Regularization of the Cauchy problem for the Laplace equation by the conjugate gradient method

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Warsaw, February 2010

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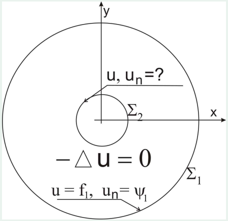
Problem settings

Inverse problem (IP) and direct problem (DP) on annular domain:

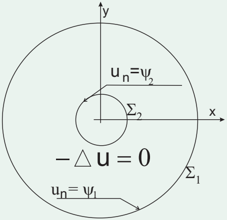
IP: Find $u|_{\Sigma_2}$, $u_n|_{\Sigma_2}$, where u is solution of the Cauchy Problem

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega \\ u = f_1 & \text{auf } \Sigma_1 \\ \frac{\partial u}{\partial n} = \psi_1 & \text{auf } \Sigma_1 \end{cases}$$



DP: Find $u|_{\Sigma_1}$, where u is weak solution of the Neumann problem

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \psi_1 & \text{auf } \Sigma_1 \\ \frac{\partial u}{\partial n} = \psi_2 & \text{auf } \Sigma_2 \end{cases}$$



Operator form of direct problem ($\psi_1 := 0$): $A : \psi_2 \mapsto u(\psi_2)|_{\Sigma_1}$

Compact operator formulation of the linear problem

$$T := R \circ Id_1 \circ A \in \mathcal{K}(L^2(\Sigma_2), L^2(\Sigma_1)),$$

$$Id_1 : \quad H^{1/2}(\Sigma_1) \to L^2(\Sigma_1) \quad \text{(compact embedding)}$$

$$R : \quad L^2(\Sigma_1) \to L^2(\Sigma_1), \quad R : u \mapsto u - \frac{1}{|\Sigma_1|} \int_{\Sigma_1} u d\Sigma_1$$

$$A : \quad L^2(\Sigma_2) \to H^{1/2}(\Sigma_1), \quad A : \psi_2 \mapsto u(\psi_2)|_{\Sigma_1}$$

where $u(\psi_2) \in V$ denotes the unique solution of variational equation

$$(\nabla u, \nabla \phi)_0 = \langle \psi_2, \phi \rangle_{L^2(\Sigma_2)} \quad \forall \phi \in V,$$

$$V = \{v \in H^1(\Omega) \mid \int_\Omega v dx = 0\}.$$

Lemma

A is linear and bounded: $\|A\psi_2\|_{H^{1/2}(\Sigma_1)} \leq c\|\psi_2\|_{L^2(\Sigma_2)}$.

Operator formulation of direct problem

Given $\psi_2 \in L^2(\Sigma_2)$, find $\tilde{f}_1$:

$$T\psi_2 = \tilde{f}_1, \quad \tilde{f}_1 = Rf_1$$

Inverse Problem: Given f_1, find $\psi_2(:= u_n|_{\Sigma_2})$: $\quad T\psi_2 = \tilde{f}_1$
is ill-posed, since $T =$ **compact.**

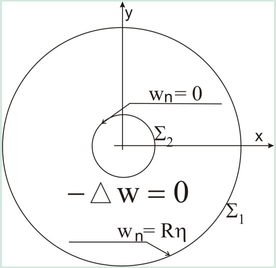
The adjoint operator:

$$T^*\eta := w|_{\Sigma_2}, \quad \eta \in L^2(\Sigma_1)$$

$$w \in V : (\nabla w, \nabla\phi)_0 = \langle R\eta, \phi\rangle_{L^2(\Sigma_1)} \quad \forall\phi \in V,$$

w is unique weak solution of

$$\begin{cases} -\triangle w = 0 \\ w_n|_{\Sigma_1} = R\eta \\ w_n|_{\Sigma_2} = 0. \end{cases}$$



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Discretization by projection

$T \in \mathcal{K}(X,Y), X := L^2(\Sigma_2),\ \ Y := L^2(\Sigma_1), f := \tilde{f}_1$

IP (operator formulation): $T\psi_2 = f,\ f \in \mathcal{R}(T)$

$P_l : X \to X_l$ and $Q_l : Y \to Y_l$ orthoprojectors with $dim(X_l) = n_l,\ dim(Y_l) = m_l$

Projected equation: $T_l\psi_{2,l} = Q_l f, \quad T_l := Q_l T P_l$ and $\psi_{2,l} \in \mathcal{R}(P_l)$

We assume only an approximation f^ε for f is available: $f^\varepsilon \in Y,\ \|f^\varepsilon - f\|_Y \leq \varepsilon$

Projected normal equation: $T_l^* T_l \psi_{2,l} = T_l^* Q_l f$

Remark: The projected problems are ill-conditioned and we regularize it using CGNE with an appropriated stopping rule.

