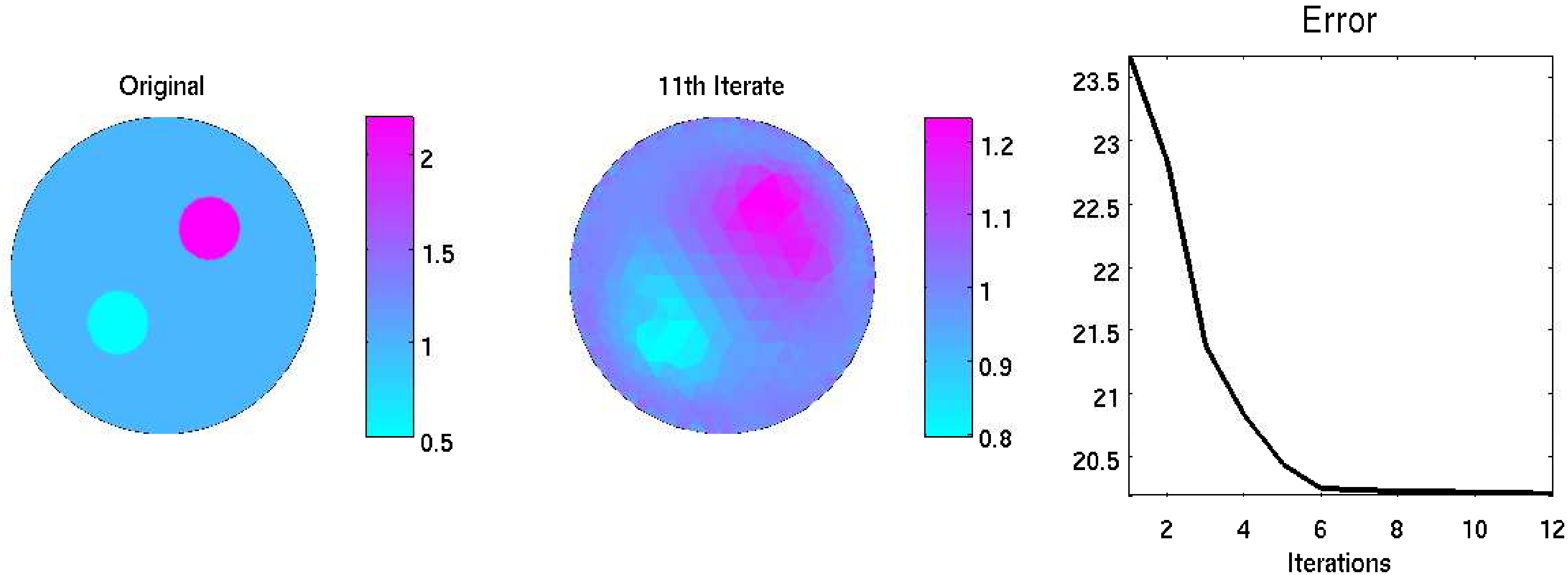
Newton Solvers for Electrical Impedance Tomography

Andreas Rieder (jointly with Armin Lechleiter, Palaiseau)

