Some property of $\mathbb{C}$-convex sets

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Let are given a set $E \subset \mathbb{C}^n$, $\theta=(0,0,\ldots,0)\in E$ and a point $z^o \in \mathbb{C}^n \setminus E$. We denote by $\Gamma(z^o)$ a set of points $w \in \mathbb{C}^n$, such that hyperplane $\{z | \langle w, z\rangle = 1\}$ pass through z° and does not cross E.

Aizenberg L. *Линейные функционалы в пространствах аналитических функций и линейная выпуклость в* $\mathbb{C}^n$ / Linear funtionals in the spaces of analytical functions and linear convexity in $\mathbb{C}^n$ //*Исследования по линейным операторам в теории функций. 99 нерешенных задач линейного и комплексного анализа: Зап. Науч. Семинаров ЛОМИ*/Studies on linear operator in functions theory. 99 unsolved problems of linear and complex analysis:.- *Л.: Наука*, 1978.- **81**.- P.29-32.

The hypothesis of Aizenberg. A bounded linearly convex domain D, $\theta \in D$, be $\mathbb{C}$- convex iff the sets $\Gamma(z)$ are connected for all $z \in \partial D$.

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We shall say that hyperplane L is tangent to the set E, if $L \cap E \subset \partial E$, $L \cap E \neq \varnothing$.

In the beginning theorem will be proved about component of conjugated set for compact K, all tangent hyperplane to which cross K on connected set. From this theorems will be received positive solution of above hypothesis.

Before proof the main result will give proof of the row of the auxiliary lemmas under more general suggestions, than us it is necessary for proof of the theorems. This is caused that lemmas present the independent interest and characterize the properties of compacts in Euclidean spaces.

Lemma 1. Let K be non connected linearly convex compact, then for arbitrary its component K_o there is hyperplane tangent simultaneously to K_o and $K \setminus K_o$.

Proof. 1. We shall show at first that for arbitrary component of $K_0 \subset K$. there is the projection on a line, such that $\pi(\partial K_0) \cap \partial\pi(K) \neq \varnothing$ ($\pi(A)$ be an image of a set A under projection π). More exactly, the projection exists on some line such that $\pi(x) \in \partial\pi(K)$.

Since K is linearly convex compact then in arbitrary neighborhood of the point x is possible to find hyperplane L, not crossing with K.. We shall project K on a line, orthogonal to L. Obviously that $\pi(L) \notin \pi(K)$.
The set of all projection on lines is compact and will assign the projective space $\mathbb{C}P^{n-1}$. Consequently, to the sequence of hyperplanes in neighborhood of the point x corresponds to the sequence of points in $\mathbb{C}P^{n-1}$.

We shall choose from it convergent subsequence. Obviously that to limit point corresponds to the direction to projections, under which $\pi(x) \in \partial\pi(K)$.

2. If for a certain projection $\mathbb{C}^n$ on a line $\pi(K \setminus K_o) \cap \pi(K_o) = \emptyset$ that obviously that exists the other projection π_1, on other line, such that $\pi_1(K \setminus K_o) \cap \pi_1(K_o) \neq \emptyset$. We shall connect in ensemble $\mathbb{C}P^{n-1}$ points, corresponding to projection π and π_1, by a continuous path. Obviously that exists the point, belonging to this path and corresponding to projection π_2, for which $\pi_2(K \setminus K_o) \cap \pi_2(K_o) = \emptyset$ and besides,

$$\mathrm{int}\,\pi_2\,(K\backslash K_o) \not\subset \pi_2(K_o) \qquad (1)$$

and

$$\mathrm{int}\,\pi_2(K_o) \not\subset \pi_2\,(K\backslash K_o) \qquad (2)$$

that is to say exists the point $y \in \partial\pi_2\,(K\backslash K_o) \cap \partial\pi_2\,(K_o)$. So hyperplane $\pi_2^{-1}(y)$ is tangent simultaneously to K_o and $K \setminus K_o$.

3. Let for all projection on analytical lines

$$\pi\,(K\backslash K_o) \cap \pi\,(K_o) \neq \varnothing. \qquad (3)$$

If for a certain projection are executed simultaneously condition (1) and (2), that, as we already saw in **2**, lemma is true.