CGNE for projected equation

$\psi_{2,l}^{(0)} := 0$ and $\psi_{2,l}^{(1)}, \psi_{2,l}^{(2)}, \ldots$, determines by

$$\psi_{2,l}^{(m)} \in \mathcal{K}_m(T_l^* T_l, T_l^* Q_l), \ \mathcal{K}_m(A^*A, r) := span\{r, (A^*A)r, \ldots, (A^*A)^{m-1} r\}$$

$$\| T_l \psi_{2,l}^{(m)} - Q_l f \|_Y = \min_{\psi \in \mathcal{K}_m(T_l^* T_l, T_l^* Q_l)} \| T_l \psi - Q_l f \|_Y$$

projected CGNE algorithm:

$r^0 := g - T_l \psi_{2,l}^{(0)}; \ p^1 := d^0 := \ T_l^* r^0; \ m := 1;$

while $(d^{m-1} \neq 0)$

$\{q^m := T_l p^m;$ $\qquad \alpha_m := \|d^{m-1}\|_X^2 / \|q^m\|_Y^2$

$\psi_{2,l}^{(m)} := \psi_{2,l}^{(m-1)} + \alpha_m p^m;$ $\qquad r^m := r^{m-1} - \alpha_m q^m;$

$d^m := T_l^* r^m;$ $\qquad \beta_m := \|d^m\|_X^2 / \|d^{m-1}\|_X^2;$

$p^{m+1} := d^m + \beta_m p^m;$ $\qquad m := m+1; \}$

- CGNE breaks down at step $m_* := m \leq dim(X_l)$ if $T_l r^m = 0$
- **Representation:** $\psi_{2,l}^{(m)} = q_{m-1}(T_l^* T_l, Q_l f) Q_l f$ with $q_{m-1} \in \Pi_{m-1}$
- $q_{m-1}(0), \ m = 1, 2, \ldots$ can be easily computed by three term recursion

A stopping criterion for projected CGNE

Stopping rule 1 Plato 90 ($\zeta_l \in \mathbb{R}$ with $\|A(I - P_l)\| \le \zeta_l < 1$ and $\tau > 1$)

Stop the iteration at step $\bar{m} = m$, if one of the following conditions is satisfied

$$\|T_l \psi_{2,l}^{(m)} - Q_l f^\varepsilon\|_Y \le \tau\varepsilon \quad \text{or} \quad q_{m-1}(0, Q_l f^\varepsilon)) \ge \zeta_l^{-2} \quad \text{or} \quad m = m_*.$$

Then define

$$\bar{\psi}_2(l, \varepsilon, \tau, f^\varepsilon) = \begin{cases} \psi_{2,l}^{\bar{m}-1} & q_{\bar{m}-1}(0) \ge \zeta_l^{-2} \\ \psi_{2,l}^{\bar{m}} & q_{\bar{m}-1}(0) < \zeta_l^{-2} \end{cases}.$$

Theorem: Assume $T \in \mathcal{K}(X, Y)$, $\mathcal{N}(A)^\perp \subset \overline{\bigcup X_h}$, $\mathcal{R}(A) \subset \overline{\bigcup Y_h}$ and MNS-Solution $\psi_2^\dagger \in X_\mu,\ \mu > 0$ and $\|f^\dagger\|_\mu \le \rho$. For projected CGNE with noisy data $f^\varepsilon \in Y,\ \|f^\varepsilon - f\|_Y \le \varepsilon$ let the iteration be stopped according to the stopping rule 1 with corr. approximation $\bar{\psi}_2(l, \varepsilon, \tau, f^\varepsilon)$. Then ex. $c_{cg} > 0$

$$\|\bar{\psi}_2(l, \varepsilon, \tau, f^\varepsilon) - \psi_2^\dagger\|_X \le c_{cg}(\varepsilon^{\frac{\mu}{\mu+1}} \rho^{\frac{1}{\mu+1}} + \rho(\zeta_l^{\min\{\mu,1\}} + \|(I - Q_l)T\|^{\min\{\mu,2\}})).$$

