa)}{(p/\kappa)^{\kappa/(p-\kappa)}\, p}(\alpha\beta_2)^{p/(p-\kappa)}.$$

Because of $0 \le \beta_1 < 1$ this yields

$$D_\xi(u_\alpha^\delta, u^*) \le \frac{2\delta^p + \frac{2(p-\kappa)}{(p/\kappa)^{\kappa/(p-\kappa)}\, p}\,(\alpha\,\beta_2)^{p/(p-\kappa)}}{\alpha\,(1-\beta_1)}$$

and

$$D_\xi(u_{\alpha(\delta)}^\delta, u^*) \,=\, \mathcal{O}\left(\delta^\kappa\right) \quad \text{as} \quad \delta \to 0$$

for an a priori parameter choice $\alpha(\delta) \asymp \delta^{p-\kappa}$.

Comparison of Hölder convergence rates

$D_\xi(u^\delta_{\alpha(\delta)}, u^*) = \mathcal{O}(\delta^\nu)$ for variational regularization with $\psi(t) = t^p$:

Low rate world $0 < \nu \le 1$: Proof ansatz $T^\delta_\alpha(u^\delta_\alpha) \le T^\delta_\alpha(u^*)$
under low order source conditions

$0 < \nu = \kappa \le 1$ obtained for arbitrary reflexive Banach spaces
U and V, $p > 1$, and diversified properties expressed by κ

with a priori choice $\alpha(\delta) \asymp \delta^{p-\nu}$

Enhanced rate world $\nu > 1$: Proof ansatz $T^\delta_\alpha(u^\delta_\alpha) \le T^\delta_\alpha(u^* - z)$
under high order source conditions

$1 < \nu \le \frac{2s}{s+1}$ obtained for s-smooth Banach space V $(s > 1)$

and a priori choice $\alpha(\delta) \asymp \delta^{(p-1)\frac{s}{s+1}}$

Upper rate limit: $\nu = \frac{4}{3}$ in Hilbert space V $(s = 2)$

Optimal rate independent of p $\ge$ 1!
(▷ sc Neubauer/Hein/H./Kindermann/Tautenhahn 2009/10)

Outline

Assumption 3

In addition to the standing assumptions we suppose here:

- Let $u^* \in \mathcal{D}$ be an Ω-minimizing solution of $(*)$.
- The operator F is Gâteaux differentiable in u^* with Gâteaux derivative $F'(u^*)$.
- The functional Ω is Gâteaux differentiable in u^* with Gâteaux derivative $\xi = \Omega'(u^*)$, i.e., the subdifferential $\partial\Omega(u^*) = \{\xi\}$ is a singleton.

The Gâteaux differentiability of F and Ω in u^* implies that there is some $t_0 > 0$ for every direction $\hat{u} \in U$ such that $u^* + t\hat{u} \in \mathcal{D}$ for all $0 \le t \le t_0$.

Structural conditions of F locally in u^* can be expressed by:

Definition (degree of nonlinearity)

Let $0 \le c_1, c_2 \le 1$ and $c_1 + c_2 > 0$. We define F to be **nonlinear of degree** (c_1, c_2) for the Bregman distance D_ξ of Ω at u^* and at $\xi \in \partial\Omega(u^*)$ if there is a constant $K > 0$ such that

$$\|F(u) - F(u^*) - F'(u^*)(u - u^*)\|_V \le K \, \|F(u) - F(u^*)\|_V^{c_1} \, D_\xi(u, u^*)^{c_2}$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

Case $\kappa > 1$:

The following proposition shows that exponents $\kappa > 1$ in the variational inequality for differentiable F and Ω in principle cannot occur ($\triangleright$ sc H./Yamamoto 2009):

Proposition 3 – exponent limitation

Under the Assumption 3 the variational inequality

$$\langle \xi, u^* - u \rangle_{U^*,U} \le \beta_1 D_\xi(u, u^*) + \beta_2 \|F(u) - F(u^*)\|_V^\kappa \quad (**)$$

cannot hold with $\xi = \Omega'(u^*) \neq 0$ and multipliers $\beta_1, \beta_2 \ge 0$ whenever $\kappa > 1$.