FAKULTÄT FÜR MATHEMATIK – INSTITUT FÜR ANGEWANDTE UND NUMERISCHE MATHEMATIK



Overview

Complete electrode model (CEM) in EIT

Inexact Newton solvers for nonlinear ill-posed problems

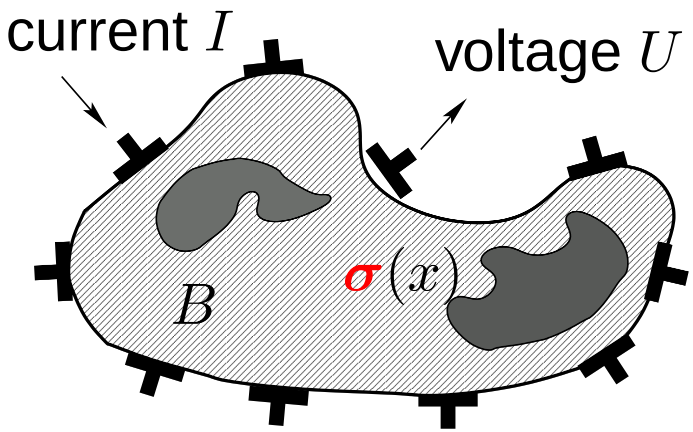
Tangential cone condition for CEM

Numerical examples

Summary

Complete electrode model (CEM) in EIT

Governing equation



electrodes E_j, $j = 1, \ldots, p$

$$\sigma \geq \sigma_0 > 0$$

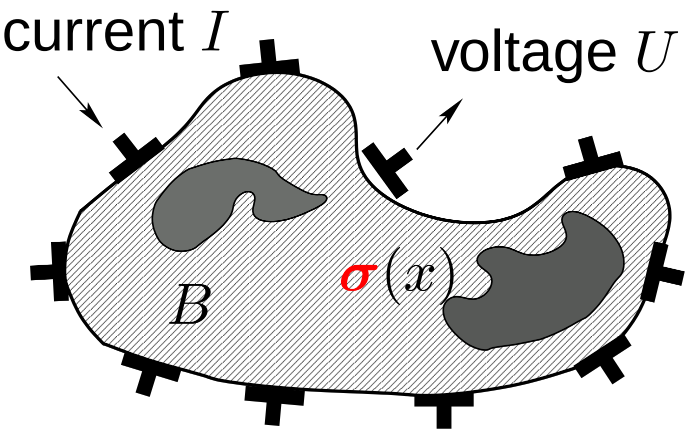
$$\mathrm{div}(\sigma \nabla u) = 0 \quad \text{in } B$$

$$u + z\,\sigma \partial_{\mathbf{n}} u = U \quad \text{on } E = \cup_j E_j$$

$$\sigma \partial_{\mathbf{n}} u = 0 \quad \text{on } \partial B \setminus \overline{E}$$

$$\frac{1}{|E_j|} \int_{E_j} \sigma \partial_{\mathbf{n}} u \, \mathrm{d}S = I|_{E_j}$$

Governing equation



electrodes E_j, $j = 1, \ldots, p$

$$\sigma \geq \sigma_0 > 0$$

$$\begin{aligned} \operatorname{div}(\sigma \nabla u) &= 0 \quad \text{in } B \\ u + z\,\sigma \partial_{\mathbf{n}} u &= U \quad \text{on } E = \cup_j E_j \\ \sigma \partial_{\mathbf{n}} u &= 0 \quad \text{on } \partial B \setminus \overline{E} \\ \frac{1}{|E_j|} \int_{E_j} \sigma \partial_{\mathbf{n}} u \,\mathrm{d}S &= I|_{E_j} \end{aligned}$$

Given $I \in \mathcal{E}_p := \operatorname{span}\{\chi_{E_1}, \ldots, \chi_{E_p}\} \cap L^2_\diamond(\partial B) \subset L^2_\diamond(\partial B)$ find $(u, U) \in H^1(B) \oplus \mathcal{E}_p$:

$$b_\sigma\big((u, U), (w, W)\big) = \int_{\partial B} I W \,\mathrm{d}S \quad \forall (w, W) \in H^1(B) \oplus \mathcal{E}_p.$$

where

$$b_\sigma\big((v, V), (w, W)\big) = \int_B \sigma \nabla v \cdot \nabla w \,\mathrm{d}x + \frac{1}{z} \int_E (v - V)(w - W) \,\mathrm{d}S$$

(Existence & Uniqueness: Somersalo, Cheney & Isaacson, 1992)

Inverse Problem

Find the conductivity σ from the observed **current-to-voltage mapping**

$$\Lambda = \Lambda_{\sigma,p} \colon \mathcal{E}_p \to \mathcal{E}_p, \quad \Lambda I = U.$$

