Finite dimensional approximation of convex regularization in nonseparable Banach spaces

Elena Resmerita

Institute of Industrial Mathematics
Johannes Kepler University
Linz, Austria

Inverse Problems: Theory and Applications
February 9-12, 2009

The talk is based on the work in progress

Discretization of convex regularization,

developed jointly with

- Christiane Pöschl (University of Innsbruck, Austria)
- Otmar Scherzer (University of Vienna and Radon Institute, Austria)

Outline

Outline

Ill-posed operator equations

Various inverse problems reduce to solving an equation

$$Fu = y,$$

where

- $F : X \to Y$ is a nonlinear compact operator;
- X is a Banach space and Y is a Hilbert space.

Such a problem is often ill-posed:

Small perturbations in the data y induce high oscillations in the solution x.

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Remedy:

One should apply some method of regularization.

Discretization issue

In order to solve the equation numerically, the space X has to be approximated by a sequence of finite dimensional subspaces X_n.

Proposition

A Banach space X is separable if and only if there exists a nested sequence of finite dimensional subspaces $\{X_n\}$ such that

$$\overline{\cup_{n\in\mathbb{N}} X_n} = X,$$

where the closure is considered with respect to the norm topology of X.

Approximation of non-separable Banach spaces

A significant complication: non-separable Banach spaces cannot be approximated by nested sequences of finite dimensional subspaces, with respect to the norm topology.

Examples: $BV(\Omega)$, $BD(\Omega)$, $L^\infty(\Omega)$

"The norm topology [of BV] is too strong for many applications."

Ambrosio, Fusco, Pallara '00

Theoretical framework

- X is a not necessarily separable Banach space.
- Z is a separable Banach space such that $X \subset Z$.
- $\mathcal{R} : X \to [0, +\infty]$ is a convex function.

Define a metric on the space X by

$$d(u, v) = \|u - v\|_Z + |\mathcal{R}(u) - \mathcal{R}(v)|.$$

Theoretical framework

Denote by $\mathcal{D}(F)$ and $\mathcal{D}(\mathcal{R})$ the domains of the operator F and of the function $\mathcal{R}$, respectively.

- $\bar{u} \in \mathcal{D}(F) \cap \mathcal{D}(\mathcal{R})$ is called an $\mathcal{R}$-minimizing solution of the equation if it solves

$$\min \mathcal{R}(u) \text{ subject to } F(u) = y.$$

- Noisy data y^δ are given such that

$$\left\| y^\delta - y \right\|_Y \leq \delta.$$

- Approximation operators F_m of F are given:
  - They have the same domain, $\mathcal{D}(F)$.
  - They satisfy

$$\|F(u) - F_m(u)\|_Y \leq \rho_m \text{ for all } u \in \mathcal{D}(F) \cap \mathcal{D}(\mathcal{R}) \,,$$

with $\lim_{m\to\infty} \rho_m = 0$.

Assumptions I

1. The Banach space X is provided with a topology τ such that
   - The three topologies satisfy

$$\tau \prec \tau_d \prec \tau_{\|\cdot\|}.$$

2. The domain $\mathcal{D}(F)$ is τ-closed and convex.
3. The operator $F : \mathcal{D}(F) \subseteq X \to Y$ is continuous from (X, τ) to Y endowed with the weak topology.

Assumptions II

4. For every $m \in \mathbb{N}$, the operator F_m is continuous from (X, τ) to Y endowed with the weak topology.
5. The function $\mathcal{R}$ is bounded from below and sequentially τ - lower semi-continuous.
6. For every $M > 0$, $\alpha > 0$ and every $m, n \in \mathbb{N}$, the sets

$$\{u \in X_n : \|F(u)\|_Y^2 + \alpha\mathcal{R}(u) \leq M\}$$

   are τ-sequentially relatively compact.
7. For every $u \in X$, there exists some $v_n \in X_n$, $n \in \mathbb{N}$, such that $d(v_n, u) \to 0$ as $n \to \infty$.
   Here (X_n) is a nested sequence of finite dimensional subspaces of X.

