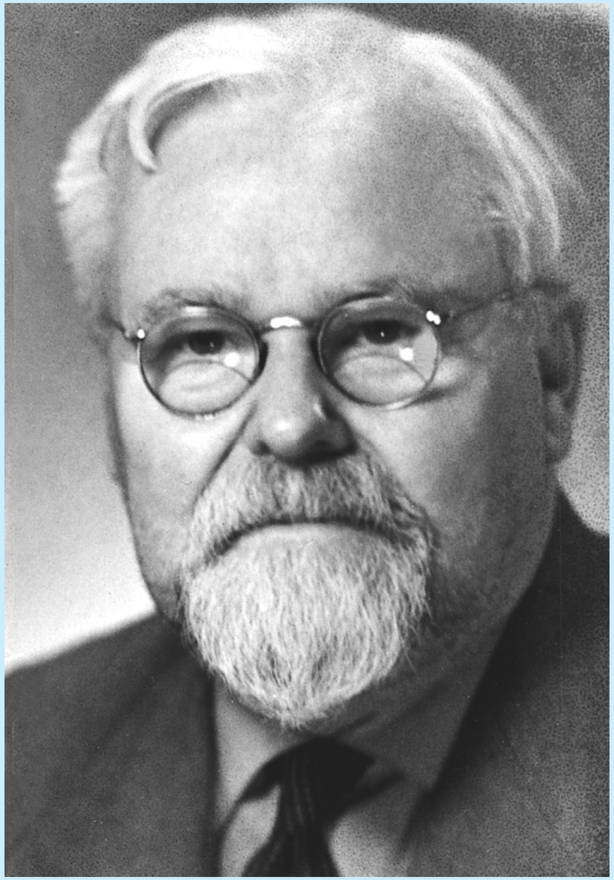
ERROR ESTIMATION FOR ILL-POSED PROBLEMS WITH A PRIORI INFORMATION

Anatoly Yagola

Professor, Dr. Sc., Department of Mathematics, Faculty of Physics, Moscow State University, Moscow 119992, Russia, e-mail: yagola@inverse.phys.msu.ru



Main publications:

1. Tikhonov, A.N., Goncharsky, A.V., Stepanov, V.V. and Yagola, A.G. (1995). *Numerical methods for the solution of ill-posed problems*. Kluwer Academic Publishers, Dordrecht.
2. Tikhonov, A.N., Leonov, A.S. and Yagola, A.G. (1998). *Nonlinear ill-posed problems*. Chapman and Hall, London.
3. Kochikov, I.V., Kuramshina, G.M., Pentin, Yu.A. and Yagola, A.G. (1999). *Inverse problems of vibrational spectroscopy*. VSP, Utrecht, Tokyo.

REFERENCES

1. Yagola A.G., Dorofeev K.Yu. Sourcewise representation and a posteriori error estimates for ill-posed problems. - In "A. G. Ramm, P. N. Shivakumar, A. V. Strauss (Eds.), Fields Institute Communications: Operator Theory and Its Applications", Providence, RI: American Mathematical Society, 2000, v. 25, pp. 543-550.
2. Titarenko V.N., Yagola A.G. Cauchy problems for Laplace equation on compact sets. – Inverse Problems in Engineering, 2002, v. 10, N 3, p. 235-254.
3. Dorofeev K.Yu., Yagola A.G. The method of extending compacts and a posteriori error estimates for nonlinear ill-posed problems. – Journal of Inverse and Ill-Posed Problems, 2004, v. 12, No 6, pp. 627-636.

4. Nikolaeva N.N., Titarenko V.N., Yagola A.G. An error estimation of a solution of Abel equation. – Numerical Functional Analysis and Optimization, 2004, v. 25, N 3-4, pp. 259-269.
5. Yagola A., Titarenko V. Using a priori information about a solution of an ill-posed problem for constructing regularizing algorithms and their applications. - Inverse Problems in Science and Engineering, 2007, v. 15, No 1, pp. 3 – 17.
6. Nikolaeva N.N., Ruchkin S.V., Rychagov M.N., Yagola A.G. Symmetric velocity profiles reconstruction in channels with a circular cross-section by ultrasonic flow measurements. – Inverse Problems in Science and Engineering, 2006, v. 14, No 7, pp. 711 – 723.
7. Titarenko V., Yagola A. Error estimation for ill-posed problems on piecewise convex functions and sourcewise represented sets. - Journal of Inverse and Ill-posed Problems, 2008, Vol. 16, No. 6, pp. 625–638.
8. Titarenko V., Yagola A. Solution of ill-posed problems on sets of functions convex along all lines parallel to coordinate axes. - Journal of Inverse and Ill-posed Problems, 2008, Vol. 16, No. 8, pp. 805–824.

Introduction

$$A\bar{z} = \bar{u} \qquad \bar{z} \in Z, \bar{u} \in U \qquad (1)$$

$A : Z \to U$ is a linear operator,

Z, U are Hilbert spaces (for simplicity).

The problem (1) is called well-posed on the class of its "admissible" data if for any pair $\{A, \bar{u}\}$ from the set of "admissible" data the solution of (1):

1) exists,
2) is unique,
3) continuously depends on errors in A and $\bar{u}$ (is stable).

Stability means that if instead of $\{A, \bar{u}\}$ we are given "admissible" $\{A_h, u_\delta\}$ such that $\|A_h - A\| \le h$, $\|u_\delta - \bar{u}\| \le \delta$, the approximate solution converges to the exact one as $h, \delta \to 0$. The numbers h and δ are error estimates for the approximate data $\{A_h, u_\delta\}$ of (1) with the exact data $\{A, \bar{u}\}$. Denote $\eta \equiv (h, \delta)$.

If at least one of the mentioned requirements is not met, then the problem (1) is called ill-posed.

APPLICATIONS

A lot of inverse problems in astrophysics, geophysics, nondestructive testing, computerized tomography, etc., etc, are Ill posed. Simplest mathematical examples are Fredholm equations of the 1^{st} kind, SLAE with perturbed matrices, numerical differentiation and very many others.

As a generalized solution, it is often taken the so-called normal pseudosolution $\tilde{z}$. It exists and is unique for any exact data of the problem (1) if $A \in L(Z,U)$, $\bar{u} \in R(A) \oplus R^{\perp}(A)$, $\tilde{z} = A^{+}\bar{u}$. Here $R(A)$ and $R^{\perp}(A)$ denote the ranges of the operator A and its orthogonal complement in U , and A^{+} stands for the operator pseudoinverse to A . Below we find $\bar{z}$ as a normal pseudosolution, i.e., $\bar{z} = \tilde{z}$.

What is to solve an ill-posed problem?

Tikhonov answered: to solve an ill-posed problem means to produce a map (regularizing algorithm) $R(A_h, u_\delta, \eta)$ such that

1) brings an element $z_\eta = R(A_h, u_\delta, \eta)$ into correspondence with any data $\{A_h, u_\delta, \eta\}$, $A_h \in L(Z,U)$, $u_\delta \in U$ of the problem (1);
2) has the convergence property $z_\eta \to \bar{z} = A^+\bar{u}$ as $\eta \to 0$, $\bar{u} \in R(A) \oplus R^\perp(A)$.

The mathematical problem is regularizable if there exists a regularizing algorithm. For well-posed problems such algorithm exists evidently (for the problem (1), $R = A^+$).

So, all mathematical problems can be classified in the following way:

1) well-posed problems;

2) ill-posed (Tikhonov) regularizable problems;

3) ill-posed nonregularizable problems.

Tikhonov's works in 1963 not only clearly define the meaning of solving ill-posed problem (1), but also give a particular RA $R(h, \delta, A_h, u_\delta)$ for solving (1). The algorithm is known as Tikhonov regularizing method and uses a parametrical family of elements $z^\alpha = z^\alpha(A_h, u_\delta) \in Z$ minimizing the Tikhonov functional $M^\alpha[z] = \|A_h z - u_\delta\|^2 + \alpha\|z\|^2$ in Z. Here $\alpha > 0$ is a regularization parameter. The regularizing procedure is actually selecting a parameter $\alpha = \alpha(h, \delta, A_h, u_\delta)$ such that it ensures the convergence of the approximate solution to z^e when $h, \delta \to 0$.

Is it possible to construct a regularizing algorithm that does not depend on h, δ?

Theorem 1: Let $R(A_h, u_\delta)$ be a map of the set $L(Z,U)\otimes U$ into Z. If $R(A_h, u_\delta)$ is a regularizing algorithm (not depending explicitly on η), then the map $P(A,\bar{u}) = A^+\bar{u}$ is continuous on its domain $L(Z,U)\otimes\left(R(A)\oplus R^\perp(A)\right)$.

Proof The second condition in the definition of RA implies in $R(A,\bar{u}) = A^+\bar{u} = P(A,\bar{u})$ valid for each $(A,\bar{u}) \in L(Z,U)\otimes\left(R(A)\oplus R^\perp(A)\right)$ and the convergence $P(A_h,u_\delta) = R(A_h,u_\delta) \to A^+\bar{u} = P(A,\bar{u})$ as $h,\delta \to 0$ valid for $(A,\bar{u}),(A_h,u_\delta) \in L(Z,U)\otimes\left(R(A)\oplus R^\perp(A)\right)$.

The map $P(A,u)$ is continuous on $L(Z,U)\otimes\left(R(A)\oplus R^\perp(A)\right) \subset L(Z,U)\otimes U$.

It is clear from Theorem 1 that a regularizing algorithm not using h and δ explicitly can only exist for problems (1) well-posed on the set of the data. The theorem generalized the assertion proved by Bakushinsky.

The necessity for the regularizing algorithm to be dependable on the data errors was mentioned in the latest works of Tikhonov related with solving unstable systems of linear algebraic equations.

Due to the trivial fact that any linear operator acting in finite-dimensional Euclidian spaces is bounded, the knowledge of δ is useful but not necessary for constructing regularizing algorithms for solving systems of linear algebraic equations.

The main result is that it is impossible to construct stable methods for ill-posed systems of linear algebraic equations without knowledge of h!!!

Very simple example is the following:

let Z = U = R^2 and we have the system of linear algebraic equations:

$$\begin{pmatrix} x + y = 1 \\ x + y = 1 \end{pmatrix}$$

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, A = \begin{pmatrix} 1 1 \\ 1 1 \end{pmatrix}, z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The normal solution of the system is (x, y) = (½, ½). Let now instead of A we have an approximate matrix For any ε ≠ 0 this system has the unique solution $(x_\varepsilon, y_\varepsilon) = (0, 1)$ that doesn't tends to (x, y) = (½, ½) when $\varepsilon \to 0$.

$$A_\varepsilon = \begin{pmatrix} 1+\varepsilon & 1 \\ 1 & 1 \end{pmatrix}, \varepsilon \neq 0.$$

For any ε ≠ 0 this system has the unique solution $(x_\varepsilon, y_\varepsilon) = (0, 1)$ that doesn't tends to (x, y) = (½, ½) when $\varepsilon \to 0$.

Let us consider now the second example:

$$\begin{pmatrix} x + y = 3/2 \\ x + y = 1/2 \end{pmatrix}$$

$$u = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}, A = \begin{pmatrix} 1\,1 \\ 1\,1 \end{pmatrix}, z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The system has no solutions, its normal pseudosolution is

(x, y) = (½, ½).

It is very easy to understand what has happened if we have errors in the matrix A. These two examples are simple illustrations of the unstability of the pseudoinversion procedure.

It is very curious that the most popular error free methods cannot solve well-posed problems also! As the first example we consider so-called the "L-curve method" (P.C. Hansen). In this method the regularization parameter in Tikhonov functional α is selected as a point maximum curvature of the L-curve $\{(\ln\|A_h z^\alpha - u_\delta\|, \ln\|z^\alpha\|): \alpha > 0\}$.

But this method cannot be used for the solution of ill-posed problems because the L-curve doesn't depend on h and δ (see the theorem). Everybody can easily prove that this method is inapplicable to solving the simplest finite-dimensional well-posed problems (e.g., equation z=1).

Another very popular "error free" method is GCV – the generalized cross-validation method (G. Wahba), where $\alpha(A_h, u_\delta)$ is found as the point of the global minimum of the function

$$G(\alpha) = \|(A_hA_h^* + \alpha I)^{-1}u_\delta\| \, [tr(A_hA_h^* + \alpha I)^{-1}]^{-1}, \; \alpha \geq 0.$$

This method is not applicable for the solution of ill-posed problems including ill-posed systems of linear algebraic equations (see the theorem above). It is possible construct well-posed systems of linear algebraic equations the GCV method failed for their solution. Let $Z = U = R^2$,

$$u = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \; A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Here $h > 0$. Very easy to calculate the GCV solution z_{gcv} and prove that it converges to $(-1/3, -1/3)^*$ instead of $z^e = (-3, 1)^*$ when $h \to 0$.

A lot of other examples could be found in a paper by

V. Titarenko and A. Yagola (2000) Vestnik Moskovskogo Universiteta, ser. 3. Fizika. Astronomia **(4)**, 15 (in Russian).

Is it possible to estimate an error of an approximate solution of an ill-posed problem?

The answer is negative. This main and very important result was obtained by Bakushinsky.

Assume $A_h = A$. Let $R(u_\delta, \delta)$ be a RA. Denote by

$$\Delta(R, \delta, \bar{z}) = \sup\left\{\left\| R(u_\delta, \delta) - \bar{z} \right\| : \forall u_\delta \in U, \left\| A\bar{z} - u_\delta \right\| \le \delta \right\}$$

the error of a solution of (1) at the point $\bar{z}$ using the algorithm R. If (1) is regularizable by a continuous map R and there is an error estimate, which is uniform on D

$$\sup\left\{\Delta(R, \delta, \bar{z}) : \bar{z} \in D\right\} \le \varepsilon(\delta) \to 0 \quad as \quad \delta \to 0$$

then the restriction of A^{-1} to $AD \subset U$ is continuous on AD .

Consider the results obtained by Vinokurov.

Let A be a linear continuous injective operator acting in Banach space Z and the inverse operator A^{-1} be unbounded on $D\left(A^{-1}\right)$. Suppose that $\varphi(\delta)$ is an arbitrary positive function such that $\varphi(\delta) \to 0$ as $\delta \to 0$, and R is an arbitrary method to solve the problem.

The following equality holds for elements $\bar{z}$ except maybe for a first category set in Z:

$$\lim_{\delta \to 0} \sup \left\{ \frac{\Delta\left(R, \delta, \bar{z}\right)}{\varphi(\delta)} \right\} = \infty$$

A uniform error estimate can only exist on a first category subset in Z.

CONCLUSIONS

- For ill-posed problems we cannot estimate an error of an approximate solution. We cannot choose the “best” RA comparing convergence rates. So we recommend at the beginning to study all physical and technical properties of an unknown solutions – to use all a priori information.

A compact set is a typical example of the first category set in a Banach space Z . For this set special regularizing algorithms may be used and a uniform error estimation may be constructed. Clearly, a uniform error estimate exists only for well-posed problems.