Inverse Problem

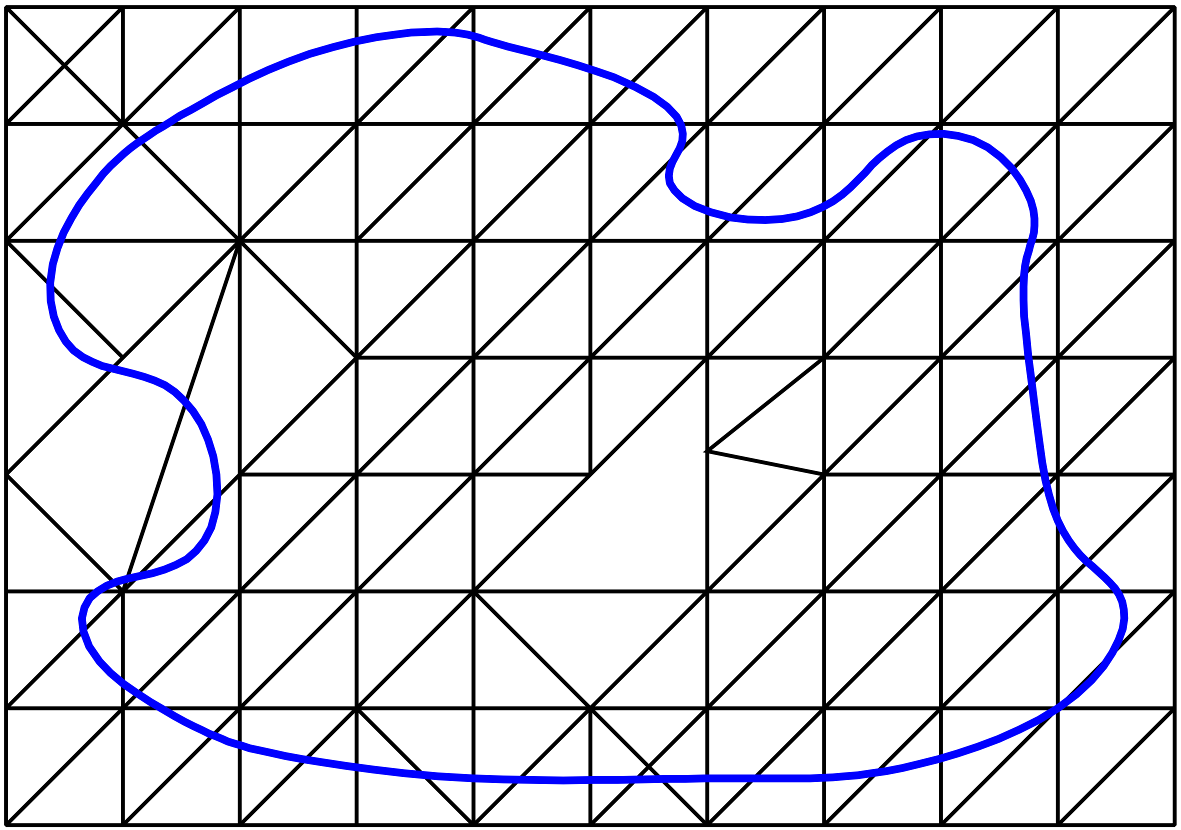
Find the conductivity σ from the observed **current-to-voltage mapping**

$$\Lambda = \Lambda_{\sigma,p} \colon \mathcal{E}_p \to \mathcal{E}_p, \quad \Lambda I = U.$$

CEM only provides $p(p-1)/2$ independent measurements.

- Therefore, we can only hope to recover conductivities whose number of DOF is at most the number of independent measurements.
- Consequently, we restrict the searched-for conductivities to a finite-dimensional space, namely to $V_{\mathcal{T}}$, a space of piecewise polynomials over a triangulation $\mathcal{T}$ of B.

Example: Constructing conductivity space $V_{\mathcal{T}}$



The conductivity σ is a polynomial on each triangle

Finite element discretization of the forward problem

For simulating numerically the action of Λ on I we use a conforming finite element discretization of the variational problem

$$b_\sigma\big((u,U),(w,W)\big) = \int_{\partial B} IW\,\mathrm{d}S \quad \forall (w,W) \in H^1(B) \oplus \mathcal{E}_p.$$

Finite element discretization of the forward problem

For simulating numerically the action of Λ on I we use a conforming finite element discretization of the variational problem

$$b_\sigma\big((u,U),(w,W)\big) = \int_{\partial B} IW \,\mathrm{d}S \quad \forall (w,W) \in H^1(B) \oplus \mathcal{E}_p.$$

Assumptions: Let

- B be a Lipschitz-domain,
- $S_{\mathcal{H}} \subset H^1(B)$ be a finite element space over a subdivsion $\mathcal{H}$ of B with discretization step size $h > 0$ ($\mathcal{T} \neq \mathcal{H}$ in general), and
- $\lim_{h\to 0} \operatorname{dist}(w, S_{\mathcal{H}}) = 0$ for all $w \in H^1(B)$.