Case $\kappa = 1$: ($\triangleright$ Monograph by SCHERZER ET AL. 2009)

As the next proposition shows the variational inequality $(**)$ is closely connected with the source condition $\xi \in \mathcal{R}(F'(u^*)^*)$.

Proposition 4 – source condition equivalence

Under Assumption 3 a variational inequality $(**)$ for $\kappa = 1$ with $\xi = \Omega'(u^*)$ and $\beta_1, \beta_2 \geq 0$ implies the benchmark source condition

$$\xi \;=\; F'(u^*)^*\, w, \qquad w \in V^*\,. \qquad (+)$$

Let F be nonlinear of degree $(0,1)$ in u^*, i.e., we have

$$\|F(u) - F(u^* - F'(u^*)(u - u^*)\|_V \;\leq\; K\; D_\xi(u, u^*)$$

for a constant $K > 0$ and all $u \in \mathcal{M}_{\bar\alpha}(\rho)$. Then conversely the source condition $(+)$ together with the smallness condition $K\,\|w\|_{V^*} < 1$ imply $(**)$ with $\xi = \Omega'(u^*)$ and multipliers $0 \leq \beta_1 = K\|w\|_{V^*} < 1,\;\; \beta_2 = \|w\|_{V^*} \geq 0$.

Case $0 < \kappa \leq 1$:

The theorem below extends the second result of Proposition 4 to a wider class of degrees of nonlinearity. The particular case $\kappa = 1$ occurs only for the complementary situation $c_1 > 0$.

Theorem 2 – utility of $c_1 > 0$

Under Assumption 3 let F be nonlinear in u^* of degree

$$(c_1, c_2) \quad \text{with} \quad 0 < c_1 \leq 1,\ 0 \leq c_2 < 1,\ c_1 + c_2 \leq 1.$$

Then without requiring any additional condition the benchmark source condition $(+)$ implies a variational inequality $(**)$ with

$$\kappa = \frac{c_1}{1 - c_2},$$

$\xi = \Omega'(u^*)$ and $0 \leq \beta_1 < 1,\ \beta_2 \geq 0$.

Extended results for a Hilbert space situation

Assumption 4

- U and V are Hilbert spaces
- $\psi(t) = t^2$
- $\Omega(u) := \|u - \bar{u}\|_U^2$ with reference element $\bar{u} \in U$

Definition (degree of nonlinearity in Hilbert space)

Let $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$. We define F to be nonlinear in u^* of degree (c_1, c_2) if there is a constant $K > 0$ such that

$$\boxed{\|F(u) - F(u^*) - F'(u^*)(u - u^*)\|_V \leq K \, \|F(u) - F(u^*)\|_V^{c_1} \, \|u - u^*\|_U^{2c_2}}$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

Proposition 5 – general source conditions

Let the operator F mapping between the Hilbert spaces U and V be nonlinear of degree (c_1, c_2) in u^* with $c_1 > 0$ and let $\xi = 2(u^* - u^*)$ satisfy the general source condition

$$\xi = (F'(u^*)^* F'(u^*))^{\eta/2} w, \qquad 0 < \eta < 1, \quad w \in U. \qquad (++)$$

Then we have the variational inequality $(**)$ with exponent

$$\kappa \;=\; \min\left\{ \frac{2\eta c_1}{1 + \eta(1 - 2c_2)}, \frac{2\eta}{1 + \eta} \right\}$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$ and multipliers $0 \le \beta_1 < 1$, $\beta_2 \ge 0$.

An exponent $\kappa = \frac{2\eta}{1+\eta}$ in Proposition 5 indicates order optimal convergence rates with respect to the general source condition $(++)$. This is the case if the condition

$$1 + \eta(1 - 2c_2 - c_1) \leq c_1$$

is satisfied. It can hold for $0 < \eta < 1$ only if either $c_1 = 1$ or for $0 < c_1 < 1$ if $c_1 + c_2 > 1$ and η is large enough.