A posteriori error estimation

For some ill-posed problems it is possible to find a so-called *a posteriori* error estimation.

Let A be an exact injective operator with closed graph and Z be a σ -compact space.

Introduce a function $\kappa(u_\delta, \delta)$ such that $\forall \bar{z} \in Z$

$\exists \delta(\bar{z}) > 0$, $\forall \delta \in (0, \delta(\bar{z})]$, $\forall u_\delta \in U$, $\|u_\delta - \bar{u}\| \le \delta$:

$$\|\bar{z} - R(u_\delta, \delta)\| \le \kappa(u_\delta, \delta)$$

The function $\kappa(u_\delta, \delta)$ is an *a posteriori* error estimation for the problem (1), if $\kappa(u_\delta, \delta) \to 0$ as $\delta \to 0$

Tikhonov variational approach

Let Z, U be Hilbert spaces, $D \subset Z$ be a closed convex set of *a priori* constraints such that $0 \in D$, A , A_h be linear operators. On a set $\{A_h, u_\delta, \eta\}$ introduce the Tikhonov's functional:

$$M^\alpha[z] = \|A_h z - u_\delta\|^2 + \alpha \|z\|^2$$

where $\alpha > 0$ is a regularization parameter.

$$\inf\{M^\alpha[z] : z \in D\} \qquad (2)$$

For any $\alpha > 0$, $u_\delta \in U$ and bounded linear operator A_h the problem (2) is solvable and has a unique solution $z_\eta^\alpha \in D$.

A priori choice of α

A regularizing algorithm using the extreme problem (2) for $M^{\alpha}[z]$: to construct $\alpha(\eta)$ such that $z_{\eta}^{\alpha(\eta)} \to \bar{z}$ as $\eta \to 0$.

If A is an injective operator, $\bar{z} \in D$ and $\alpha(\eta) \to 0$, $\frac{(h+\delta)^2}{\alpha(\eta)} \to 0$ as $\eta \to 0$, then $z_{\eta}^{\alpha(\eta)} \to \bar{z}$ as $\eta \to 0$, i.e., there is the *a priori* choice of α .

A posteriori choice of α

The incompatibility measure of (1) on D:

$$\mu_\eta(u_\delta, A_h) = \inf\left\{\|A_h z - u_\delta\| : z \in D\right\}$$

Let it can be computed with an error $\kappa > 0$, i.e., instead of $\mu_\eta(u_\delta, A_h)$ there is $\mu_\eta^\kappa(u_\delta, A_h)$ such that

$$\mu_\eta(u_\delta, A_h) \le \mu_\eta^\kappa(u_\delta, A_h) \le \mu_\eta(u_\delta, A_h) + \kappa$$

The generalized discrepancy:

$$\rho_\eta^\kappa(\alpha) = \|A_h z_\eta^\alpha - u_\delta\|^2 - \left(\delta + h\|z_\eta^\alpha\|\right)^2 - \left(\mu_\eta^\kappa(u_\delta, A_h)\right)^2$$

The generalized discrepancy $\rho_\eta^\kappa(\alpha)$ is continuous and monotonically non-decreasing for $\alpha > 0$.

The generalized discrepancy principle to choose the regularization parameter:

1) If the condition $\|u_\delta\|^2 > \delta^2 + \left(\mu_\eta^\kappa(u_\delta, A_h)\right)^2$ is not just, then $z_\eta = 0$ is an approximate solution of (1);
2) If the condition $\|u_\delta\|^2 > \delta^2 + \left(\mu_\eta^\kappa(u_\delta, A_h)\right)^2$ is just, then the generalized discrepancy has a positive zero α^* and $z_\eta = z_\eta^{\alpha^*}$.

If A is an injective operator, then $\lim_{\eta \to 0} z_\eta = \bar{z}$. Otherwise, $\lim_{\eta \to 0} z_\eta = z^*$, where z^* is the normal solution of (1), i.e., $\|z^*\| = \inf\{\|z\| : z \in D, Az = \bar{u}\}$.

If A, A_h are bounded linear operators, D is a closed convex set, $0 \in D$, $\bar{z} \in D$, the generalized discrepancy principle are equivalent to the generalized discrepancy method:

find

$$\inf \left\{ \|z\| : z \in D, \|A_h z - u_\delta\|^2 \le \left(\delta + h\|z\|\right)^2 + \left(\mu_\eta^\kappa(u_\delta, A_h)\right)^2 \right\}$$

Inverse problem for the heat conduction equation

$$\begin{cases} w_t = a^2 w_{xx} & x \times t \in (0,l) \times (0,T) \\ w(0,t) = 0 \\ w(l,t) = 0 \end{cases}$$

There is a function $u_\delta(\xi) \equiv w(\xi,T) \in L^2[0,l]$, we want to find $z(x) \equiv w(x,0) \in W_2^1[0,l]$ such that $z(x) \to \bar{z}(x)$ as $\eta \to 0$.

We can write that

$$\|u(\xi)\|^2 = \int_0^l |u(\xi)|^2 d\xi, \quad \|z(x)\|^2 = \int_0^l \left(|z(x)|^2 + \left| \frac{\partial z(x)}{\partial x} \right|^2 \right) dx$$

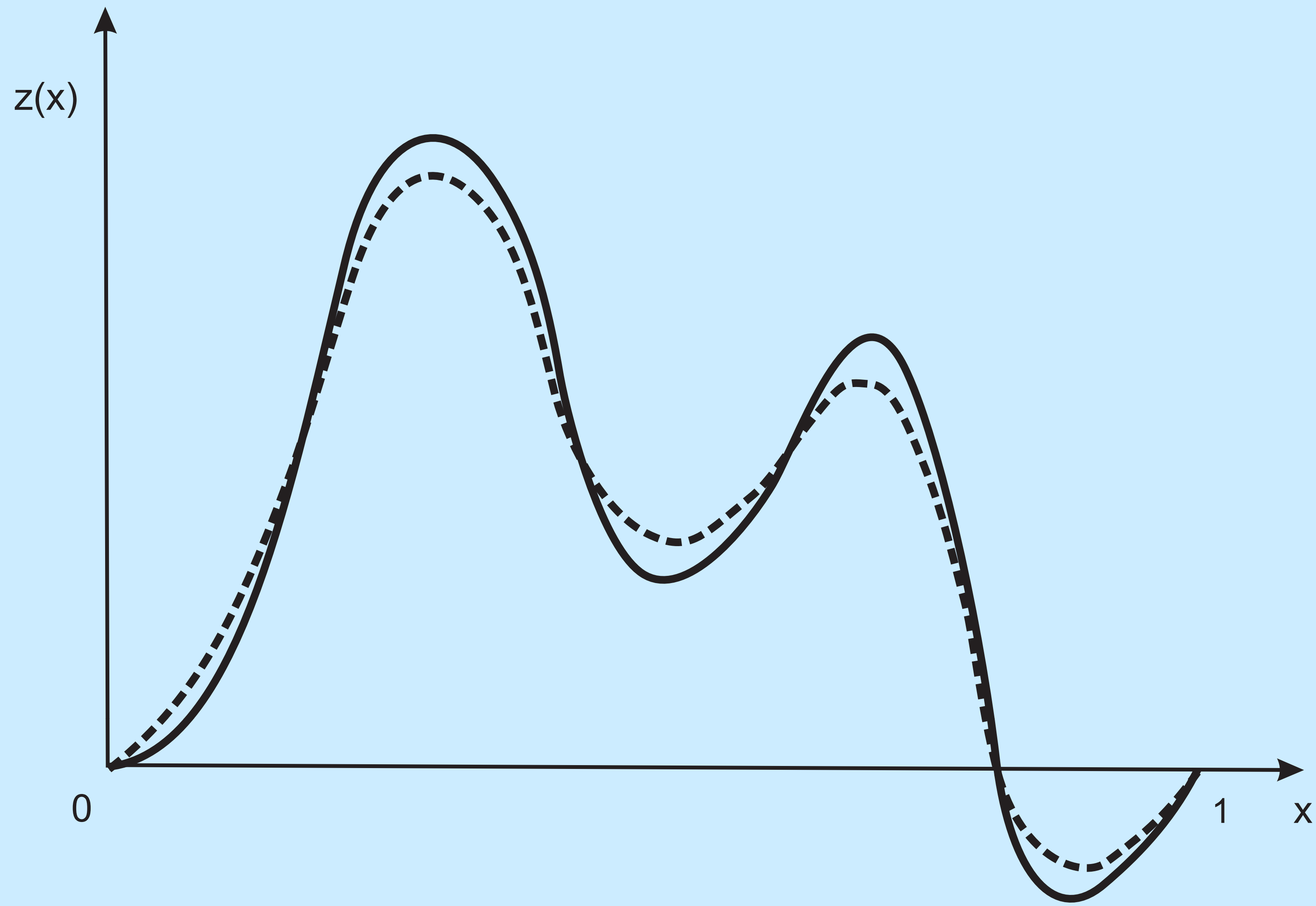
The problem may be written in the form of integral equation

$$u(\xi) = \int_0^l G(\xi, x, T) z(x) dx$$

where $G(\xi, x, t)$ is the Green function:

$$G(\xi, x, t) = \tfrac{2}{l} \sum_{n=1}^{+\infty} \sin\left(\frac{\pi n \xi}{l}\right) \sin\left(\frac{\pi n x}{l}\right) \exp\left(-\left(\frac{\pi n a}{l}\right)^2 t\right)$$

The problem is solved for the parameters $a = 1.0, T = 0.1, l = 1.0$, the function $u_\delta(\xi)$ is taken such that $\delta = 0.05 \cdot \|\overline{u}(\xi)\|$.



The exact solution $\bar{z}(x)$ (——) and the approximate solution $z_\eta(x)$ (- - - -).

The Euler equation

The Tikhonov's functional $M^{\alpha}[z]$ is a strongly convex functional in a Hilbert space.

The necessary and sufficient condition for z_{η}^{α} to be a minimum point of $M^{\alpha}[z]$ on a set D of *a priori* constraints is

$$\left(\left(M^{\alpha}\left[z_{\eta}^{\alpha}\right]\right)', z - z_{\eta}^{\alpha} \right) \geq 0 \quad \forall z \in D$$

If z_{η}^{α} is an interior point of D, then $\left(M^{\alpha}\left[z_{\eta}^{\alpha}\right]\right)' = 0$, or

$$A_h^* A_h z_{\eta}^{\alpha} + \alpha z_{\eta}^{\alpha} = A_h^* u_{\delta}$$

We obtain the Euler equation.

Sourcewise represented sets

$$A\bar{z} = \bar{u} \qquad (1)$$

$A: Z \to U$ is a linear injective operator.

Assume the next *a priori* information: $\bar{z}$ is sourcewise represented with a linear compact operator $B: V \to Z$:

$$\bar{z} = B\bar{v} \qquad (3)$$

Here V is a reflexive Banach space.

Suppose B is injective, A is known exactly, $\|u_\delta - \bar{u}\| \le \delta.$

Set $n = 1$ and define the set

$$Z_n = \{z \in Z : z = Bv, v \in V, \|v\| \le n\}$$

Minimize the discrepancy $F(z) = \|Az - u_\delta\|$ on Z_n.

If $\min\{\|Az - u_\delta\| : z \in Z_n\} \le \delta$, then the solution is found. Denote $n(\delta) = n$. Otherwise, we change n to $n+1$ and reiterate the process.

If $n(\delta)$ is found, then we define the approximate solution $z_{n(\delta)}$ of (1) as an arbitrary solution of the inequality

$$\|Az - u_\delta\| \le \delta \quad z \in Z_{n(\delta)}$$

Theorem 1: The process described above converges: $n(\delta) < \infty$. There exists $\delta_0 > 0$ (generally speaking, depending on $\bar{z}$) such that $n(\delta) = n(\delta_0)$ for $\forall \delta \in (0, \delta_0]$. Approximate solutions $z_{n(\delta)}$ strongly converge to $\bar{z}$ as $\delta \to 0$.

Proof The ball $V_n = \{v \in V : \|v\| \leq n\}$ is a bounded closed set in V. The set Z_n is a compact in Z for any n, since B is a compact operator. Due to Weierstrass theorem the continuous functional $F(z)$ attains its exact lower bound on Z_n.

Clearly, $\bar{z} = B\bar{v} \in Z_N$, where

$$N = \begin{cases} \|\bar{v}\| & \|\bar{v}\| \text{ is a positive integer} \\ [\|\bar{v}\|] + 1 & \text{otherwise} \end{cases}$$

$[\cdot]$ is the integer part of a number.

Therefore $n(\delta)$ is a finite number and there is δ_0 such that $n(\delta) = n(\delta_0)$ for any $\delta \in (0, \delta_0]$. The inequality $n(\delta) \geq N$ for any $\delta > 0$ is evident. Thus, for all $\delta \in (0, \delta_0]$ the approximate solutions $z_{n(\delta)}$ belong to the compact set $Z_{n(\delta_0)}$, and the method coincides with the quasisolutions method for all sufficiently small positive δ. The convergence $z_{n(\delta)} \to \bar{z}$ follows from the general theory of ill-posed problems.

Remark 1: The method is a variant of the method of extending compacts.

Theorem 2: For the method described above there exists an *a posteriori* error estimate. It means that a functional $\kappa(u_\delta, \delta)$ exists such that $\kappa(u_\delta, \delta) \to 0$ as $\delta \to 0$ and $\left\| z_{n(\delta)} - \bar{z} \right\| \le \kappa(u_\delta, \delta)$ at least for all sufficiently small positive δ .

Remark 2: The existence of the a posteriori error estimation follows from the following. If by $\bar{Z} \in Z$ we denote the space of sourcewise represented with the operator B solutions of (1), then $\bar{Z} = \bigcup_{n=1}^{\infty} Z_n$. Since Z_n is a compact set, then $\bar{Z}$ is a σ-compact space.

An *a posteriori* error estimate is not an error estimate in general meaning that is impossible in principle for ill-posed problems. But it becomes an upper error estimate of the approximate solution for "small" errors $\delta < \delta_0$, where δ_0 depends on the exact solution $\bar{z}$.

The operators A and B are known with errors. Let there be linear operators A_{h_A}, B_{h_B} such that $\|A_{h_A} - A\| \le h_A$, $\|B_{h_B} - B\| \le h_B$. Denote the vector of errors by $\eta \equiv (\delta, h_A, h_B)$. For any integer n define a compact set $Z_{n,h_B} \equiv \{z \in Z : z = B_{h_B} v, v \in V, \|v\| \le n\}$.

Find a minimal positive integer number $n = n(\eta)$ such that the inequality

$$\|A_{h_A} z - u_\delta\| \le \delta + \left(h_A \|B_{h_B}\| + h_B \|A_{h_A}\| + h_A h_B\right) \cdot n(\eta)$$

has a nonempty set of solutions.