A modified stopping criterion for projected CGNE

Stopping rule 2 ($\zeta_l \in \mathbb{R}$ with $\|A(I-P_l)\| \leq \zeta_l < 1$ and $\tau > 1$)

Stop the iteration at step $\bar{m} = m$, if one of the following conditions is satisfied

$$\|T_l\psi_{2,l}^{(m)} - Q_l f^\varepsilon\|_Y \leq \tau\varepsilon \text{ or } q_{m-1}(0, Q_l f^\varepsilon) \geq P(\varepsilon,\rho,\zeta_l) \text{ or } m = m_*.$$

Then define $\bar{\psi}_2(l,\varepsilon,\tau,f^\varepsilon) = \begin{cases} \psi_{2,l}^{\bar{m}-1} & q_{\bar{m}-1}(0) \geq P(\varepsilon,\rho,\zeta_l) \\ \psi_{2,l}^{\bar{m}} & q_{\bar{m}-1}(0) < P(\varepsilon,\rho,\zeta_l) \end{cases},$

where $P(\varepsilon,\rho,\zeta_l) := \begin{cases} \zeta_l^{-2}, & \text{if } \left(\frac{\varepsilon}{\rho}\right)^{\frac{1}{\mu+1}} \leq \zeta_l \\ \left(\frac{\varepsilon}{\rho}\right)^{-\frac{2}{\mu+1}} & \text{if } \left(\frac{\varepsilon}{\rho}\right)^{\frac{1}{\mu+1}} > \zeta_l. \end{cases}$

Theorem: Assume $T \in \mathcal{K}(X,Y)$, $\mathcal{N}(A)^\perp \subset \overline{\bigcup X_h}$, $\mathcal{R}(A) \subset \overline{\bigcup Y_h}$ and MNS-Solution $\psi_2^\dagger \in X_\mu,\ \mu > 0$ and $\|f^\dagger\|_\mu \leq \rho$. For projected CGNE with noisy data $f^\varepsilon \in Y,\ \|f^\varepsilon - f\|_Y \leq \varepsilon$ let the iteration be stopped according to the stopping rule 2 with corr. approximation $\bar{\psi}_2(l,\varepsilon,\tau,f^\varepsilon)$. Then ex. $\widehat{c}_{cg} > 0$

$$\|\bar{\psi}_2(l,\varepsilon,\tau,f^\varepsilon) - \psi_2^\dagger\|_X \leq \widehat{c}_{cg}(\varepsilon^{\frac{\mu}{\mu+1}}\rho^{\frac{1}{\mu+1}} + \rho(\zeta_l^{\min\{\mu,1\}} + \|(I-Q_l)T\|^{\min\{\mu,2\}})).$$

Conclusions

- On the annular domain with $h_1 = \frac{2\pi R}{n_l}$ on $L^2(\Sigma_1)$ and $h_2 = \frac{2\pi r}{m_l}$ on $L^2(\Sigma_2)$ holds: $\zeta_l = \mathcal{O}(h_2), \|(I - Q_l)T\| = \mathcal{O}(h_1)$
- CGNE with stopping rule 2 possibly breaks down earlier than with stop. rule 1
- If $(\frac{\varepsilon}{\rho})^{\frac{1}{\mu+1}} > \zeta_l$ we can formulate stopping rule 2 without the discrepancy constraint $\|T_l\psi_{2,l}^{(m)} - Q_l f^\varepsilon\|_Y < \tau\varepsilon$
- If $(\frac{\varepsilon}{\rho})^{\frac{1}{\mu+1}} > \zeta_l$ CGNE with stop. rule 2 provides more precise approximations

Disadvantange:

- For stopping rule 2 we additionally need to know the source representation data ρ, μ