Converse assertions concluding from $(**)$ with exponents $0 < \kappa < 1$ to Hölder source conditions of type $(++)$ are of interest. We have some result for $F := A \in \mathcal{L}(U, V)$ **linear** using $\triangleright$ NEUBAUER 1987:

Proposition 6 – converse result

Let $F := A \in \mathcal{L}(U, V)$ be a bounded linear operator with non-closed range mapping between the Hilbert spaces U, V and let $\xi = 2(u^* - \bar{u})$ satisfy a variational inequality

$$\langle \xi, u^* - u \rangle_U \le \beta_1 \|u - u^*\|_U^2 + \beta_2 \|A(u - u^*)\|_V^\kappa$$

with some

$$0 < \kappa < 1$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$ and multipliers $0 \le \beta_1 < 1$, $\beta_2 \ge 0$, then a Hölder source condition

$$\xi = (A^* A)^{\eta/2} w, \qquad w \in U,$$

is valid for all

$$0 < \eta < \frac{\kappa}{2 - \kappa} < 1.$$

Outline

Now we return to the Banach space setting!

If there is no $c_1 > 0$ such that F is nonlinear of degree (c_1, c_2) for the Bregman distance D_ξ of Ω at u^* and at $\xi \in \partial\Omega(u^*)$, we can **moderate** as follows:

Boundary layer condition for the nonlinearity of F at u^*

$$\left\|F'(u^*)(u-u^*)\right\|_V \le K\,\sigma(\|F(u)-F(u^*)\|_V) \qquad (BLC)$$

for some **concave** index function σ, $K > 0$, and all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

$\sigma(t) = t^{c_1}$ $(0 < c_1 \le 1)$: (BLC) implies degree of nonlinearity $(c_1, 0)$.
Interesting (BLC) case: $t^\nu = o(\sigma(t))$ as $t \to 0$ for all $\nu > 0$.

An adaption of $(**)$ with respect to (BLC) is the

extended variational inequality

$$\langle \xi, u^* - u \rangle_{U^*,U} \le \beta_1 D_\xi(u,u^*) + \beta_2 \varphi(\|F(u) - F(u^*)\|_V)\,. \quad (***)$$

with $0 \le \beta_1 < 1$, $\beta_2 \ge 0$, and some **concave** index function φ.

Assumption 5

- There exist $\bar{a}, \bar{b} > 0$ such that

$$\psi(z_1 + z_2) \le \bar{a}\,\psi(z_1) + \bar{b}\,\psi(z_2) \quad (z_1, z_2 \in [0,\infty)).$$

- There is an index function f such that

$$\psi(s) = \int_0^{\varphi(s)} f(t)dt \qquad (s \ge 0).$$

The existence of an index function f can be ensured for **strictly convex** ψ with $\lim_{s\to 0} \psi'(s) = 0$ and **concave** φ whenever both functions are twice differentiable for positive arguments:

$$f(0) = 0, \quad f(s) = \left[\frac{\psi'}{\varphi'} \circ \varphi^{-1}\right](s) = \left[\psi \circ \varphi^{-1}\right]'(s) \quad (s > 0).$$

Theorem 3 – rates for extended variational inequalities

Under Assumption 5 assume that an extended variational inequality $(***)$ is valid with $0 \le \beta_1 < 1$, $\beta_2 \ge 0$, and concave index function φ for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

Then we have the convergence rate

$$D_\xi(u^\delta_{\alpha(\delta)}, u^*) \;=\; \mathcal{O}\left(\varphi(\delta)\right) \quad \text{as} \quad \delta \to 0$$

for an a priori parameter choice $\alpha(\delta) = \frac{1}{\bar{a}\beta_2} f(\varphi(\delta))$.