Then the *a posteriori* error estimation is

$$\kappa\left(u_\delta, A_{h_A}, B_{h_B}, \eta\right) \equiv h_B n(\eta) + \max\{\|z - z_{n(\eta)}\| : z \in Z_{n(\eta),h_B},$$
$$\|A_{h_A} z - u_\delta\| \le \delta + \left(h_A \|B_{h_B}\| + h_B \|A_{h_A}\| + h_A h_B\right) \cdot n(\eta)\}$$

Let A be a linear injective completely continuous operator, and Z and U be Hilbert spaces. Consider the case when $\bar{z} = (A^*A)^{p/2}\bar{v}$, ($\bar{v} \in Z$, $p = \text{const.} > 0$).

LEMMA 1 *The operator* $(A^*A)^{p/2}$ *is completely continuous and injective from* Z *into* Z *for any* $p > 0$.

For the case when Z and U are Hilbert spaces, $V = Z$, $A : Z \to U$ is a linear injective completely continuous operator, $B = (A^*A)^{p/2}$, and $p = \text{const.} > 0$, the following theorem holds.

THEOREM 3 (see [14,15]) *For the case under consideration, the extending compacts method is the optimal regularizing algorithm in terms of the order of accuracy.*

It is clear that, in the extending compacts method, we can replace the sequence $n = 1, 2, \ldots$ with any increasing sequence of positive numbers $r_1, r_2, \ldots, r_n, \ldots$ such that $\lim_{n\to\infty} r_n = +\infty$.

Inverse problem for the heat conduction equation

For any moment of time $t_\varepsilon > 0$ there is

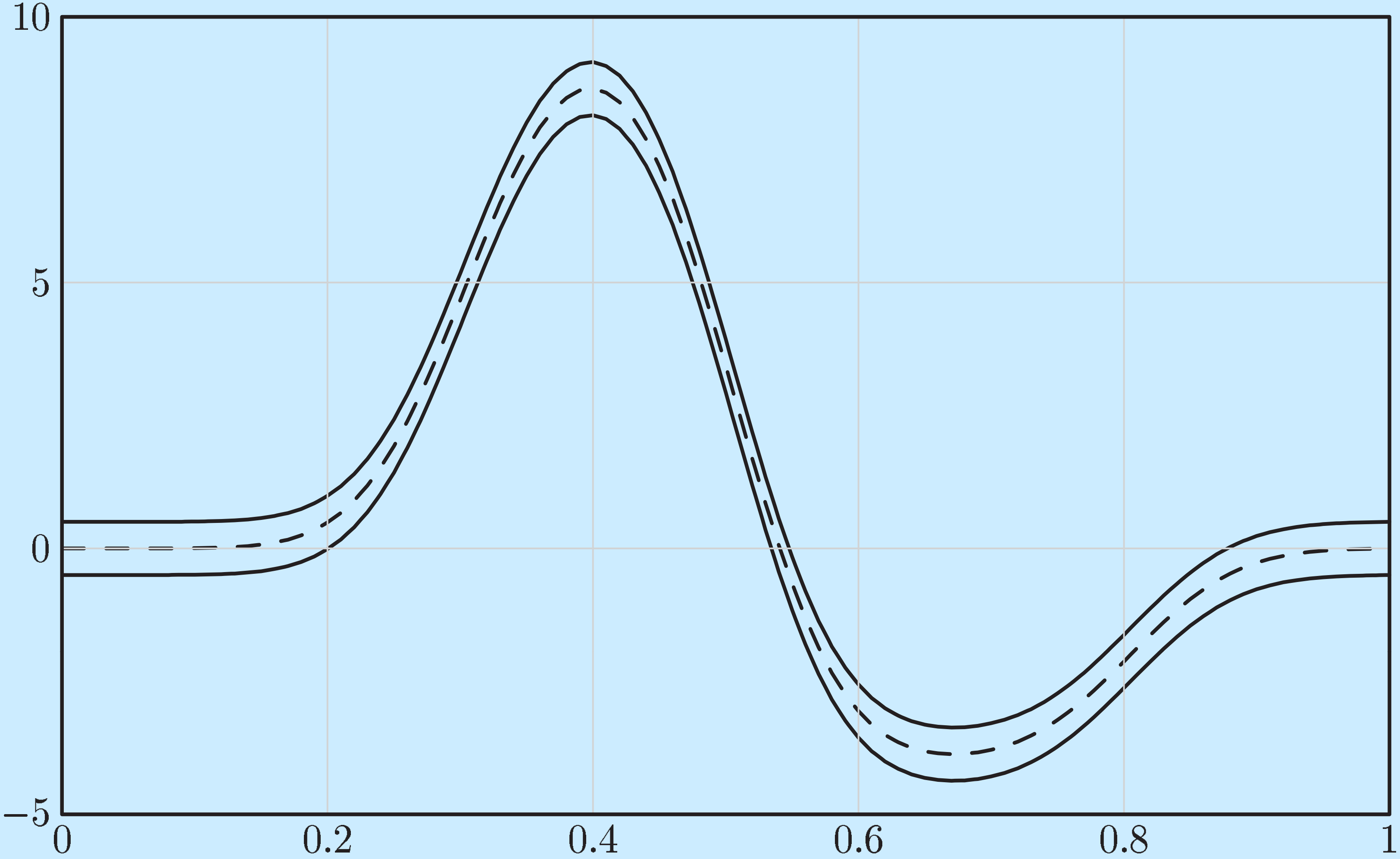
$$z(\xi) = Bv(x) = \int_0^l G(\xi, x, t_\varepsilon)\, v(x)\, dx$$

where $v(x) = w(x,0)$. Suppose $V = Z = U = L^2[0,l]$.

We solve the problem using the method of extending compacts.

Let $a = 1.0$, $l = 1.0$, $t_\varepsilon = 0.02$, $T = 0.1$, $\delta = 0.03 \cdot \|\overline{u}\|$.

$$\overline{v}(x) = \begin{cases} 10 & 0.3 < x < 0.5 \\ -4 & 0.5 < x < 0.8 \\ 0 & \text{otherwise} \end{cases}$$



The approximate solution $z_\eta(x)$ and its *a posteriori* error estimation. We obtain $n(\delta) = 5$.

Compact sets

There is the additional *a priori* information: the exact solution $\bar{z}$ of (1) belongs to a compact set M and A is a linear continuous injective operator.

As a set of approximate solutions of (1) it is possible to accept

$$Z_M^\eta \equiv \left\{ z \in M : \|A_h z - u_\delta\| \le h\|z\| + \delta \right\}$$

Then $z_\eta \to \bar{z}$ as $\eta \to 0$ in Z for any $z_\eta \in Z_M^\eta$.

After finite dimensional approximation we obtain that $\hat{Z}_M^\eta \equiv \hat{M} \cap \hat{Z}^\eta$, where $\hat{M}$ is a convex polyhedron for convex or monotonic functions and

$$\hat{Z}^\eta = \left\{ \hat{z} \in \hat{Z} : \left\| \hat{A}\hat{z} - \hat{u}_\delta \right\| \leq \Delta(\eta) \right\}$$

$\hat{A}$ is a matrix, $\hat{z}$ and $\hat{u}_\delta$ are vectors.

To find $\hat{z}_\eta$ it is possible to use the method of conditional gradient or the method of projection conjugated gradients.

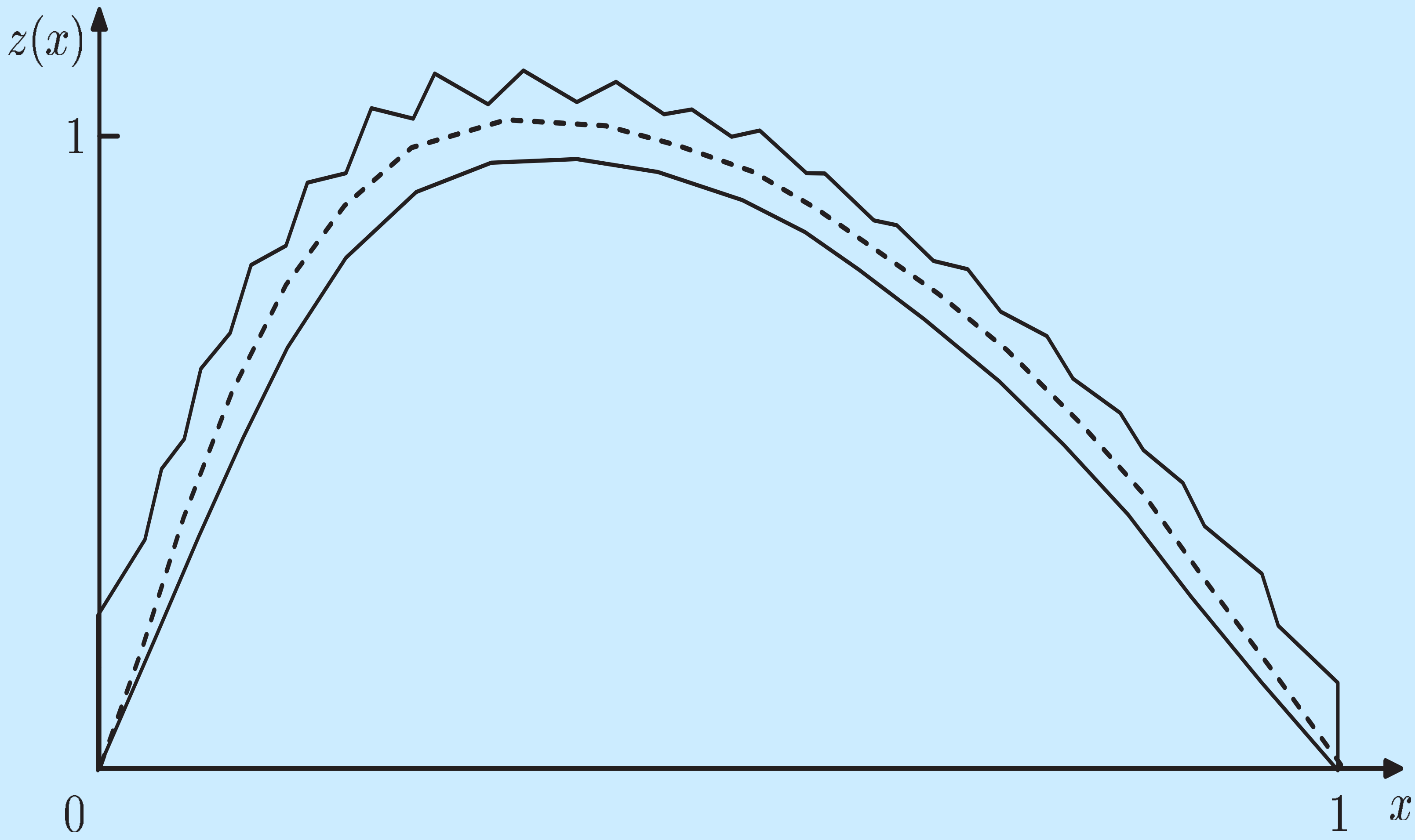
Error estimation

1) Find the minimum and the maximum values for each coordinate of $\hat{Z}_M^\eta$. Denote them by z_i^l , z_i^u, $i = \overline{1, n}$.
2) Secondly, using the found $\hat{z}^l, \hat{z}^u$ we construct functions $z^l(x)$ and $z^u(x)$ close to Z_M^η such that $\forall z \in Z_M^\eta : z^l(x) \le z(x) \le z^u(x)$ for each $x \in [a, b]$.

Therefore, we should minimize a linear function on a convex set. We may approximate the set by a convex polyhedron and solve a linear programming problem. The simplex-method or the method to cut convex polyhedrons may be used.

Inverse problem for the heat conduction equation

Let M be a set of convex upward functions $z(x)$ such that $0 \le z(x) \le C$. Assume that $a = 1.0$, $l = 1.0$, $T = 1.0$, $C = 1.2$, the number of nodes 20.



The exact solution $\bar{z}(x)$ (----), the functions $z^l(x)$, $z^u(x)$.