Finite element discretization of the forward problem

For simulating numerically the action of Λ on I we use a conforming finite element discretization of the variational problem

$$b_\sigma\big((u,U),(w,W)\big) = \int_{\partial B} IW \,\mathrm{d}S \quad \forall (w,W) \in H^1(B) \oplus \mathcal{E}_p.$$

Assumptions: Let

- B be a Lipschitz-domain,
- $S_{\mathcal{H}} \subset H^1(B)$ be a finite element space over a subdivsion $\mathcal{H}$ of B with discretization step size $h > 0$ ($\mathcal{T} \neq \mathcal{H}$ in general), and
- $\lim_{h\to 0} \mathrm{dist}(w, S_{\mathcal{H}}) = 0$ for all $w \in H^1(B)$.

Define $\Lambda_h : \mathcal{E}_p \to \mathcal{E}_p$, a computable approximation to Λ, by $\Lambda_h I = U_h$ where $(u_h, U_h) \in S_{\mathcal{H}} \oplus \mathcal{E}_p$ solves

$$b_\sigma\big((u_h,U_h),(w,W)\big) = \int_{\partial B} IW \,\mathrm{d}S \quad \forall (w,W) \in S_{\mathcal{H}} \oplus \mathcal{E}_p.$$

Inverse Problem as operator equation

Forward mapping:

$$F_{p,h}\colon V_{\mathcal{T}}^{+} \subset V_{\mathcal{T}} \to \mathcal{L}(\mathcal{E}_p), \quad \sigma \mapsto \Lambda_h,$$

where $V_{\mathcal{T}}^{+} = \{\gamma \in V_{\mathcal{T}}\colon \gamma \geq \sigma_0\}$.

Inverse Problem as operator equation

Forward mapping:

$$F_{p,h}\colon V_{\mathcal{T}}^{+} \subset V_{\mathcal{T}} \to \mathcal{L}(\mathcal{E}_p), \quad \sigma \mapsto \Lambda_h,$$

where $V_{\mathcal{T}}^{+} = \{\gamma \in V_{\mathcal{T}}\colon \gamma \geq \sigma_0\}$.

Inverse Problem:
Find $\sigma \in V_{\mathcal{T}}^{+}$ from the observed current-to-voltage mapping $\widetilde{\Lambda}$ such that

$$F_{p,h}(\sigma) = \widetilde{\Lambda}.$$

Uniqueness:
Under which assumptions on $\mathcal{E}_p$, $V_{\mathcal{T}}$, and $S_{\mathcal{H}}$ is $\sigma \in V_{\mathcal{T}}^{+}$ uniquely determined by Λ_h?

Local answer:
The solution is locally unique whenever $\mathcal{E}_p$ and $S_{\mathcal{H}}$ are rich enough.

Remark: Borcea, Druskin and Vasquez (*Inverse Problems* 24(3), 2008) have shown recently injectivity for a radial FD discretization.

Inexact Newton solvers for nonlinear ill-posed problems

Abstract setting

$$F\colon \mathrm{D}(F) \subset X \to Y, \quad X, Y \text{ Hilbert spaces}$$

$$F(x) = y^\delta$$

where $\|y - y^\delta\|_Y \le \delta$, $y = F(x^+)$, and $F(x) = y$ locally ill-posed in x^+.

Let x_n be an approximation to x^+: $\quad x_{n+1} = x_n + s_n^{\mathrm{N}}$

The exact Newton step $s_n^{\mathsf{e}} = x^+ - x_n$ satisfies $(A_n := F'(x_n))$

$$A_n s_n^{\mathsf{e}} = y - F(x_n) - E(x^+, x_n)$$

Abstract setting

$$F\colon \mathsf{D}(F) \subset X \to Y, \quad X, Y \text{ Hilbert spaces}$$

$$F(x) = y^\delta$$

where $\|y - y^\delta\|_Y \le \delta$, $y = F(x^+)$, and $F(x) = y$ locally ill-posed in x^+.