The proof is essentially based on Young's inequality in the form

$$a\,b \le \int_0^a f(t)dt + \int_0^b f^{-1}(\tau)d\tau \qquad (a, b \ge 0)\,.$$

▷ BOŢ/H. JIEA 2010

Outline

Extended variational inequalities based on the benchmark source condition and on approximate source conditions

Here we are going to formulate **sufficient conditions** for extended variational inequalities:

Theorem 4 – benchmark source condition case

Let for u^* and $\xi = \Omega'(u^*)$ the benchmark source condition

$$\xi \;=\; F'(u^*)^*\, w, \qquad w \in V^* \qquad (+)$$

and the nonlinearity condition (BLC) with some index function σ be satisfied. Then an extended variational inequality $(***)$ holds with two multipliers $0 \le \beta_1 < 1,\ \beta_2 > 0$ and with the index function $\varphi = \sigma$.

Applying the method of approximate source conditions

The **distance function**

$$d(R) := \min_{w \in V^*:\, \|w\|_{V^*} \le R} \|\xi - F'(u^*)^* w\|_{U^*}$$

measures the **degree of violation** of ξ with respect to the benchmark source condition $\xi = F'(u^*)^* w, \; w \in V^*$.

Proposition 9 – decay of distance function

Let ξ satisfy the requirements

$$\xi \notin \mathcal{R}(F'(u^*)^*)$$

and

$$\xi \in \overline{\mathcal{R}(F'(u^*)^*)}^{\|\cdot\|_{U^*}}.$$

Then $d(R)$ $(0 \le R < \infty)$ is a non-increasing positive function tending to zero as $R \to \infty$.

Theorem 5 – approximate source condition case

Let u^* and $\xi = \Omega'(u^*)$ satisfy the nonlinearity condition (BLC) with some index function σ , but fail to satisfy the benchmark source condition $(+)$, i.e., $d(R)$ is a positive function for all $R \geq 0$. If $F'(u^*)$ is injective and the Bregman distance is q-coercive with $2 \leq q < \infty$ and some constant $c_q > 0$ such that

$$D_\xi(u, u^*) \geq c_q \, \|u - u^*\|_U^q \, ,$$

then an extended variational inequality $(* * *)$ holds with two multipliers $0 \leq \beta_1 < 1, \; \beta_2 > 0$ and with the index function

$$\varphi(0) = 0, \;\; \varphi(t) = \left[d\left(\Psi^{-1}(\sigma(t)) \right) \right]^{q^*} \;\; (t > 0) \, ,$$

where $\frac{1}{q} + \frac{1}{q^*} = 1$ and $\Psi(R) := \frac{d(R)^{q^*}}{R}$.

Example: Get logarithmic rates over two different ways

Consider in extended variational inequality $(***)$

$$\varphi(t) = \begin{cases} 0 & (t = 0) \\ C\,[\log(1/t)]^{-\mu} & (0 < t \le e^{-\mu-1}) \end{cases}$$

By Theorem 3 we obtain a convergence rate

$$D_\xi(u^\delta_{\alpha(\delta)}, u^*) = \mathcal{O}\left([\log(1/\delta)]^{-\mu}\right) \quad \text{as} \quad \delta \to 0\,.$$

(I) Case $\sigma = \varphi$ characterizes a very weak logarithmic structural condition (BLC).
It gives by Theorem 4 such φ in $(***)$ whenever the benchmark source condition $(+)$ is satisfied.

(II) Case $\sigma(t) = t$ characterizes $c_1 = 1$ and hence a strong nonlinearity condition. The logarithmic function φ is obtained by Theorem 5 if with

$$d(R) = (\log R)^{-\nu} \qquad (\nu > 0)$$

the distance function decay to zero as $R \to \infty$ is very slow.
However, since we have for large R and for $\varepsilon > 0$ a constant $K > 0$ with

$$\Psi(R) = \frac{1}{R(\log R)^{\nu q^*}} \geq \frac{K}{R^{1+\varepsilon}},$$

this implies $\Psi^{-1}(t) \geq \hat{K} t^{-1/(1+\varepsilon)}$ for sufficiently small $t > 0$.
Hence we have that φ with $\mu = \nu q^*$.

Some conjecture:

We conjecture that **no convergence rates** can be proven when the structure of nonlinearity at u^* is **too rough and** $(+)$ **is violated**, i.e., $d(R) > 0$ ($R \geq 0$).

What does roughness mean here?
(BLC) cannot be satisfied for any index function σ ?