Examples and Applications

ONE INVERSE PROBLEM OF QUANTITATIVE ELECTRON PROBE MICROANALYSIS

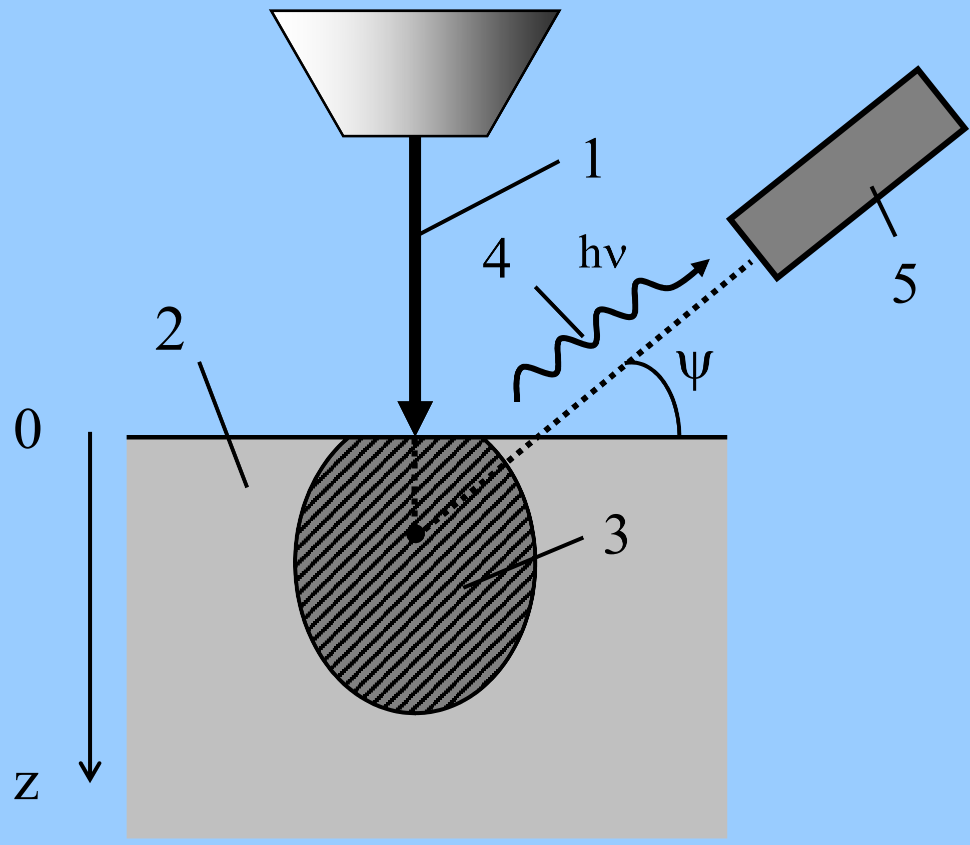


Fig.1

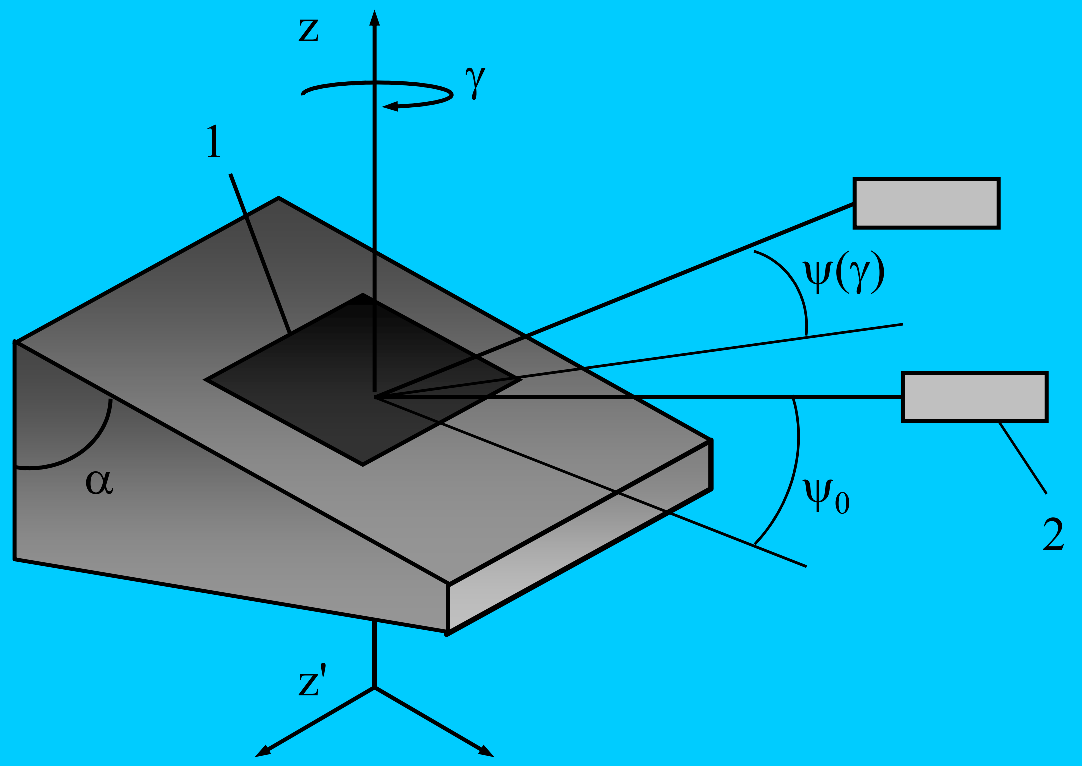


Fig.2

$$A\varphi = I_{\delta}^{om}, \qquad (1)$$

where

$$A\varphi = \int_{0}^{\rho r} K(\rho z, \gamma)\varphi(\rho z)\, d\rho z,$$

δ>0 – error of assignment of the right part of the equation (1), i.e. $\left\| I_{\delta}^{rel} - \bar{I}^{rel} \right\| \le \delta$, $A\bar{\varphi} = \bar{I}^{rel}$, $I^{rel} = \frac{I}{I(0)}$ - relative intensity.

$$F[\varphi] = \left\| A\varphi - I_\delta \right\|^2_{L_2}, \qquad (2)$$

At that it is enough to find such an element φ_δ, that

$F\left[\varphi_\delta\right] \le \delta^2$.

$$f(\varphi) = \sum_{j=1}^{N_\gamma} \left(\sum_{i=1}^{N_{\rho z}} K_{ij} \varphi_i h_{\rho z} - I_j^{om} \right)^2 h_\gamma \qquad (3)$$

At finite-difference approximation set Z transforms into set

$$\hat{Z} = \left\{ \varphi : \begin{array}{ll} \varphi_{i-1} - 2\varphi_i + \varphi_{i+1} \le 0, & i = 2,\ldots,N_{\rho z} - 1 \\ \varphi_i \ge 0, & i = 1,2,\ldots,N_{\rho z} \end{array} \right\}, \qquad (4)$$

Let $T^{(j)}$, $j = 0,1,\ldots,m\,(m = N_{\rho z})$ – apexes of a convex limited polyhedron $\hat{Z}$.

Lemma. *Let* $\varphi \in \hat{Z}$. *Then the unique representation is correct*

$$\varphi = \sum_{j=1}^{m} a_j T^{(j)},$$

at that $a_j \geq 0, j = 1,2,\ldots,m$.

Let us examine operator T from R^m in R^m, determined by the formula

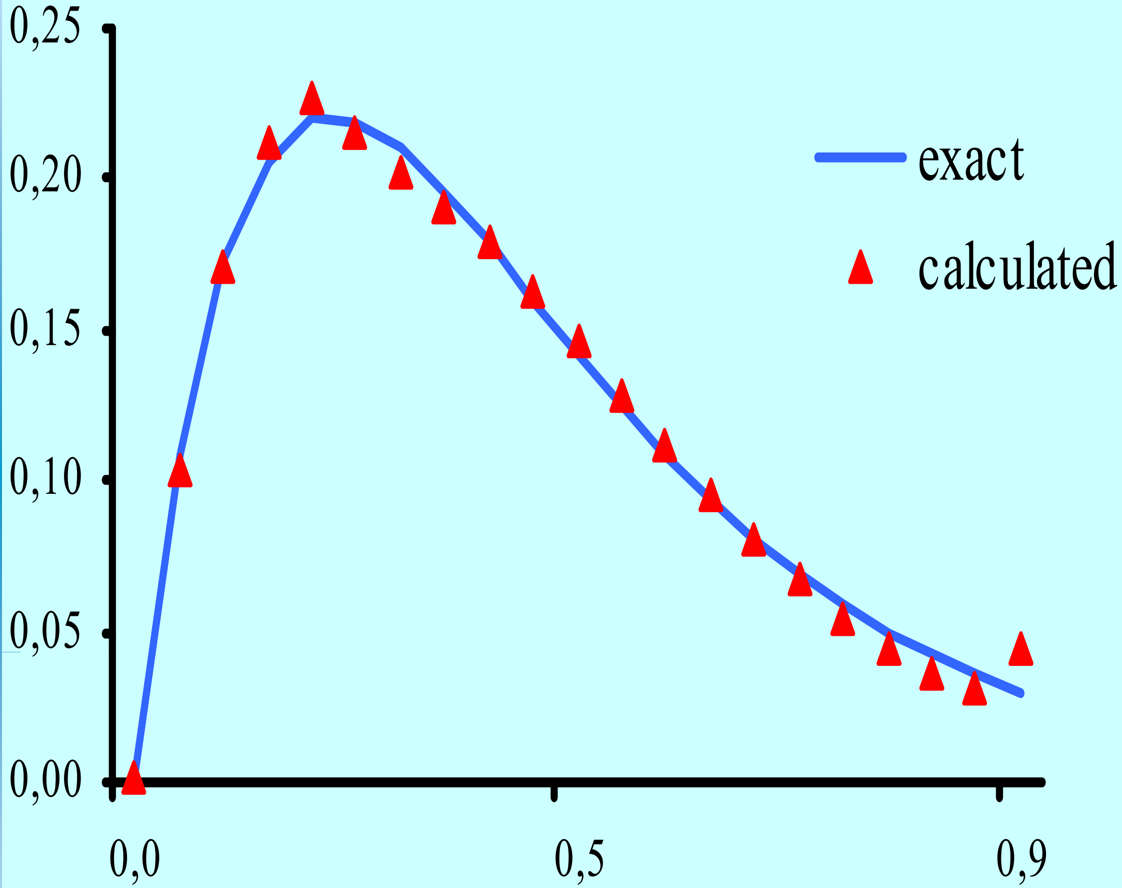
$$T\xi = \sum_{j=1}^{m} \xi_j T^{(j)}, \qquad \xi \in R^m .$$

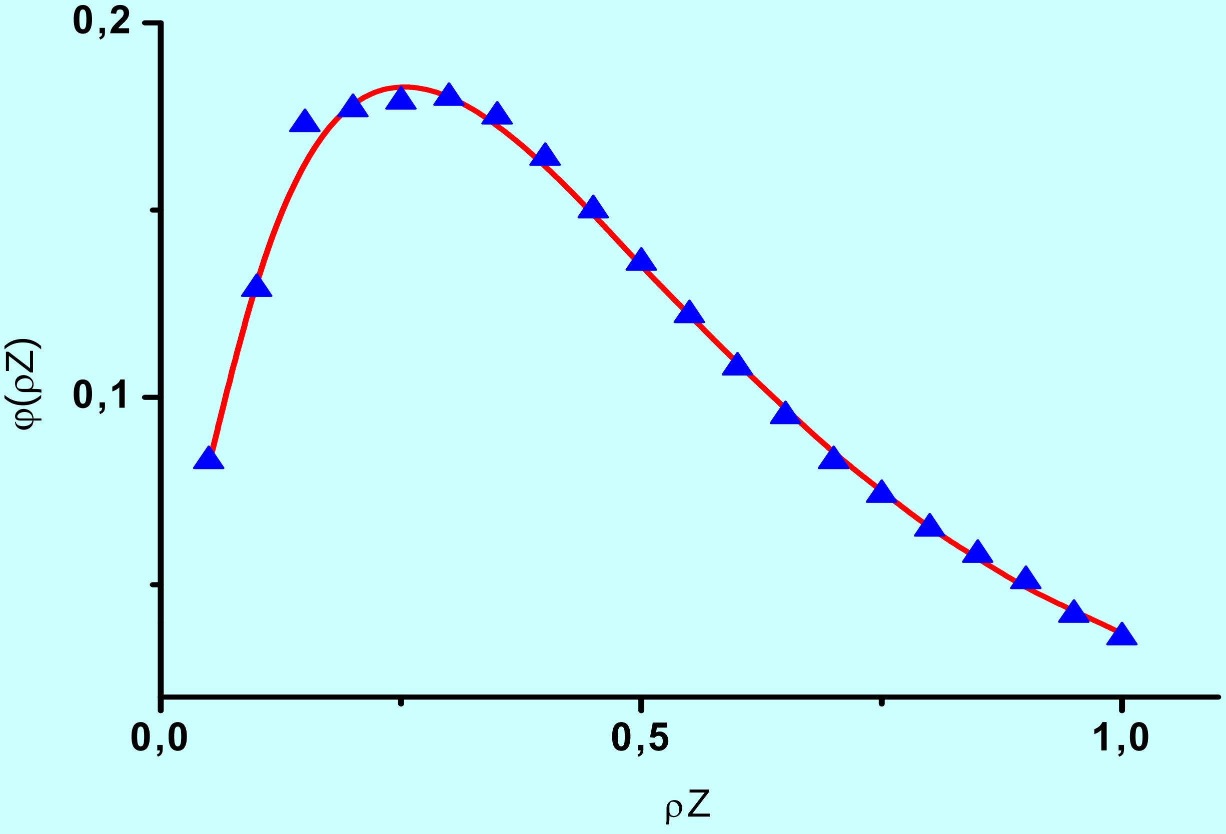
It is obvious, that $T\mathrm{R}_+^m = \hat{\mathrm{Z}}$ and $T^{-1}\hat{\mathrm{Z}} = \mathrm{R}_+^m$,

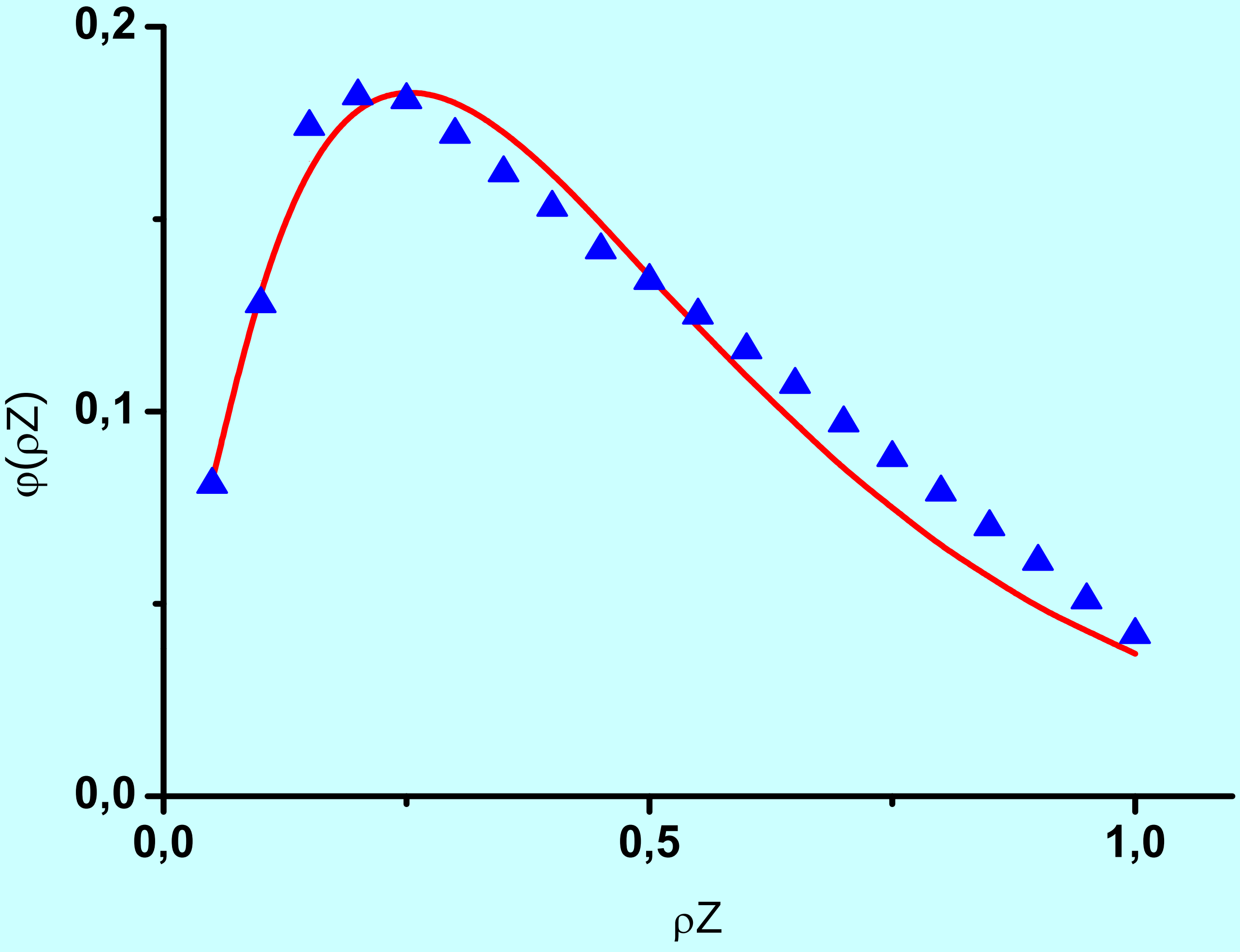
Where R_+^m - set of vectors $\mathrm{R}_+^m \subset \mathrm{R}^m$, that have all non-negative coordinates $\xi \in \mathrm{R}_+^m$, if $\xi_j \geq 0, j = 1,2,\ldots,m$.