Let now condition (3) exists, but nor for what projections is not executed simultaneously (1) and (2). According to proved in 1, exists the projection $\pi 1$ for which is met the condition (1) and similarly exists the projection $\pi 2$, for which is met the condition (2). The Projection $\pi 1$ and $\pi 2$ correspond to two points in ℂPn-1. On suggestion, for projection $\pi 1$ we have inclusion

$$\text{int } \pi_1(K_o) \supset \pi_1 (K \backslash K_o). \qquad (4)$$

The set of projection, satisfying condition (4), is open in $\mathbb{C}P^{n-1}$

Similarly for projection π_2

$$\text{int } \pi_2(K \backslash K_o) \supset \pi_2 (K_o). \qquad (5)$$

The set of projection, satisfying condition (5), is open too in $\mathbb{C}P^{n\text{-}1}$. According to suggestions made, for each point from $\mathbb{C}P^{n\text{-}1}$ is executed one of the conditions (4) and (5), but obviously that for arbitrary point can be executed only one of the conditions (4) or (5) because of compactness K_o and $K \backslash K_o$. Consequently, connected space $\mathbb{C}P^{n\text{-}1}$ is presented as union of two open disjoint nonempty sets. From got contradictions follows that is the projection π, for which are executed simultaneously both conditions (1) and (2). But then according to **2** there exists hyperplane, tangent simultaneously to K_o and $K \setminus K_o$**.** The Lemma is proved.

Corrolary 1. Let $K \subset \mathbb{C}^n$ be arbitrary compact and exists the projection of K on line such that image of K. under this projections is non connected. That there is hyperplane, tangent simultaneously the least to two components from K..

Proof is similar to **2** of lemma 1.

Definition 1. The map $f: X \to Y$ is called monotone if preimage $f^{-1}(y)$ of each point $y \in Y$ is connected.

Lemma 2. Let f be a monotone mapping of compact K.. Than induced homomorphism of cohomology groups

$$f^*: H^1(f(K)) \to H^1(K)$$

is a monomorphism.

The Lemma presents itself part of the theorem Sklyarenko .

Lemma 3. Let D be bounded domain in real Euclidean space R^n. Then for arbitrary $j \geq 0$ in exact sequence of the cohomology groups

$$H^j(\overline{D}, \partial D) \to H^j(\overline{D}) \xrightarrow{\tau^*} H^j(\partial D) \to H^{j+1}(\overline{D}, \partial D)$$

τ^* is a monomorphism.

Proof. From Aleksander's duality theorem we have isomorphisms

$$D^*\colon H^j(\overline{D}) \approx H_{n-j-1}(R^n \backslash \overline{D})$$

$$D^*\colon H^j(\partial D) \approx H_{n-j-1}(R^n \backslash \partial D) \approx H_{n-j-1}(R^n \backslash \overline{D}) \oplus$$

$$\oplus \quad H_{n-j-1}(D, \partial D)$$

and diagram

$$\begin{array}{ccc} H^j(\overline{D}) & \xrightarrow{\tau^*} & H^j(\partial D) \\ D^* \downarrow & & \downarrow D^* \\ H_{n-j-1}(R^n \backslash \overline{D}) & \xrightarrow{\tau_*^!} & H_{n-j-1}(R^n \backslash \partial D) \end{array}$$

is commutative because of compactness pair $(\overline{D}, \partial D)$.

The embedding $\tau:\partial D\rightarrow\overline{D}$ give embedding of complements $\tau^{1}: R^{n}\backslash\overline{D}\rightarrow R^{n}\backslash\partial D$. Consequently, induced homomorphism $\tau_{*}^{1}= D^{*}\tau^{*}D^{*-1}$ will assign embedding

$$\tau_{*}^{1}: H_{n-j-1}(R^{n}\backslash\overline{D})\xrightarrow{\tau_{*}^{1}} H_{n-j-1}(R^{n}\backslash\partial D)$$

and is a monomorphism. But then $\tau^{*}=D^{*-1}\tau_{*}^{1}D^{*}$ is a monomorphism too.