Let x_n be an approximation to x^+: $\quad x_{n+1} = x_n + s_n^{\mathrm{N}}$

The exact Newton step $s_n^{\mathsf{e}} = x^+ - x_n$ satisfies ($A_n := F'(x_n)$)

$$A_n s_n^{\mathsf{e}} = y - F(x_n) - E(x^+, x_n)$$

$\Longrightarrow$ Determine s_n^{N} as regularized solution of

$$A_n s = b_n^\delta, \quad b_n^\delta := y^\delta - F(x_n)$$

Abstract setting

$$F\colon \mathrm{D}(F) \subset X \to Y, \quad X, Y \text{ Hilbert spaces}$$

$$F(x) = y^\delta$$

where $\|y - y^\delta\|_Y \leq \delta$, $y = F(x^+)$, and $F(x) = y$ locally ill-posed in x^+.

Let x_n be an approximation to x^+: $\quad x_{n+1} = x_n + s_n^{\mathrm{N}}$

The exact Newton step $s_n^{\mathrm{e}} = x^+ - x_n$ satisfies $(A_n := F'(x_n))$

$$A_n s_n^{\mathrm{e}} = y - F(x_n) - E(x^+, x_n)$$

$\Longrightarrow$ Determine s_n^{N} as regularized solution of

$$A_n s = b_n^\delta, \quad b_n^\delta := y^\delta - F(x_n)$$

Let $\{s_{n,m}\}_{m\in\mathbb{N}}$ be a regularizing sequence. Then, $s_n^{\mathrm{N}} = s_{n,m_n}$.

Inexact Newton scheme

$\texttt{REGINN}(x_{N(\delta)}, R, \{\mu_n\})$

$n := 0;\ \ x_0 := x_{N(\delta)};$

$\texttt{while}\ \ \|b_n^\delta\|_Y > R\delta\ \ \texttt{do}$

$\{\quad m := 0,\ \ s_{n,0} = 0;$

$\qquad \texttt{repeat}$

$\qquad\quad m := m + 1;$

$\qquad\quad \texttt{compute}\ \ s_{n,m}\ \ \texttt{from}\ \ A_n s = b_n^\delta;$

$\qquad \texttt{until}\ \ \|A_n s_{n,m} - b_n^\delta\|_Y < \mu_n \|b_n^\delta\|_Y$

$\qquad x_{n+1} := x_n + s_{n,m};$

$\qquad n := n + 1;$

$\}$

$x_{N(\delta)} := x_n;$

$$m_n = \min\left\{m \in \mathbb{N} : \|A_n s_{n,m} - b_n^\delta\|_Y < \mu_n \|b_n^\delta\|_Y\right\}$$

Assumptions on $\{s_{n,m}\}$

For the analysis of `REGINN` we restrict ourselves to the following four iterative regularization schemes

- Landweber iteration

$$s_{n,m} = s_{n,m-1} + \lambda A_n^*(b_n^\delta - A_n s_{n,m-1}), \quad s_{n,0} = 0$$

- Implicit iteration

$$s_{n,m} = \big(A_n^* A_n + \lambda I\big)^{-1}(\lambda s_{n,m-1} + A_n^* b_n^\delta), \quad s_{n,0} = 0$$

- steepest decent

$$s_{n,m} = s_{n,m-1} + \frac{\|A_n^* r_{m-1}\|_X^2}{\|A_n A_n^* r_{m-1}\|_Y^2} A_n^*(b_n^\delta - A_n s_{n,m-1}), \quad s_{n,0} = 0$$

- cg-method

$$s_{n,m} = s_{n,m-1} + A_n^* P_m(A_n A_n^*, b_n^\delta) b_n^\delta, \quad s_{n,0} = 0$$

Structural assumption on the nonlinearity

Assume (modified) tangential cone condition

$$\|F(v) - F(w) - F'(w)(v - w)\|_Y \leq L\|F'(w)(v - w)\|_Y$$

for one $L < 1$ and for all $v, w \in B_r(x^+) \subset \mathsf{D}(F)$.

Monotonicity and Convergence

Theorem: Let L, R and $\{\mu_n\}$ be "suitable" and start `REGINN` with $x_0 \in B_r(x^+)$. Then, all iterates $\{x_1, \ldots, x_{N(\delta)}\}$ are well-defined and

$$\|x^+ - x_n\|_X < \|x^+ - x_{n-1}\|_X, \quad n = 1, \ldots, N(\delta),$$

and, if x^+ is unique in $B_r(x^+)$,

$$\lim_{\delta \to 0} \|x^+ - x_{N(\delta)}\|_X = 0.$$

- Norm convergence with rates can be shown under stronger assumptions.
- The tolerances $\{\mu_n\}$ are adapted dynamically during the iteration incorporating information on the local degree of ill-posedness.