Let us examine function $Y(\xi) = f(T\xi)$, determined on set R_+^m.

We need to find such an element $\xi_\delta \in \mathrm{R}_+^m$, that $Y(\xi_\delta) \leq \delta^2$. The approximate solution of the original problem is found then by the formula $\varphi_\delta = T\xi_\delta$.


0,25
0,20
0,15
0,10
0,05
0,00
0,0
0,5
0,9
exact
calculated

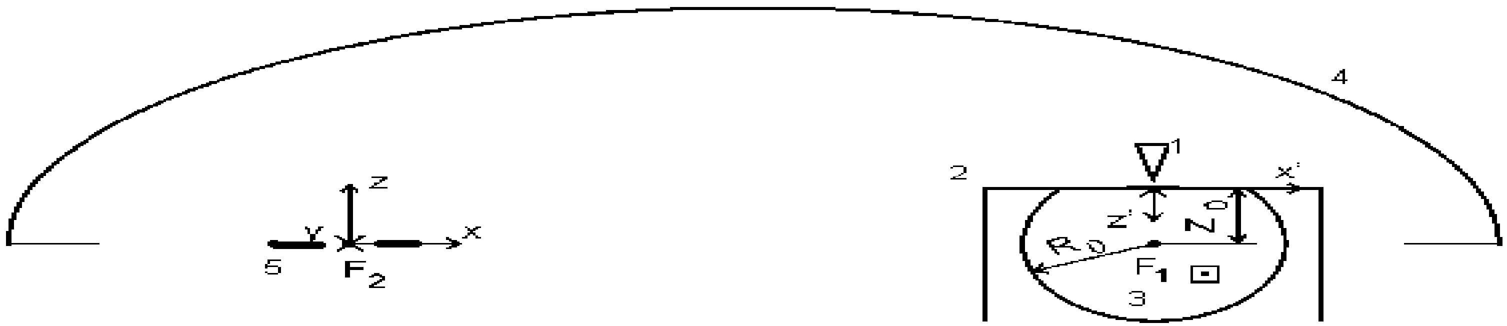

0,2
0,1
φ(ρZ)
0,0
0,5
1,0
ρZ



0,2
0,1
0,0
φ(ρZ)
0,0
0,5
1,0
ρZ

INVERSE PROBLEM OF CATHODOLUMINESCENCE MICROTOMOGRAPHY

The Scheme of Installation



1. Focused electrical probe
2. Object under investigation
3. Region of generation of nonequilibrium carriers
4. Ellipsoidal mirror
5. Diaphragm with detector

Problem

Develop method for determination of optoelectrical local properties of cathodoluminescence objects with resolution of micrometer part, having at our disposal the set of measurements of intensity values. Describe the scheme of experiment, mathematical statement and the method of solution of the problem, which is ill-posed.

The interaction of focused electrical probe with cathodoluminescence substance was modulated. An alternative method of microtomography in cathodolumenscence mode is presented. The solution is based on confocal ellipsoidal mirror [Phang J.C.H, Chan D.C.H.].
The photon rays transport in luminescence volume of specimen and ellipsoid are calculated.

We have to solve the next inverse problem:
define the internal quantum yield of the material

$$\eta(s), \quad s \in [0, R_0]$$

from Fredholm integral equation of the first kind:

$$I(x) = \int_0^{R_0} K_1(x,s)\eta(s)\,ds \;, \quad x - \textit{the deflection of the object in respect to the mirror} \quad I(x) \in L_2[x_{\min}, x_{\max}] \quad \eta(s) \in [0, R_0]$$

where $I(x)$ - intensity, measured in experiment, as function of deflection of the object in vertical direction,

s -the distance from the surface of the object,

R_0 -maximal depth of penetration of electrons into the object,

$K_1(x,s)$-some continuous function, which was calculated by numerical methods (the physical sense of is that $K_1(x,s)\,ds$ is the contribution into the total intensity the layer with center on the depth *s* and thickness *ds*).

A Priori Information

Let it is known that the solution of the problem is sourcewise represented with help of completely continuous integral operator:

$$\eta(s) = \int_0^{R_0} K_2(s,\xi)\eta_0(\xi)d\xi,\ s \in [0, R_0]$$

where

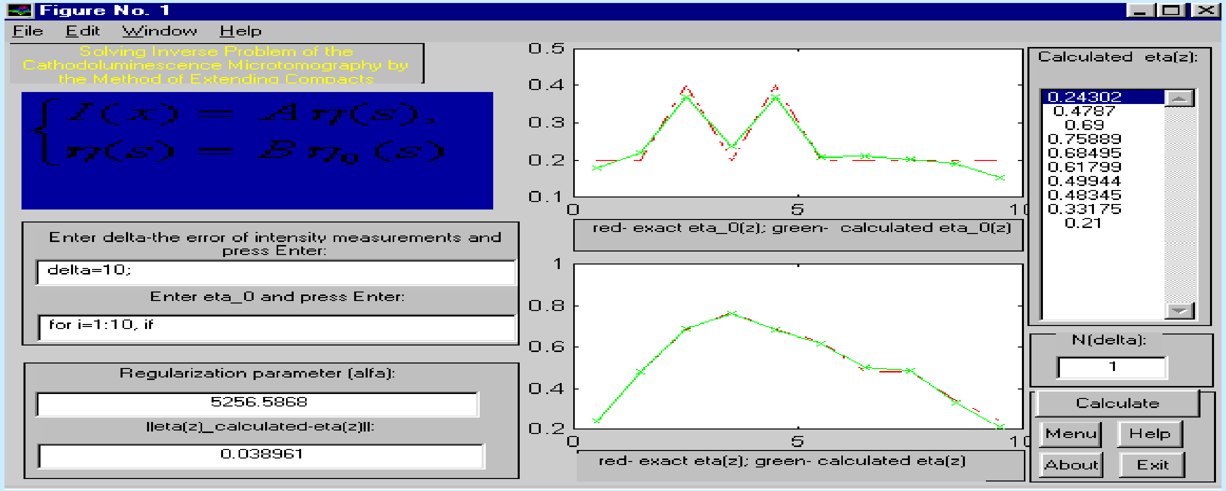
$$K_2(s,\xi) = \begin{cases} \cos((s-\xi)1/2*\pi/2, & |s-\xi| \leq 2, \\ 0, & otherwise \end{cases}$$

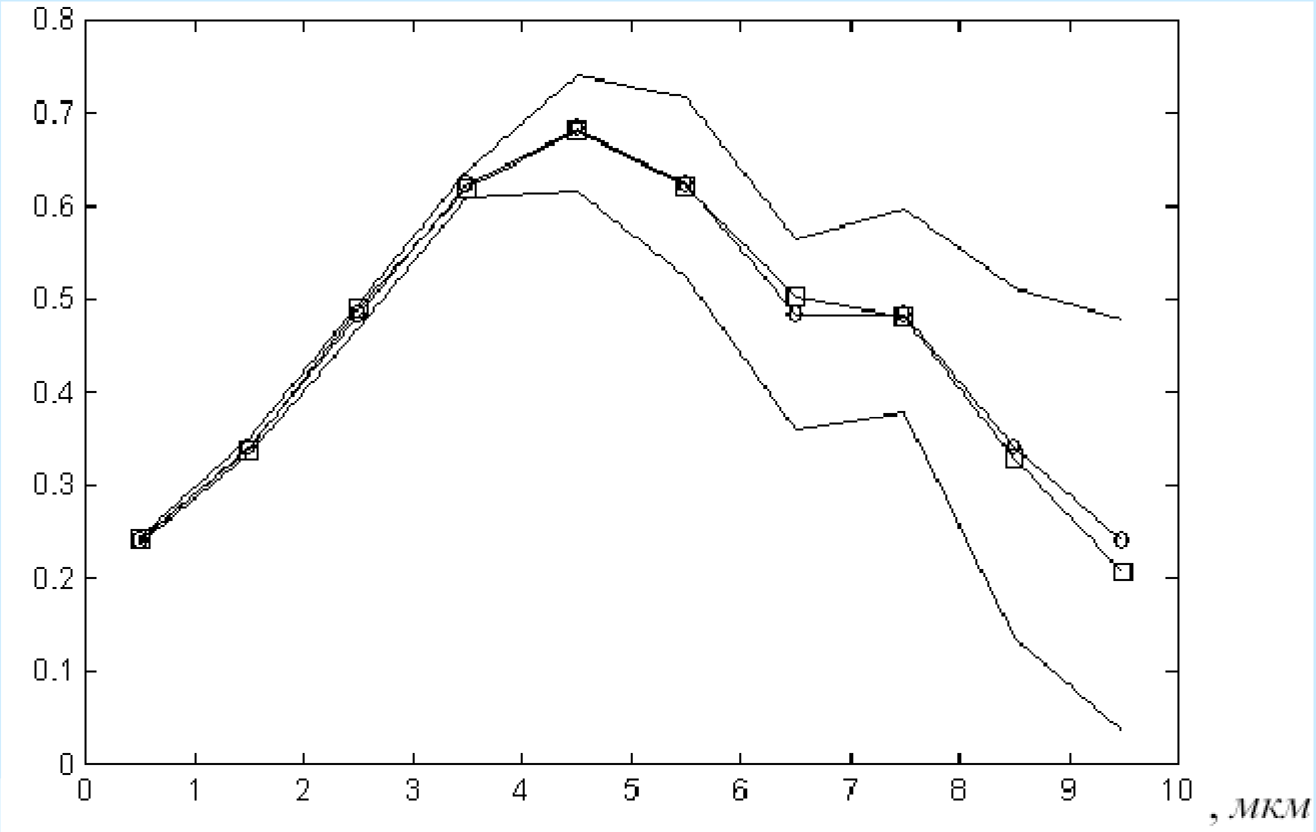
We shall consider that:

$\eta_0(s) \in L_2[0,R_0]$ $\eta_0(s) \in L_2[0,R_0]$

For solving the problem under such a priori information the method of extending compacts, which was described above, is used.

Model Calculations Results





0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10
, мкм

An Inverse Problem of Nuclear Physics

- **Experiment:**

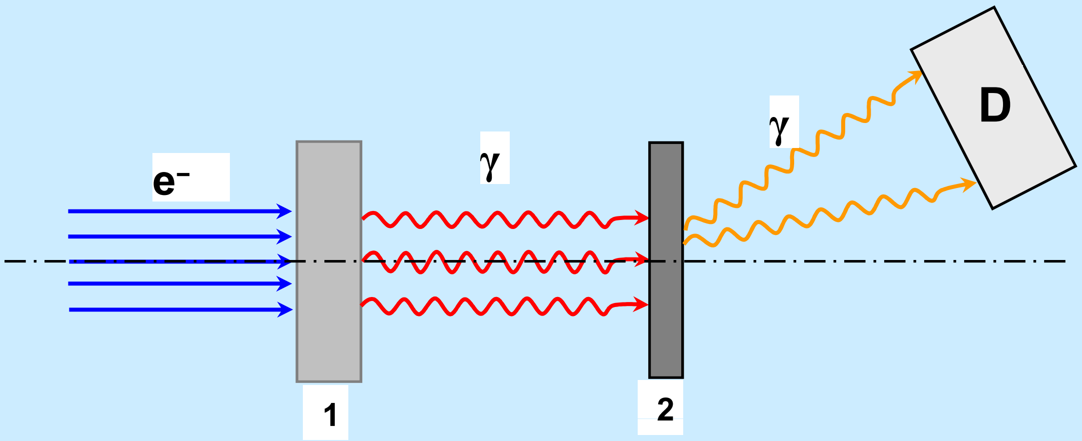


Fig.1: 1 - the target for producing bremsstrahlung beam, 2 - the sample under consideration, D – detector.

Passing through the first target the accelerated electrons produce the bremsstrahlung beam (γ-rays). The bremsstrahlung spectrum is continuous. The sample 2 is bombarded by the γ-rays. The scattered γ-rays are detected.

- **Nuclear reaction:** $\gamma + {}^{63}_{29}Cu \rightarrow {}^{62}_{29}Cu + n$
- **Constraints:**
  - *A priori :* $0 \le \sigma(E_\gamma) \le 90, \; E_\gamma \in [10, 24.1]$
  - *A posteriori:*
    - $\sigma(E_\gamma), \; E_\gamma \in [10,16]$ **is a monotone nondecreasing function**
    - $\sigma(E_\gamma), \; E_\gamma \in [16,18]$ **is a convex upwards function**
    - $\sigma(E_\gamma), \; E_\gamma \in [18, 24.1]$ **is a monotone nonincreasing function**

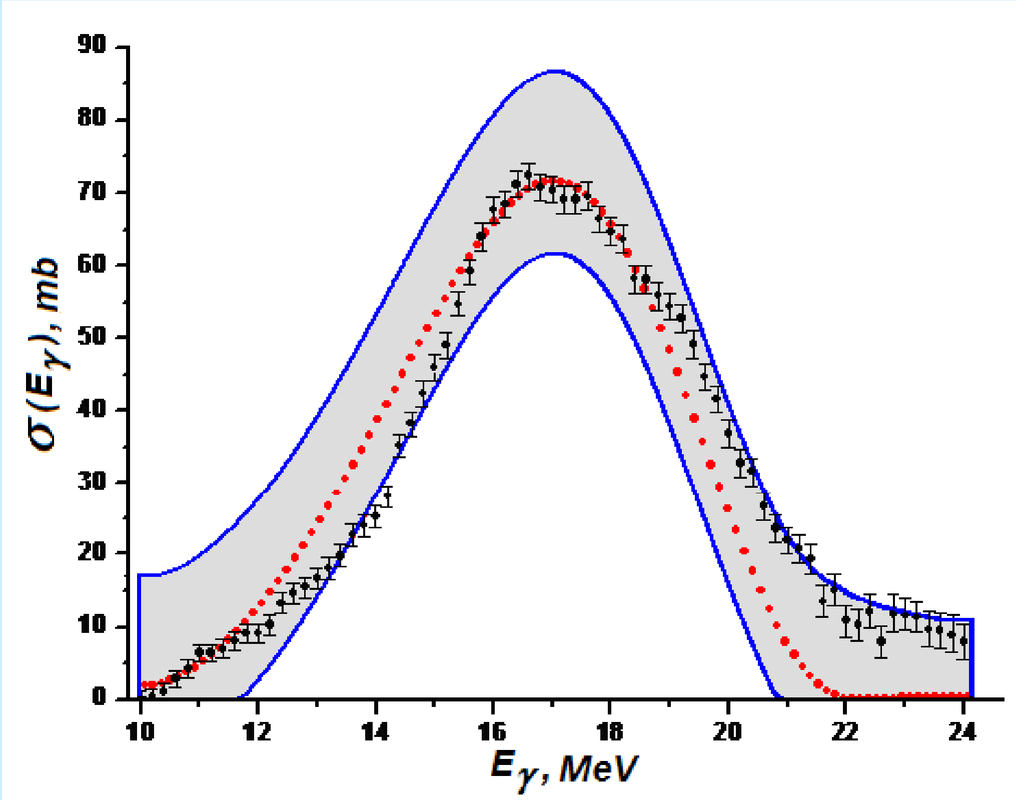


Fig.2: (• • •) – the approximate cross section from the Center of Data of Photonuclear experiments (http://depni.sinp.msu.ru/cdfe/);

(• • •) – the approximate solution found by Tikhonov regularization;

(–) – the functions $\sigma^{low}(E_\gamma)$, $\sigma^{upper}(E_\gamma)$

bounded the set of approximate solutions from below and from above .