Theorem 1. Let $K\subset\mathbb{C}^{n}$, $\Theta\in K$,be such compact that all sections K by tangent hyperplanes are connected. Then each connected component of the set, K^{*} is $\mathbb{C}$-convex domain.

Proof. For projection K on a line the preimage of arbitrary point $y \in \pi(K)$ will be a section K by hyperplane $\pi^{-1}(y)$, which under projection of $\mathbb{C}^n$ maps to point y. According to assessed condition preimages of points from $\partial\pi(K)$ are connected. Except this, corrolary 1 provides connectednece of $\pi(K)$. We shall suppose that $H^1(\pi(K)) \neq 0$. Then by lemma 2 $\tau^* : H^1(\pi(K)) \to H^1(\partial\pi(K))$ is a monomorphism, but according to lemma 3 $\pi_1^* : H^1(\partial\pi(K)) \to H^1(\pi^{-1}(\partial\pi(K) \cap K))$ is a monomorphism too - мономорфизм by connectednece of all preimages of points from $\partial\pi(K)$.

We shall consider commutative diagram

$$\begin{array}{ccc} \pi^{-1}(\partial\pi(K)\cap K) & \subset & K \\ \downarrow & \tau & \downarrow \\ \partial\pi(K) & \subset & \pi(K) \end{array}$$

To it corresponds commutative diagram of the cohomology groups

$$\begin{array}{ccc} H^1(\pi(K)) & \xrightarrow{\tau^*} & H^1(\partial\pi(K)) \\ \pi^*\downarrow & & \downarrow\pi_1^* \\ H^1(K) & \rightarrow & H^1(\pi^{-1}(\partial\pi(K)\cap K)). \end{array}$$

From commutativity of diagram and monomorphity of $\pi_1^*\tau*$ follows that π^* is monomorphism.

To each cocycle $c \in H^1(\pi(K))$ corresponds by Aleksander's duality dual cycle in $\bar{H}_0$ $(\mathbb{C} \setminus (K))$, where $\overset{o}{\mathbb{C}} = \mathbb{C} \cup \infty$. Accordingly, to pair of points (x,∞) (x lies in a certain bounded component of the set $\overset{o}{\mathbb{C}} \setminus \pi(K)$, which is a cycle in $\bar{H}_0\left(\overset{0}{\mathbb{C}} \setminus \pi(K)\right))$ one-to-one corresponds the dual cocycle in $H^1(\pi(K))$. For arbitrary point $x \in \overset{0}{\mathbb{C}} \setminus \pi(K)$ we shall consider hyperplane $\pi^{1}(x)$ and ($2n$ - 2)-dimensional cycle $\pi^{-1}(x) \cup \infty = S^{2n-2}$ (homeomorphe to ($2n$ - 2)- dimensional sphere).

Let $(x,\ \infty)$ and (x_1,∞) be two cycles, dual to not cohomological between itself on $\pi(K)$ cocycles c and c_1. Then from monomorphity of $\pi*$ the cycles S^{2n-2} and S_1^{2n-2}, dual to not cohomological between itself on K cocycles $\pi*(c)$ and $\pi*(c_1)$ accordingly,are not homological between itself in $\overset{0}{\mathbb{C}}{}^n \setminus K$, where $\overset{0}{\mathbb{C}}{}^n = \mathbb{C}^n \bigcup \infty$.

At first, this signifies that the set *K*. is not connected, since points in *K*., assigning hyperplanes $\pi^{-1}(x)$and $\pi^{-1}(x_1)$, must lie in different cjnnected component of *K*. In second, to projection π on line corresponds to certain section of open set $K*$ by line *l,* passing through begin coordinates. Exists homeomorphism $\overset{0}{\mathbb{C}} \setminus \pi(K) \approx l \bigcap K*$.