Tangential cone condition for CEM

Tangential cone condition

Forward mapping: $F_{p,h} : V_{\mathfrak{T}}^{+} \subset V_{\mathfrak{T}} \to \mathfrak{L}(\mathcal{E}_p), \quad \sigma \mapsto \Lambda_h,$

Geometric assumption on the electrodes $\{E_j^p\}_{j=1}^p$:

$$\lim_{p\to\infty} \sum_{j=1}^{p} \left|E_j^p\right| = \left|\partial B\right| \quad \text{sufficiently fast.}$$

Theorem: If $\sigma \in \operatorname{int}(V_{\mathfrak{T}}^{+})$ then there is an open ball $B_r(\sigma) \subset \operatorname{int}(V_{\mathfrak{T}}^{+})$ about σ of radius $r > 0$ such that, for all τ, $\gamma \in B_r(\sigma)$,

$$\left\|F_{p,h}(\tau) - F_{p,h}(\gamma) - F'_{p,h}(\gamma)[\tau-\gamma]\right\|_{\mathfrak{L}(\mathcal{E}_p)} \lesssim \|\tau-\gamma\| \left\|F'_{p,h}(\gamma)[\tau-\gamma]\right\|_{\mathfrak{L}(\mathcal{E}_p)}$$

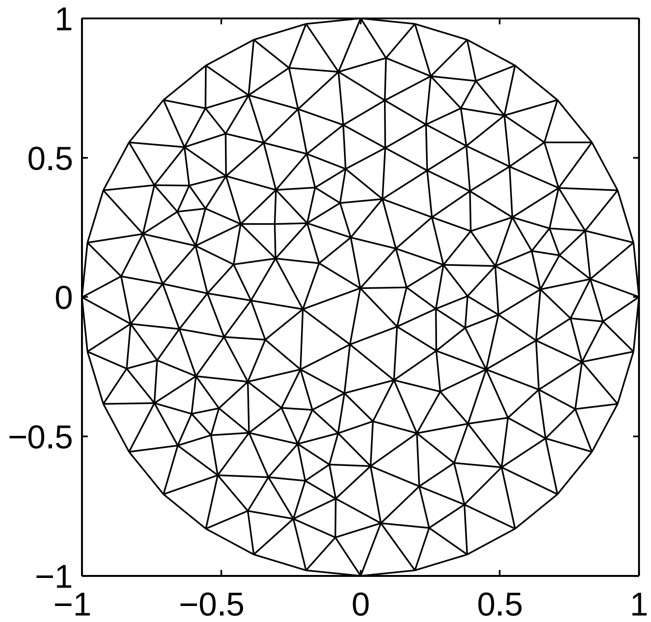
uniformly for all $p \geq \mathbf{p}_{\mathfrak{T}}(r + \|\sigma\|)$ and all $0 < h \leq h_{\max}(r + \|\sigma\|)$, that is, neither the involved constant nor the radius r depend on p or on h.

Consequence

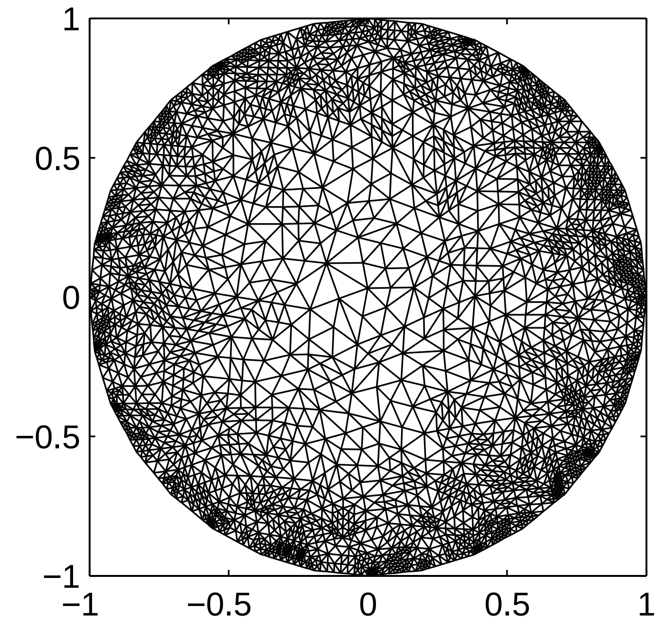
The inexact Newton solver `REGINN` converges locally when applied to the inverse EIT problem under CEM (provided the spaces $\mathcal{E}_p$ and $S_{\mathcal{H}}$ are rich enough).