- **Nuclear reaction:** $\gamma + {}^{34}_{16}S \rightarrow {}^{33}_{15}P + p$
- **Constraints:**
  - *A priori:* $0 \le \sigma(E_\gamma) \le 45,\ E_\gamma \in [12.3, 25.3]$
  - *A posteriori:*
    - $\sigma(E_\gamma),\ E_\gamma \in [12.3, 16]$ **is a monotone nondecreasing function**
    - $\sigma(E_\gamma),\ E_\gamma \in [16, 17]$ **is a convex upwards function**
    - $\sigma(E_\gamma),\ E_\gamma \in [17, 18.5]$ **is a monotone nonincreasing function**
    - $\sigma(E_\gamma),\ E_\gamma \in [18.5, 20]$ **is a convex downwards function**
    - $\sigma(E_\gamma),\ E_\gamma \in [20, 22]$ **is a monotone nondecreasing function**
    - $\sigma(E_\gamma),\ E_\gamma \in [22, 23]$ **is a convex upwards function**
    - $\sigma(E_\gamma),\ E_\gamma \in [23, 25.3]$ **is a monotone nonincreasing function**

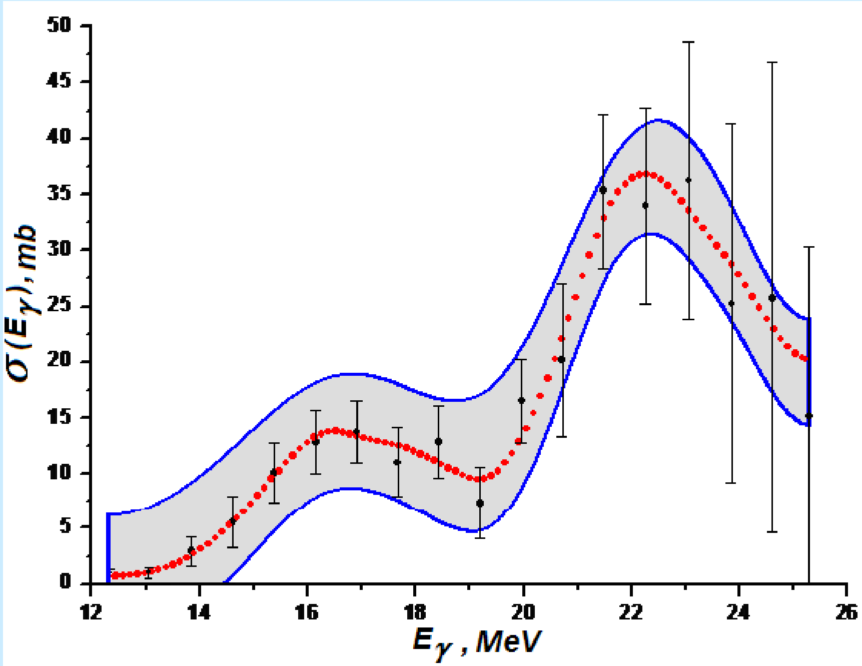


Fig.3: (• • •) – the approximate cross section from the Center of Data of Photonuclear experiments;

(• • •) – the approximate cross section found by Tikhonov regularization;

(–) – the functions $\sigma^{low}(E_\gamma),\ \sigma^{upper}(E_\gamma)$

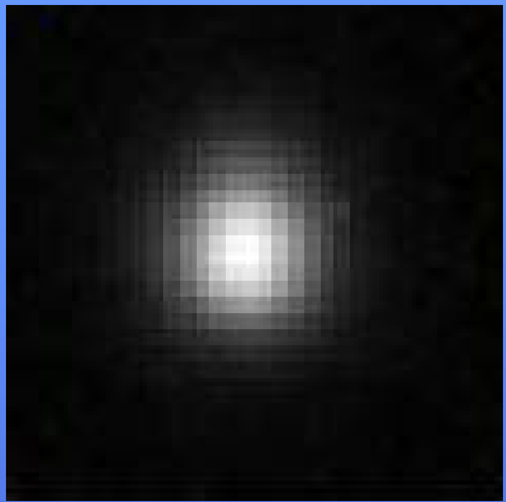
bounded the set of approximate solutions from below and from above .

Image reconstruction for gravitational lens

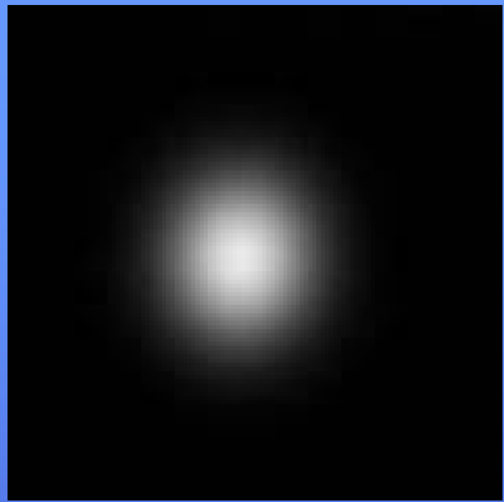
The system QSO 2237+0305, known as the "Einstein Cross": 4 quasar images against the background of the lensing galaxy.

Several observation were carried out using the Huble Space Telescope and Nordic Optical Telescope.

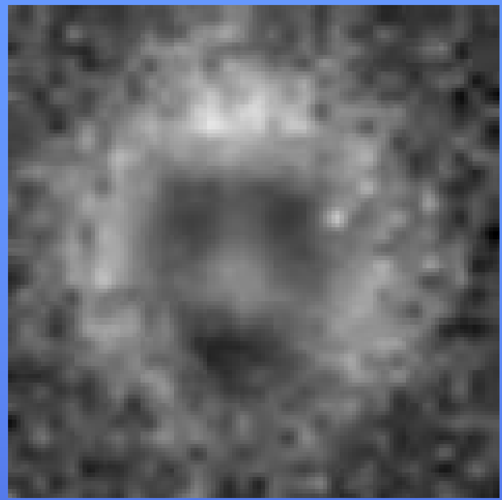
Model of Kernel



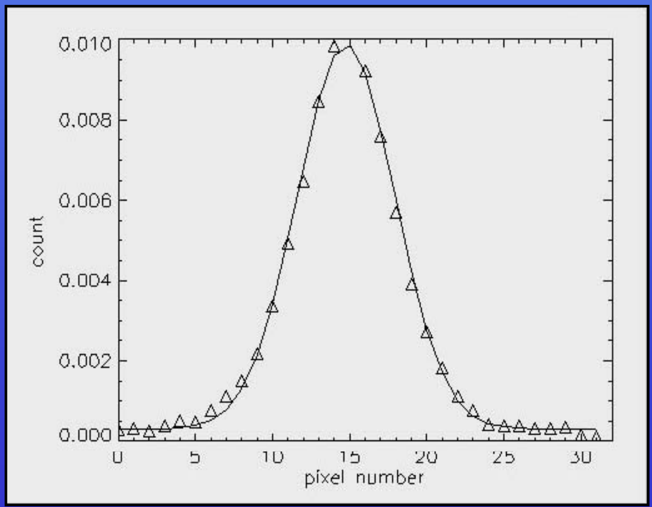
Star



Kernel



Residuals



PSF (Kernel) profile

Approximation of the star from the frame with 2-dimensional Gauss profile

FWHM ~ 5 pixels

Tikhonov Regularization

- Ill-posed problem
- Smoothing function:

$$M^{\alpha}[z] = \|k * z - u_{\delta}\|_{U}^{2} + \alpha \cdot \Omega(z)$$

- Solution z^{α} :

$$M^{\alpha}[z^{\alpha}] = \inf\{M^{\alpha}[z] : z \in Z\}$$

- Regularization parameter α from discrepancy principle:

$$\left\| k * z^{\alpha} - u_{\delta} \right\|_{U} \cong \delta, \alpha > 0$$

A priori information

True Image = Galaxy + Quasar Components

$$z(x,y) = g(x,y) + \sum_{k=1}^{K} I_k \delta(x-a_k, y-b_k)$$

$K=4$, number of quasar components

$K=5$, number of quasar components + galaxy nuclear

A priori information

- Nonnegativity of the solution, $z_{ij} \geq 0$
- Galaxy: assumption about smoothness

$$\Omega(g) = \|g\|_G^2 \quad ; G \equiv \{L_2, W_{21}, BV\}$$

- Galaxy model

$$\Omega(g) = \|g - g_{\mathrm{model}}\|_G^2$$

generalized de Vaucouleurs profile (Sersic's model)

$$g_{\mathrm{model}}(r) = I(0)\exp\{-b_n (r/r_e)^{1/n}\}$$

$b_n = 2n - 0.324$ for $1 \leq n \leq 4$

A priori information

Sourcewise representation:

$$z = R[z'] \equiv r * z'$$

Total PSF = *Source* PSF * *Final* PSF

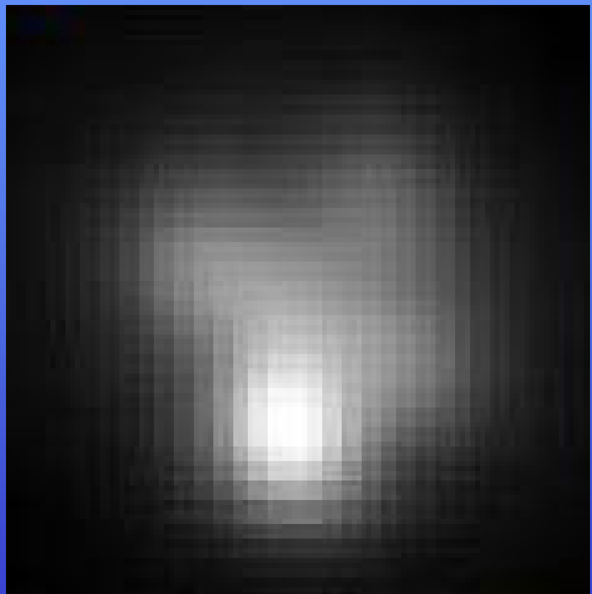
$$\Updownarrow$$

$$k = s * r$$

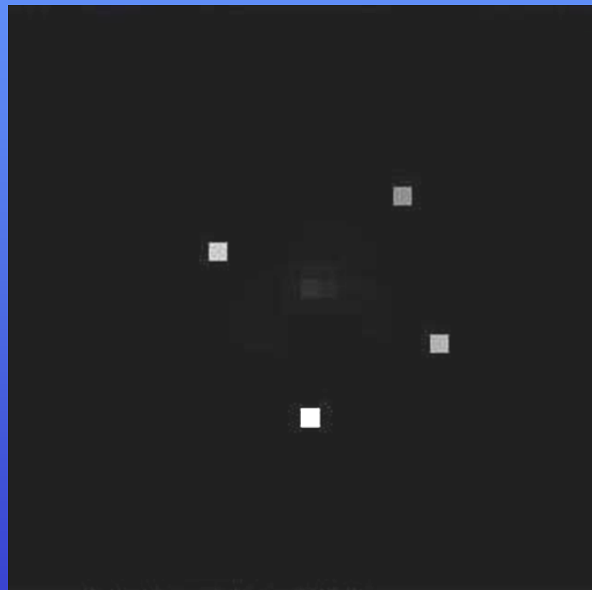
$$z(x, y) = \sum_{k=1}^{K} a_k r(x - b_k, y - c_k) + g(x, y)$$