Outline

Variational regularization

Let

$$D_n := \mathcal{D}(F) \cap X_n \cap \mathcal{D}(\mathcal{R}) \neq \emptyset, n \in \mathbb{N},$$

We are interested in approximating $\mathcal{R}$-minimizing solutions of equation $F(u) = y$ by solutions $u_{m,n}^{\alpha,\delta} \in D_n$ of the problem

$$\min \left\{ \left\| F_m(u) - y^\delta \right\|_Y^2 + \alpha \mathcal{R}(u) \right\} \text{ subject to } u \in D_n.$$

Stability

Proposition

Let $m, n \in \mathbb{N}$ and $\alpha, \delta > 0$ be fixed and let the assumptions be satisfied.
Then, for every $y^\delta \in Y$, there exists at least one minimizer u of the regularization problem.
Moreover, the minimizers are stable with respect to the data y^δ in the following sense:
If $\lim_{k\to\infty} y_k = y^\delta$, then every sequence $\{u_k\}_{k\in\mathbb{N}}$ of minimizers of the regularization problem with y_k instead of y^δ has a subsequence $\{u_l\}_{l\in\mathbb{N}}$ which converges to a minimizer $\tilde{u}$ corresponding to y^δ, as follows:

$$u_l \xrightarrow{\tau} \tilde{u} \text{ and } \mathcal{R}(u_l) \to \mathcal{R}(\tilde{u}), \quad \text{as } l \to \infty.$$

Convergence analysis

Theorem

Let the assumptions on $X, Y, Z, F, F_m, \mathcal{R}$ be satisfied. Moreover,
(i) An $\mathcal{R}$-minimizing solution $\bar{u}$ is in $int\,(\mathcal{D}(\mathcal{R}) \cap \mathcal{D}(F))$;
(ii) $v_n \in \mathcal{D}(F)$ for n sufficiently large, where $v_n \in X_n$ and $\lim_{n\to\infty} d(v_n, \bar{u}) = 0$;
(iii) The parameter $\alpha = \alpha(m, n, \delta)$ is such that $\alpha \to 0$, $\frac{\delta^2}{\alpha} \to 0$, $\frac{{\rho_m}^2}{\alpha} \to 0$ and

$$\frac{\|F(v_n) - y\|}{\sqrt{\alpha}} \to 0, \;\text{ as }\; \delta \to 0, \; m, n \to \infty.$$

Then every sequence of minimizers $\{u_k\}$, with $u_k := u_{m_k,n_k}^{\alpha_k,\delta_k}$ and $\alpha_k := \alpha(m_k, n_k, \delta_k)$ where $\delta_k \to 0$, $m_k, n_k \to \infty$, as $k \to \infty$, has a subsequence $\{u_l\}$ which converges to an $\mathcal{R}$-minimizing solution $\tilde{u}$,

$$u_l \xrightarrow{\tau} \tilde{u} \;\text{ and }\; \mathcal{R}(u_l) \to \mathcal{R}(\tilde{u}), \quad \text{as } l \to \infty.$$

Remark

In our setting:

$$d(u_k, u) = \|u_k - u\|_Z + |\mathcal{R}(u_k) - \mathcal{R}(u)| \to 0$$

$$\Downarrow$$

$$\left(u_k \xrightarrow{\tau} u \ \text{ and } \ \mathcal{R}(u_k) \to \mathcal{R}(u)\right).$$

Metric convergence is stronger than 'Kadec-Klee' ('Radon-Nikodym') convergence.

Assumptions towards convergence rates I

F is Fréchet differentiable around $\bar{u} \in int\,(\mathcal{D}(\mathcal{R}) \cap \mathcal{D}(F))$.