Numerical examples

FE meshes for 16 electrodes

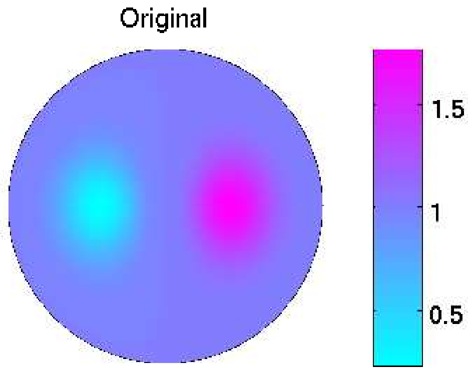


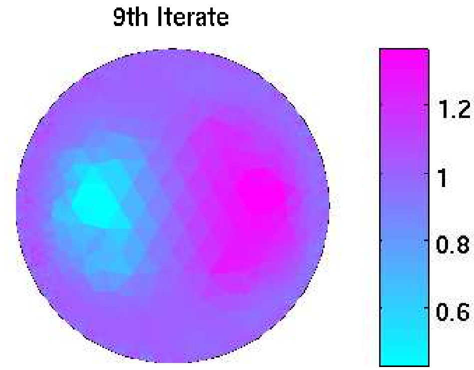
Finite element mesh $\mathcal{T}$

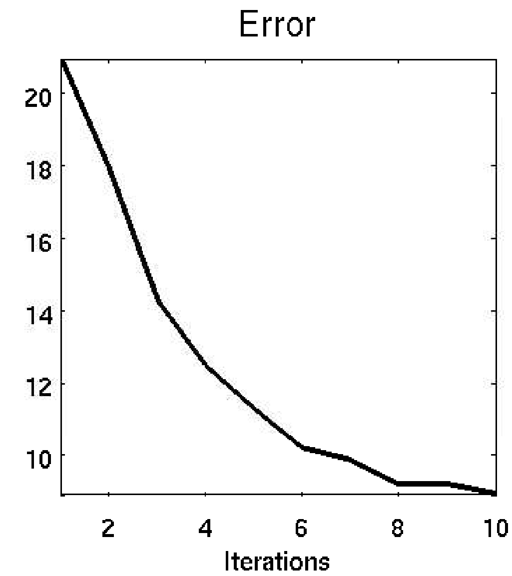


Finite element mesh $\mathcal{H}$

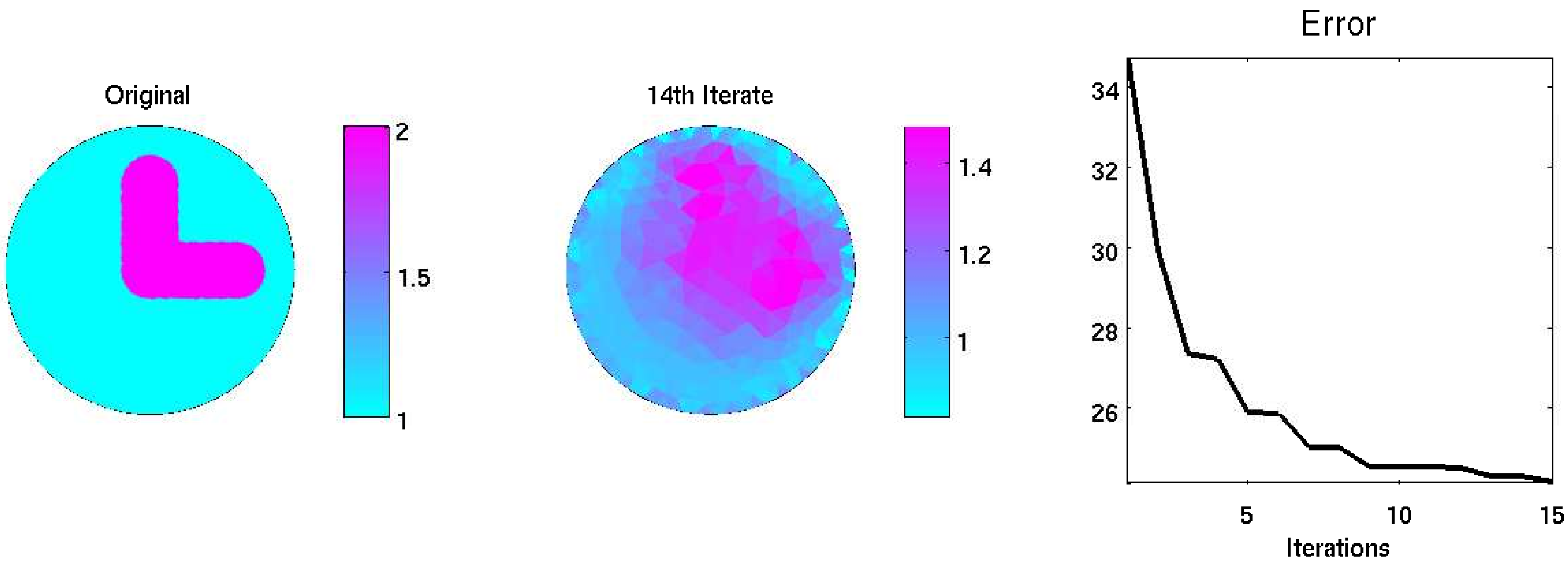
64 electrodes, 1% artificial noise







32 electrodes, 0.5% artificial noise



n	i_n	μ_n	L^2-error in %	n	i_n	μ_n	L^2-error in %
0	0	—	34.72	8	6	0.938	24.52
1	2	0.799	29.92	9	1	0.989	24.52
2	4	0.799	27.35	10	2	0.958	24.50
3	3	0.899	27.20	11	1	0.978	24.50
4	6	0.871	25.88	12	5	0.948	24.30
5	2	0.935	25.84	13	1	0.989	24.29
6	6	0.906	25.01	14	5	0.958	24.15
7	1	0.968	25.00	15	1	0.991	24.15

Bibliographical notes

- A. Lechleiter, A. R.
  Towards a general convergence theory for inexact Newton regularizations
  Numer. Math. 114 (2010).
- A. Lechleiter, A. R.
  Newton regularization for impedance tomography: convergence by local injectivity
  Inverse Problems 24 (2008).
- A. Lechleiter, A. R.
  Newton regularizations for impedance tomography: a numerical study,
  Inverse Problems 22 (2006).

Summary

What to remember from this talk

- We presented the inverse problem of EIT under CEM. Since CEM provides only finitely many independent measurements for reconstructing the conductivity, we considered a fully discrete setting.
- For solving nonlinear ill-posed problems we introduced a Newton algorithm and gave a local convergence analysis under TCC.
- Finally, we argued that TCC holds asymptotically for the fully discrete CEM. Hence, the presented convergence results apply to CEM. Numerical results illustrated the performance of our algorithm.

What to remember from this talk