Results: L_2

$$\Omega(g) = \left\| g - g_{sersic} \right\|_{L_2}^2$$

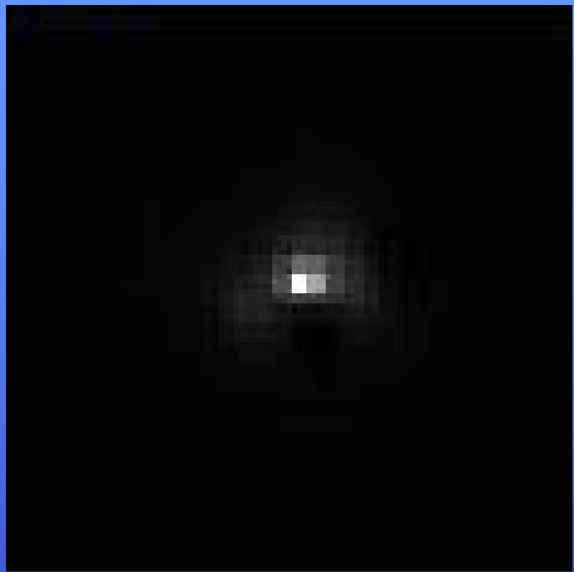


Observed image

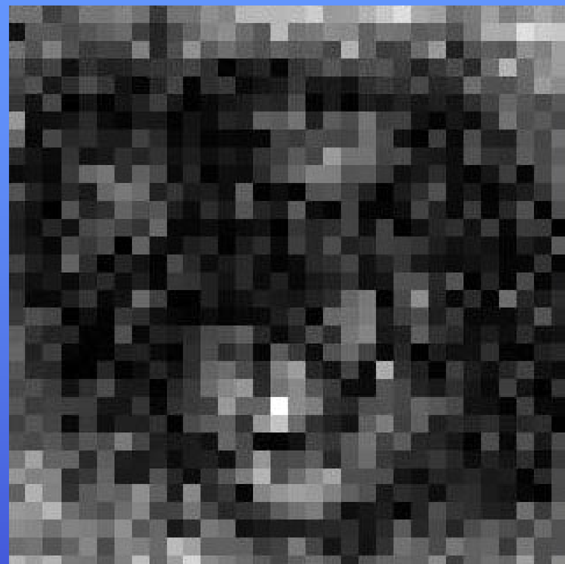


Reconstructed image

Results: L_2



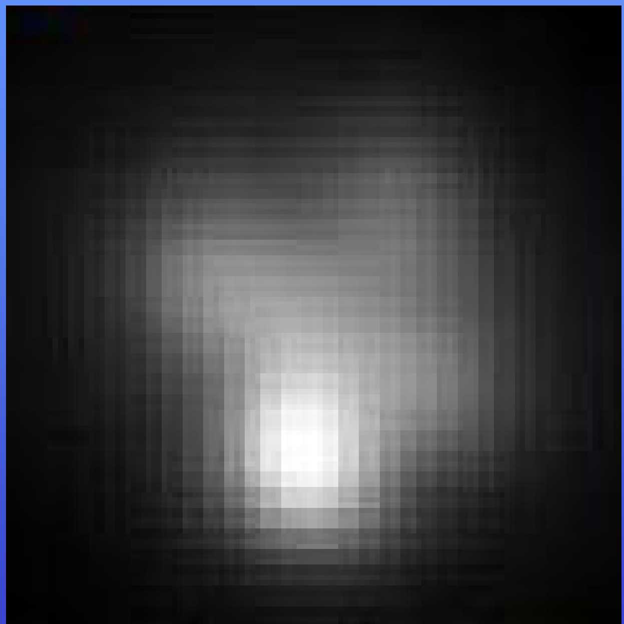
Galaxy



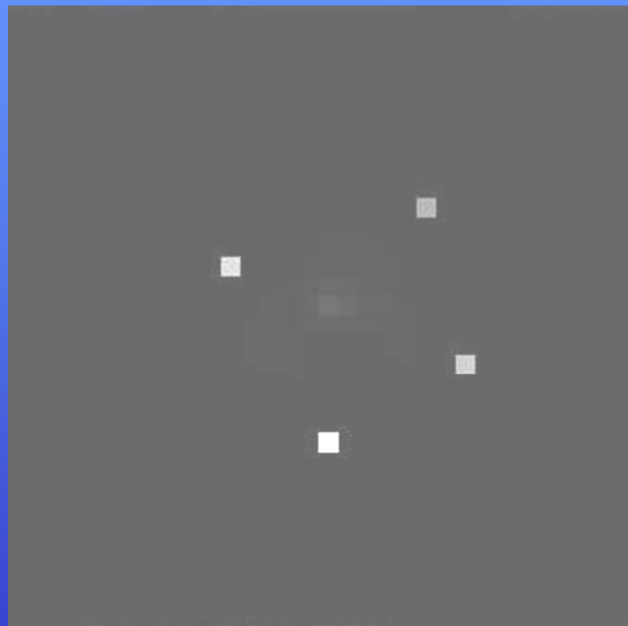
Error distribution

Results: W_{21}

$$\Omega(g) = \left\| g - g_{sersic} \right\|_{W_{21}}^{2}$$

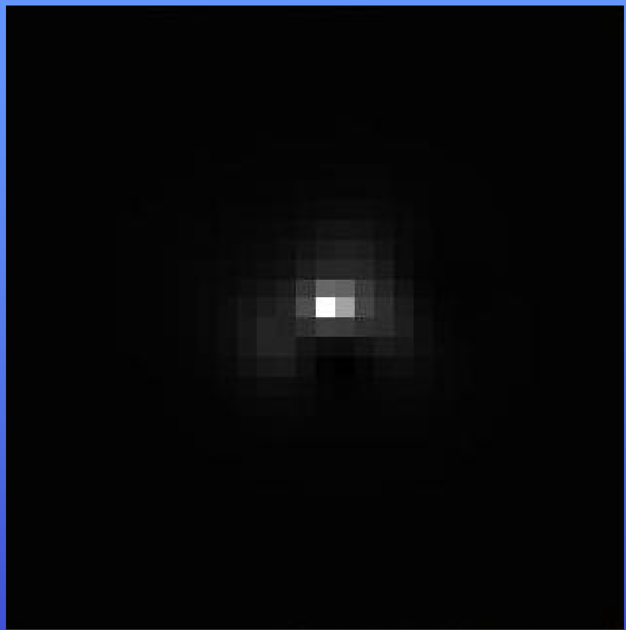


Observed image

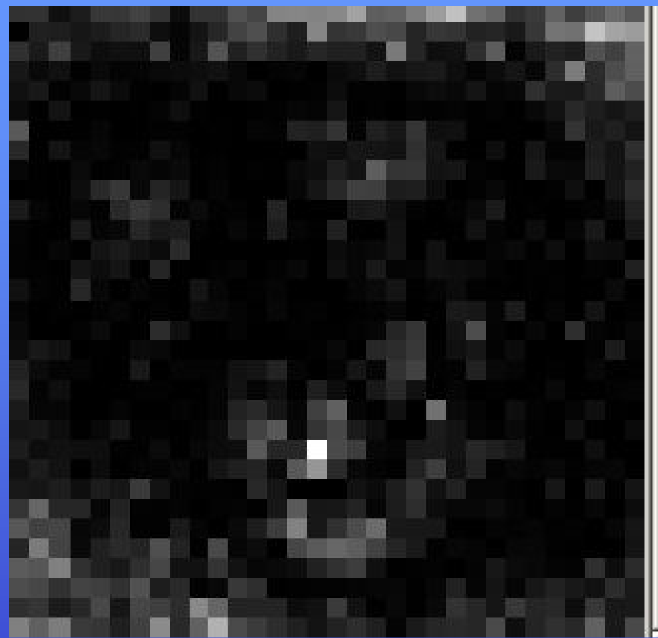


Deconvolved image

Results: W_{21}



Galaxy



Error distribution

Results: MCS

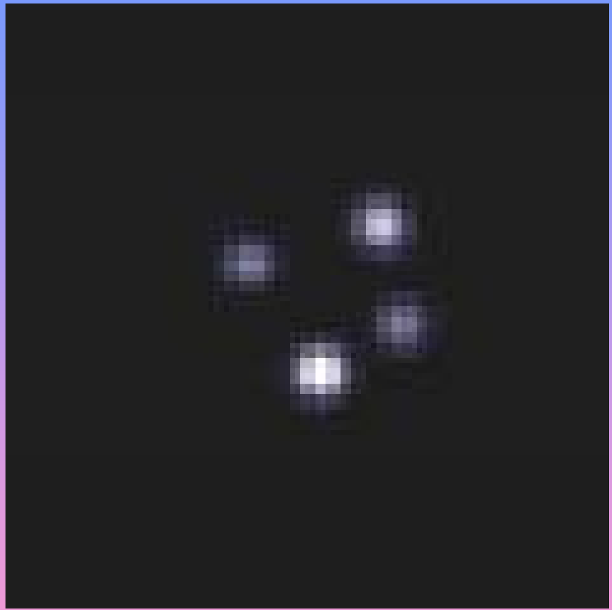
$$\text{kernel} = s * r$$

$$\Omega(g) = \|g - r * g\|_{L_2}^2$$

Observed image

Deconvolved image

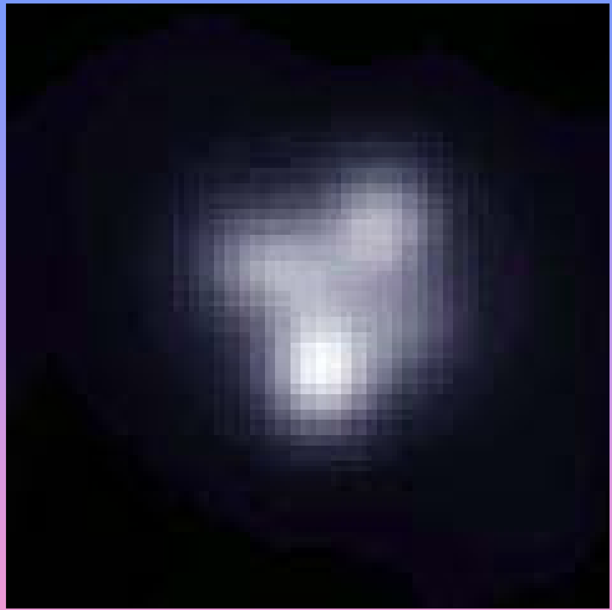
Results: MCS



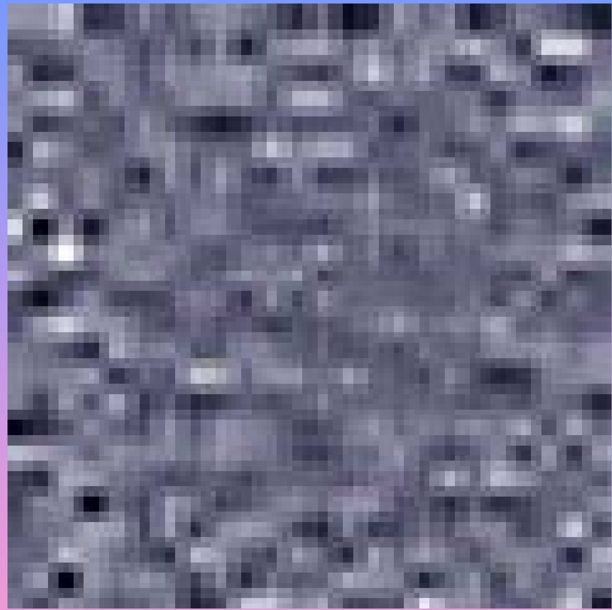
Quasar components

Galaxy

Results: MCS



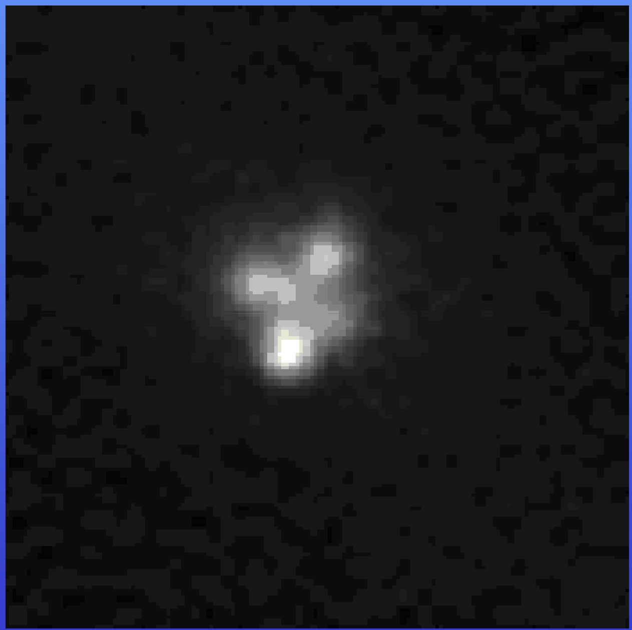
Quasar components



Error distribution

Results: TV

$$\Omega(g) = \sum_{m=1}^{N_1 - 1} \sum_{n=1}^{N_2 - 1} \left| g_{m+1,n+1} - g_{m+1,n} - g_{m,n+1} + g_{m,n} \right|$$

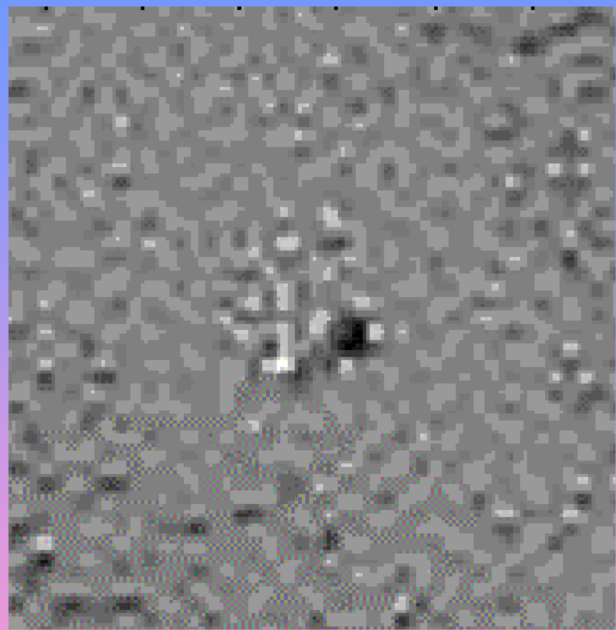


Observed image

Deconvolved image

Results: TV

Quasar components



Galaxy

Functions convex along lines parallel to coordinate axes.

Consider an n-dimensional Euclidean space R^n, $n < \infty$.

A set $\Omega \subset R^n$ is convex along all lines parallel to coordinate axes if $\forall i \in [1,n]$ $\forall x_1, x_2 \in \Omega$ such that

$$x_1=(a^1,\dots, a^{i-1}, x_1^i, a^{i+1},\dots, a^n),$$
$$x_2=(a^1,\dots, a^{i-1}, x_2^i, a^{i+1},\dots, a^n)$$

and $\forall \lambda \in (0,1)$: $x_3=\lambda x_1 +(1-\lambda) x_2 \in \Omega$.

A cross is an example of a set convex along coordinate axes

A function $z(x)$ on Ω is convex downwards along all lines parallel to an i-th coordinate axis if $\forall x_1, x_2 \in \Omega$ such that

$$x_1=(a^1,\ldots, a^{i-1}, x_1^i, a^{i+1},\ldots, a^n),$$

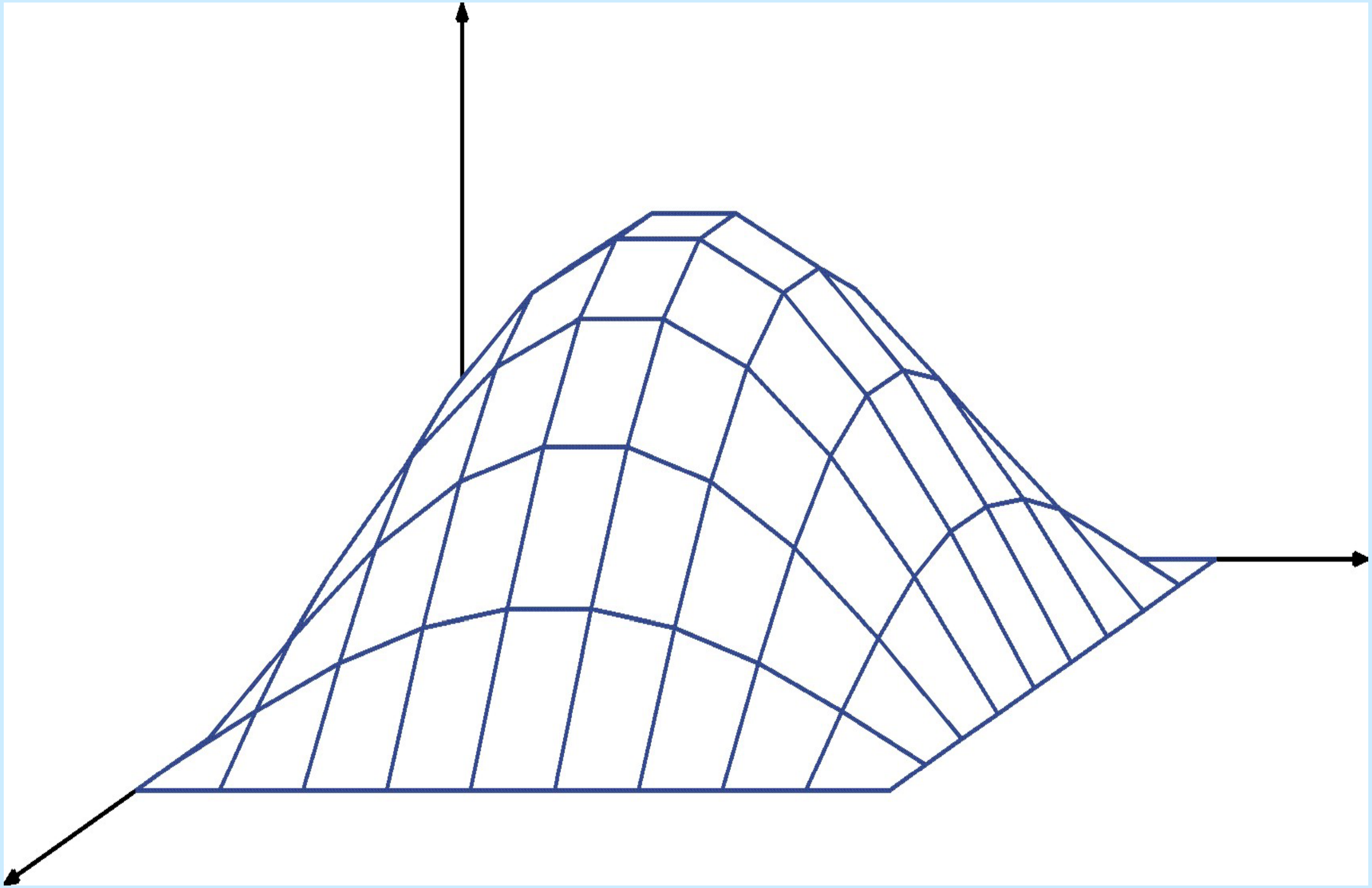
$$x_2=(a^1,\ldots, a^{i-1}, x_2^i, a^{i+1},\ldots, a^n)$$