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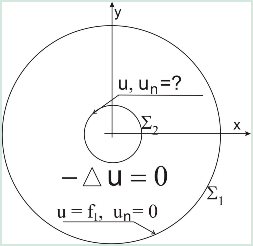
$$f_1 := v_m(x,0) = \frac{\cos(m\pi x)}{(m\pi)^2},$$

$$\psi_2^\dagger := \frac{\partial v_m}{\partial n}(x,0.2) = \frac{\cos(m\pi x)\sinh(0.2m\pi)}{m\pi}.$$

$\psi_2^\dagger \in R(T^*) = X_1$ (source representation)



Noisy data: $f_1^\varepsilon = \sum_i (f_1(x_i) + \delta(x_i))\phi_i \quad (\varepsilon = \mathcal{O}(h^2 + \delta))$

Reconstruction Error(Stopping rule 1+2):

$$\|\bar{\psi}_2 - \psi_2^\dagger\|_{L^2(\Sigma_2)}) \leq C(\psi_2^\dagger)((h_1^2 + \delta)^{1/2} + h_1 + h_2),$$

With $h_1 = h_2 = h$ and $\delta = h^2$:

$$\|\bar{\psi}_2 - \psi_2^\dagger\|_{L^2(\Sigma_2)} = \mathcal{O}(h) \text{ for } (h \to 0)$$

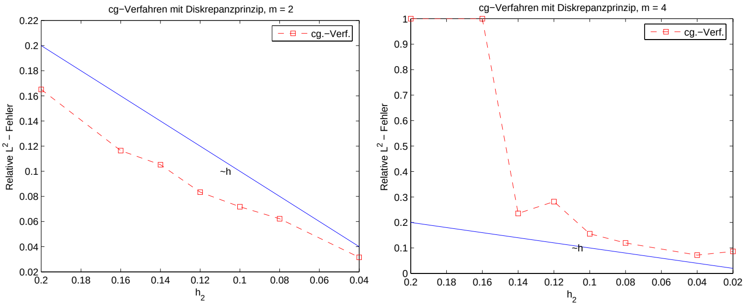


Figure: Order optimality of CGNE with stopping rule 2

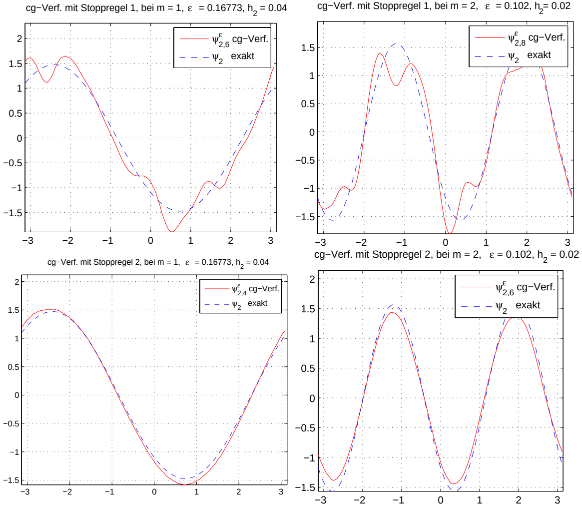


Figure: Top: CGNE with stop. rule 1, Bottom: CGNE with stop. rule 2 $((\varepsilon/\rho)^{\frac{1}{\mu+1}} > \zeta_h)$.

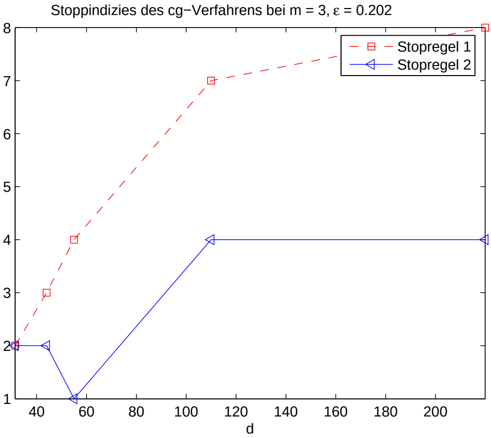


Figure: Stopping index: stopping rule 1 and 2.

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