- We presented the inverse problem of EIT under CEM. Since CEM provides only finitely many independent measurements for reconstructing the conductivity, we considered a fully discrete setting.
- For solving nonlinear ill-posed problems we introduced a Newton algorithm and gave a local convergence analysis under TCC.
- Finally, we argued that TCC holds asymptotically for the fully discrete CEM. Hence, the presented convergence results apply to CEM. Numerical results illustrated the performance of our algorithm.

What to remember from this talk

- We presented the inverse problem of EIT under CEM. Since CEM provides only finitely many independent measurements for reconstructing the conductivity, we considered a fully discrete setting.
- For solving nonlinear ill-posed problems we introduced a Newton algorithm and gave a local convergence analysis under TCC.
- Finally, we argued that TCC holds asymptotically for the fully discrete CEM. Hence, the presented convergence results apply to CEM. Numerical results illustrated the performance of our algorithm.

What to remember from this talk

- We presented the inverse problem of EIT under CEM. Since CEM provides only finitely many independent measurements for reconstructing the conductivity, we considered a fully discrete setting.
- For solving nonlinear ill-posed problems we introduced a Newton algorithm and gave a local convergence analysis under TCC.
- Finally, we argued that TCC holds asymptotically for the fully discrete CEM. Hence, the presented convergence results apply to CEM. Numerical results illustrated the performance of our algorithm.

Thank you for your attention!