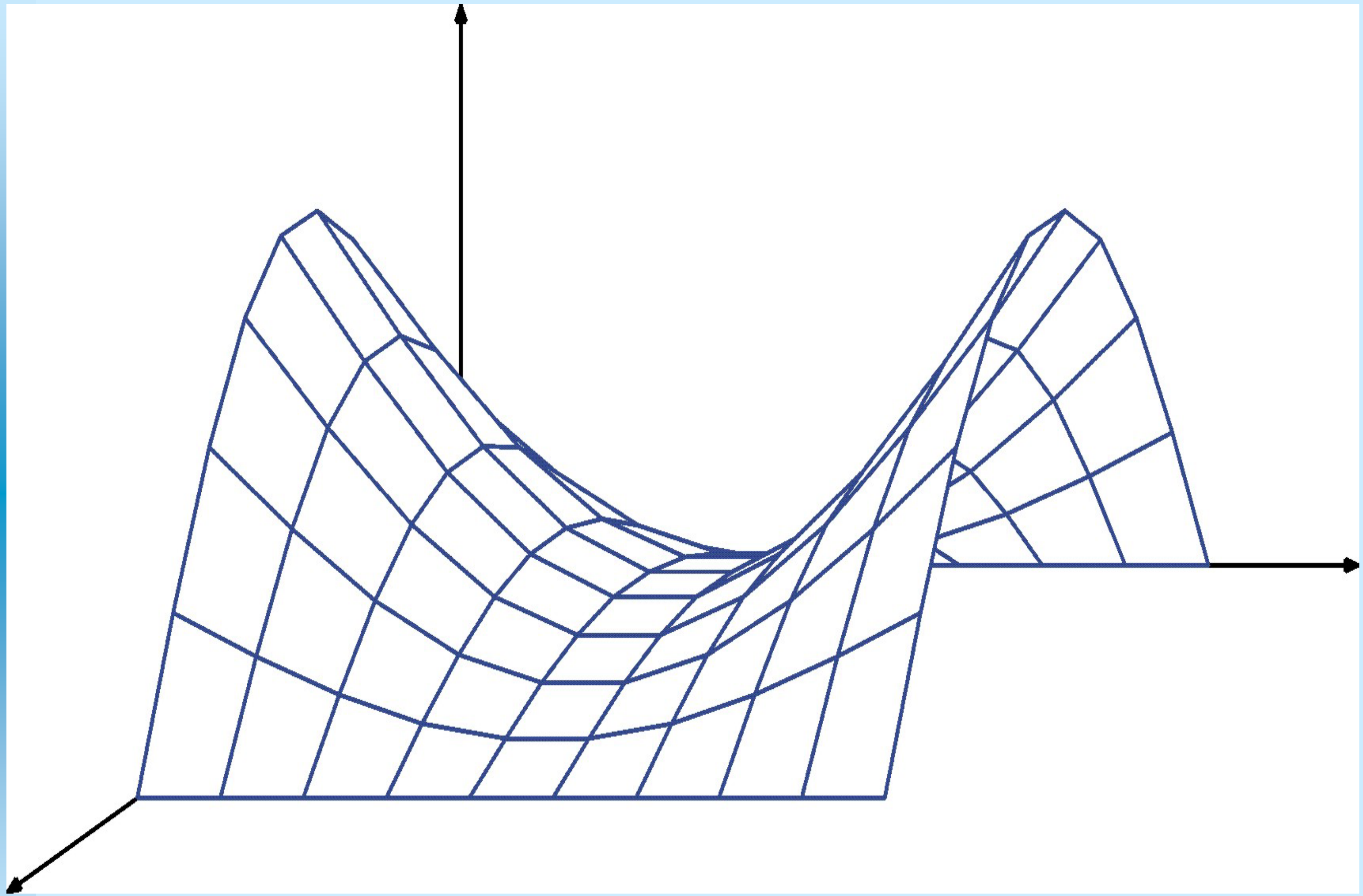
and $\forall \lambda \in (0,1)$:

$$z(\lambda x_1 + (1-\lambda) x_2) \leq \lambda z(x_1) + (1-\lambda) z(x_2)$$

Let $n^* \in [0,n]$. Consider functions $z(x)$ given on Ω. By $M_{n^*}^n(\Omega)$ define the set of functions $z(x)$ that are convex downwards along all lines parallel to n^* first coordinate axes and convex upwards along all lines parallel to $(n-n^*)$ last coordinate axes.

Assume there exist finite numbers C^L and C^U such that $\forall x \in \Omega$ and $\forall z(x) \in M_{n^*}^n(\Omega)$: $C^L \leq z(x) \leq C^U$.





Theorem 1.: Let there be a sequence $\{z_m\}$ and an element z such that $\forall m \in 1,\ldots,+\infty$: $z_m \in M^n{}_{n*}(\Omega)$, $z \in L^p(\Omega)$, $p \geq 1$, $\|z_m - z\|_{Lp(\Omega)} \to 0$ as $m \to +\infty$, where Ω is an open bounded set. Then from the sequence $\{z_m\}$ a subsequence $\{z_m{}^{(k)}\}$ may be taken that converges to a function $\check{z} \in M^n{}_{n*}(\Omega)$ at any point of Ω and $\check{z} = z$ in $L^p(\Omega)$.

Corollary 1.: $M^n{}_{n*}(\Omega)$ is a compact set in $L^p(\Omega)$.

Corollary 2.:The sequence $\{z_m(x)\}$ considered in Theorem 2.1 converges to the function $\check{z}(x)$ at any point of Ω.

Theorem 2.: Let $\|z_m - z\|_{Lp(\Omega)} \to 0$ as $m \to \infty$, where $z_m, z \in M^n{}_{n*}(\Omega)$, $p \geq 1$ and Ω is an open bounded set. Then the sequence $\{z_m\}$ converges to z uniformly on any closed set $\upsilon \subset \Omega$.

Let $D=[a_1,b_1]\times[a_2,b_2]\times\ldots\times[a_n, b_n]$. On each segment $[a_i ,b_i]$, we define a grid $X_i=\{x_i^j\}_{j=1}^{ni}$ such that $a_i = x_i^1 < x_i^2 <\ldots< x_i^{ni} = b_i$. Let $X=X_1\times X_2\times\ldots\times X_n$. A vector of indices $J=(j_1, j_2,\ldots, j_n)$ for a grid point with coordinates $(x_1^{j1},x_2^{j2},\ldots,x_n^{jn})$. Then the point is written as x_J.

For any $x\in D$ there is a set $B_J=[x_1^{j1}, x_1^{j1+1}]\times\ldots\times[x_n^{jn}, x_n^{jn+1}]$: $x \in B_J$. As an approximation of a function $z(x)$ we use a function $z_N(x)$ that is linear on grid values of $z(x)$ at vertices of B_J.

After finite dimensional approximation we obtain a set $\check{Z}_M$ which is a polytope.

If x_1, x_2, x_3 are grid points that belong to a line parallel to an i-th coordinate axis and there is no another grid point between them, then for a uniform grid X_i: $-z_1 + 2z_2 - z_3 \leq 0$ ($i \leq n^*$) or $z_1 - 2z_2 - z_3 \leq 0$ (otherwise). ($z_k = z(x_k)$)

Error estimation

1) Find the minimum and the maximum values for each coordinate of $\check{Z}_M^\eta$. Denote them by z^l_i and z^u_i, $1 \le i \le n$. They form vectors $\check{z}^l$, $\check{z}^u$.
2) Secondly, using $\check{z}^l$, $\check{z}^u$ we construct functions $z^l(x)$ and $z^u(x)$ close to Z_M^η such that $\forall z \in Z_M^\eta$: $z^l(x) \le z(x) \le z^u(x)$.

Therefore, we should minimize a linear function on a convex set. We may approximate the set by a convex polyhedron and solve a linear programming problem. The simplex-method or *the method to cut convex polyhedra* may be used. We also may construct the sequence $W_0 \supset W_1 \supset \ldots \supset W_m$ of convex polyhedrons contained the point of minimum.

Let $D=[0,d_1]\times[0,d_2]$, $d_1, d_2<+\infty$, and for $w(x,y,t)$ there are the heat conduction equation and zero boundary conditions:

$$\frac{\partial w}{\partial t} = a^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$w(0,y,t) = 0 \quad w(d_1,y,t) = 0$$

$$w(x,0,t) = 0 \quad w(x,d_2,t) = 0$$

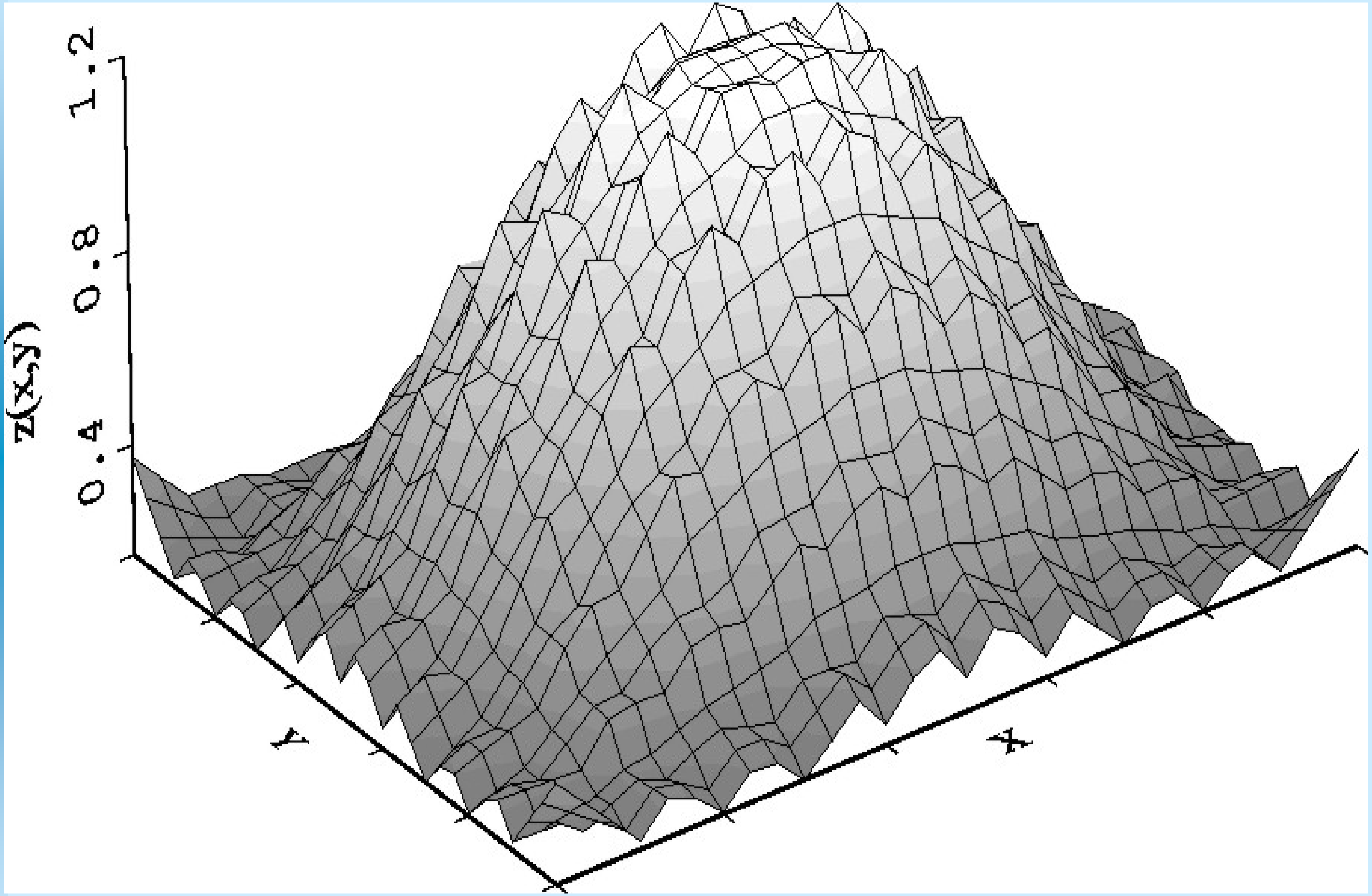
Denote $z(x,y)=w(x,y,0)$, $u(x,y)=w(x,y,T)$, $0<T<+\infty$. Therefore

$$u(x,y)=\iint G(x, y,\xi,\eta, T)\, z(\xi,\eta)\, d\xi d\eta.$$

Assume the exact solution $z \in M^2_0(D)$. We set $n_1 = n_2 = 11$, $d_1 = d_2 = 1.0$, the grids are uniform, $a = 1.0$, $T=0.001$. As the exact solution the function $z(x,y) = \sin(\pi x) \cdot \sin(\pi y)$ is taken. The approximate right-hand side we take as $u_\delta = \bar{u}$. The error of finite dimensional approximation $\Delta = 0.01 \cdot \|\bar{u}\| \approx 0.005$.

In the figure there is an upper function $z^U(x,y)$ that bounds all approximate solutions. To construct it we use additional grid values.

We find that $\|z^U - z^L\| = 0.212$ $(\approx 0.424 \cdot \|z\|)$.


1.2
0.8
0.4
z(x,y)
y
x