Source condition

There exists $\omega \in Y$ such that

$$(SC) \quad \xi = F'(\bar{u})^* \omega \in \partial \mathcal{R}(\bar{u}).$$

Nonlinearity condition

There exist $\varepsilon, c > 0$ such that

$$\left\| F(u) - F(\bar{u}) - F'(\bar{u})(u - \bar{u}) \right\|_Y \leq c D_{\mathcal{R}}(u, \bar{u}),$$

for all $u \in \mathcal{D}(F) \cap U_\varepsilon(\bar{u})$ with $c\|\omega\|_Y < 1$ and

$$D_{\mathcal{R}}(u, \bar{u}) = \mathcal{R}(u) - \mathcal{R}(\bar{u}) - \langle F'(\bar{u})^* \omega, u - \bar{u} \rangle.$$

Convergence rates

Let $v_n \in X_n$ and $v_n \in \mathcal{D}(F)$ for n sufficiently large, with

$$\lim_{n\to\infty} d(v_n, \bar{u}) = 0.$$

Denote

$$\gamma_n := \|F'(\bar{u})(v_n - \bar{u})\|_Y, \quad \lambda_n := D_{\mathcal{R}}(v_n, \bar{u}).$$

Observe that

$$\lim_{n\to\infty} \gamma_n = 0 \text{ and } \lim_{n\to\infty} \lambda_n = 0.$$

Theorem

Let the assumptions on $X, Y, Z, F, F_m, \mathcal{R}$ hold. Moreover, assume that $\rho_m = O(\delta + \lambda_n + \gamma_n)$.
If $\alpha \sim \max\{\delta, \lambda_n, \gamma_n\}$, then

$$D_{\mathcal{R}}(u_{m,n}^{\alpha,\delta}, \bar{u}) = O(\delta + \lambda_n + \gamma_n).$$

Previous results

X and Y are Hilbert spaces and $\mathcal{R} = \| \cdot \|^2$

- Linear equations (a priori strategy)

  Neubauer '89

- Nonlinear equations
  Neubauer, Scherzer '90 (a priori)

  Qi-nian '99 (a posteriori)

Outline

The space of bounded variation functions

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain, $N \in \mathbb{N}$.

$$BV(\Omega) = \{w \in L^1(\Omega) : \int_\Omega |Dw|_p < \infty\},$$

where

$$\int_\Omega |Dw|_p = \sup\left\{\int_\Omega w(x)\, div\psi(x) dx : \psi \in \mathcal{C}_0^\infty(\Omega)^N, |\psi(x)|_{p'} \leq 1,\ x \in \Omega\right.$$

Here, $|\cdot|_{p'}$ denotes the $l_{p'}$ vector norm, and $p' = p/(p-1)$.

In particular we are interested in the cases $p = 1, 2$.

The space of bounded variation functions

Several properties of $BV(\Omega)$

- It is the dual of a separable Banach space when provided with the norm
$$\|u\|_{BV} = \|u\|_{L^1} + \int_\Omega |Du|_p \,.$$
It has a weak* topology; bounded sets in $BV(\Omega)$ are sequentially relatively compact.
-
$$u_k \xrightarrow{w^*} u \iff \left(\|u_k - u\|_{L^1} \to 0 \text{ and } \{\|u_k\|_{BV})\} \text{ bounded}\right).$$

The space of bounded variation functions

- Consider $X = BV(\Omega)$, $Z = L^1(\Omega)$ with $\tau = w^*$, $\mathcal{R}(u) = \int_\Omega |Du|_p$ and

$$d(u,v) = \|u - v\|_{L^1(\Omega)} + \left| \int_\Omega |Du|_p - \int_\Omega |Dv|_p \right| .$$

The metric d gives the so-called *strict convergence*.

Ambrosio, Fusco, Pallara '00

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$$d(u_k, u) \to 0 \;\Leftrightarrow\; \left(u_k \xrightarrow{w^*} u \;\text{ and }\; \int_\Omega |Du_k|_p \to \int_\Omega |Du|_p \right) .$$

Approximation by piecewise constant functions

Theorem

Let $\{\Omega_j\}$ be a decomposition of Ω into parallelepipeds with $h_n \to 0$ as $n \to \infty$, where h_n is the maximal length of a parallelepiped . Consider X_n the space of piecewise constant functions on $\{\Omega_j\}$. Then for every $u \in BV(\Omega)$, there exist functions $u_n \in X_n$, where , such that

$$\|u_n - u\|_{L^1} + \left| \int_\Omega |Du_n|_1 - \int_\Omega |Du|_1 \right| \to 0 \quad \text{as } n \to 0 \, .$$

Casas, Kunisch, Pola '99

Further approximations of the BV space

A similar result holds in the cases:

- $\mathcal{R}(u) = \int_\Omega |Du|_1$ and X_n consisting of piecewise polynomial functions which are continuous on $\bar{\Omega}$.

  Casas, Kunisch, Pola '99

- $\mathcal{R}(u) = \int_\Omega |Du|_2$, Ω a polygonal domain and $\{\Omega_j\}$ a triangulation of Ω.

  Belik, Luskin '03

The regularization result in the BV space

Let $X = BV(\Omega)$, $Z = L^1(\Omega)$ and

$$d(u, v) = \|u - v\|_{L^1(\Omega)} + \left| \int_\Omega |Du|_p - \int_\Omega |Dv|_p \right| .$$

Then for every $u \in BV(\Omega)$ there exists an approximating sequence of piecewise constant functions. Consequently, minimization of the discretized regularized problem is well–posed, stable, and convergent.

The piecewise constant regularizers approximate the $\mathcal{R}$-minimizing solution $\bar{u}$ on subsequences in the sense of the metric d.

Previous work on discretization of BV regularization

Consider

$$u_n = argmin\left\{\|Au - y\|_Y^2 + \alpha\mathcal{R}(u)\right\} \quad \text{subject to} \quad u \in X_n.$$

For fixed α, it is shown strong convergence in L^1 (weak convergence in L^p, $p \in (1, \infty)$) of subsequences of $\{u_n\}$, as $n \longrightarrow \infty$ to

$$v = argmin\left\{\|Au - y\|_Y^2 + \alpha\mathcal{R}(u)\right\} \quad \text{subject to} \quad u \in X.$$

Fitzpatrick, Keeling '97

Casas, Kunisch, Pola '99

Belik, Luskin '03

Neubauer '07

The space of bounded deformation functions

Let $\Omega = (0,1)^N$ and denote

$$BD(\Omega) := \left\{ \mathbf{u} \in L^1(\Omega; \mathbb{R}^N), E_{ij}(\mathbf{u}) \in M_1(\Omega), i, j = 1, \dots, N \right\},$$

with $\mathbf{u} = (u^1, \dots, u^N)$, where

$$E_{ij}\mathbf{u} := \frac{1}{2}(D_i u^j + D_j u^i)$$

is a (matrix-valued) measure with finite total variation in Ω.

Here $M_1(\Omega)$ denotes the space of bounded measures.

This space is useful in the mathematical theory of plasticity.

Temam, Strang '80

The space of bounded deformation functions

Several properties of $BD(\Omega)$

- It is the dual of a separable Banach space when provided with the norm

$$\|\mathbf{u}\|_{BD} = \|\mathbf{u}\|_{L^1(\Omega;\mathbb{R}^N)} + \underbrace{\sum_{i,j}^{N} \int_\Omega |E_{ij}(\mathbf{u})|}_{=:\int_\Omega |E\mathbf{u}|} .$$

- $BD(\Omega)$ is strictly larger than $BV(\Omega;\mathbb{R}^N)$.

The space of bounded deformation functions

- Consider the setting $X = BD(\Omega)$, τ the weak* topology on $BD(\Omega)$ and $Z = L^1(\Omega; \mathbb{R}^N)$.
- Let

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|_{L^1(\Omega; \mathbb{R}^N)} + \left| \int_\Omega |E\mathbf{u}| - \int_\Omega |E\mathbf{v}| \right|,$$

One cannot consider approximations by piecewise constant functions in the metric d.
Counterexample:

$$\mathbf{u}(x, y) = (-2y, x), \;\; \Omega = (0, 1) \times (0, 1).$$

Then

$$\int_\Omega |E\mathbf{u}| \neq \lim \int_\Omega |E\mathbf{u}_n| \,.$$

- One should try another type of approximation!

The space of essentially bounded functions

Assume that $\{\Omega_j\}$ is a decomposition of Ω in parallelepipeds. Consider

$$X_n = \left\{ u_n = \sum_{j=1}^{n} u^j \chi_{\Omega_j} : u^j \in \mathbb{R}\,, 1 \leq j \leq n \right\} .$$

Theorem

Assume that $h_n \to 0$ when $n \to \infty$. Then for every $u \in L^\infty(\Omega)$ one can find $u_n \in X_n$ such that

$$\lim_{n\to\infty} \left(\|u_n - u\|_{L^p} + \left| \|u_n\|_\infty - \|u\|_\infty \right| \right) = 0, \quad p \in [1, +\infty).$$

The regularization result in the L^∞ space

Let $X = L^\infty(\Omega)$, $Z = L^p(\Omega)$, $p \in (1, +\infty)$ and

$$d(u, v) = \|u - v\|_{L^p(\Omega)} + \left| \|u\|_\infty - \|v\|_\infty \right| .$$

Then, for every $u \in L^\infty(\Omega)$ there exists an approximating sequence of piecewise constant functions. Consequently, minimization of the discretized regularized problem is well–posed, stable, and convergent.

The piecewise constant regularizers approximate the $\mathcal{R}$-minimizing solution $\bar{u}$ on subsequences in the sense

$$u_l \overset{w^*}{\to} \bar{u}, \text{ and } \|u_l\|_\infty \to \|\bar{u}\|_\infty, \quad \text{as } l \to \infty.$$

Convergence in τ_d is stronger than in (τ and $\mathcal{R}$)

Proposition

If $\{u_k\} \subset L^\infty(\Omega)$ is such that

$$d(u_k, u) = \|u_k - u\|_{L^p} + |\|u_k\|_\infty - \|u\|_\infty| \to 0$$

then $u_k \xrightarrow{w^*} u$ and $\mathcal{R}(u_k) \to \mathcal{R}(u)$.

Remark

The converse implication is not true.

Counterexample in L^∞

Consider the Rademacher functions $f_n : [0,1] \to \{-1,1\}$,

$$f_n(t) = (-1)^{i+1} \text{ if } x \in \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right), \quad 1 \leq i \leq 2^n.$$

$f_n \stackrel{w^*}{\to} 0$ in $L^\infty([0,1])$, but $f_n \stackrel{L^1}{\nrightarrow} 0$.
Consider $g_n : [0,2] \to \mathbb{R}$,

$$g_n(t) = f_n(t), \text{ if } t \in [0,1]$$

and $g_n(t) = 1$ for $t \in [1,2]$.
Then $g_n \stackrel{w^*}{\to} \chi_{[1,2]}$ in $L^\infty([0,2])$ and $\|g_n\|_\infty = \|\chi_{[1,2]}\|_\infty = 1$.
However,
$\lim_{n\to\infty} \left(\left\|g_n - \chi_{[1,2]}\right\|_{L^1} + \left| \|g_n\|_\infty - \|\chi_{[1,2]}\|_\infty \right| \right) \neq 0.$
Cooper '09

Outline

Conclusions

- We investigate discretization of convex variational regularization in Banach spaces.
- Non-separable Banach spaces are of special interest: $BV(\Omega)$, $BD(\Omega)$, $L^\infty(\Omega)$, $W^{1,\infty}$.
- It is useful to consider a metric topology when approximating the non-separable Banach space by finite dimensional subspaces, rather than the norm topology.