Since, as it was seen above, points in different bounded components $\overset{0}{C} \setminus \pi(K)$ will assign the non homological to each other in $\overset{0}{\mathbb{C}}{}^{n} \setminus K$ cycles S^{2n-2} and S_1^{2n-2}, that this signifies that different components of the section $l \cap \bar{K}$ belongs to the miscellaneous a component of the set K^*. If l be a line, not passing through begin coordinates then $l \cap \bar{K}$ homeomorphely corresponds to the set $\mathbb{C}P^1 \setminus \pi_1(K)$, where CP^1 is a projective space, formed by ensemble hyperplanes $A = \{w | \langle w, z \rangle = 1, z \in l\}$ and hyperplane L, which containing the begin of coordinates and the general for family A $(n-2)$-plane T, π_1 be a projection on this space. In other words, $\mathbb{C}P^1$ be a set of hyperplanes containing $(n - 2)$-plane T, $T \cap K = \varnothing$.

The reasons, similar called on above, show that π_1 induce monomorphism $\pi_1^*: H^1(\pi_1(K)) \to H^1(K)$ and that different components of the section $l \cap K^*$ belongs to the different components of the set K^*. This signifies that sections of arbitrary components of the set K^* by lines are connected. The set K is connected , so section of the set K^* by lines are simple connected. Then for arbitrary component of a set K^* its sections by lines are connected and simple connected. But then every component of the set K^* is $\mathbb{C}$-convex domain.

Corrolary 2. Let $K \subset \mathbb{C}^n$ be connected compact lying on a line, that;

1) $H^1(K) \approx \bar{H}_0(K^*)$;

2) every component of the set K^* is $\mathbb{C}$-convex domain.

The First statement follows from Vietoris-Begle theorem and duality. The Second follows from theorem 1, since arbitrary line, not complying with the line containing *K*, can cross *K* no more than on one point. This corrolary shows the structure of conjugated set.

Now we shall go to consideration of the hypothesis, quoted at the beginning.

At first we shall find the description of set $\Gamma(z^\circ)$ by means of conjugated set D^*. Obviously that to hyperplane $\{z | \langle w^0, z \rangle = 1\}$ got through point z^0, necessary and it is enough to $\langle w^0, z^0 \rangle = 1$. Consequently, set of points, which define hyperplanes, containing point *w,* fills hyperplane $\{w | \langle w, z^0 \rangle = 1\}$. On the other hand is known that for bounded domain $D, \Theta \in D$, point w° belongs to the boundary ∂D^* of compact D^* iff , when hyperplane $\{z | \langle w^0, z \rangle = 1\}$ pass through some point of the boundary ∂D, but does not cross domain D. Thereby following statements .are true

Sentence 1. Set $\Gamma(z^\circ)$, $z^\circ \in \partial D$, in accuracy complies with intersection of hyperplane $\{w | \langle w, z^0 \rangle = 1\}$ with boundary ∂D^*.

Sentence 2. An intersection of compact D^* with tangent plane will be set $\Gamma(z^\circ)$ for a certain point $z^0 \in \partial D$.

We shall notice that sets $\Gamma(z^0)$ and $\Gamma(z^1)$ for pair of the miscellaneous point can, generally speaking, coincide.

Теорема 2. Bounded domain $D \subset \mathbb{C}^n, \Theta \in D$, *is* $\mathbb{C}$-convex iff, when sets $\Gamma(z)$ are non empty and connected for all point $z \in \partial D$.

Proof. Need of the theorem obvious on the strength of sentence 1 and $\mathbb{C}$-convexity of compact D^*. We shall show its sufficiency. We shall consider compact D^*. It is linearly convex. Consequently, the set $\mathbb{C}^n \setminus D^*$ is connected. Is easy see that for compact $K = D^*$ are true all condition of the theorem 1 since sections K by tangent hyperplanes are connected in view of sentence 2. By theorem 1 each component of et D^{**} is $\mathbb{C}$-convex domain. Easy check that if set $\Gamma(z)$ is not empty for each point $z \in \partial D$, that domain D , will be one of the component of a linearly convex set D^{**}. But then $K = D^*$ $\mathbb{C}$- convex compact and $D^{**} = D$.The Theorem is proved

This theorem gives the geometric criterion of $\mathbb{C}$-convexity of domain.

Corrolary *3*. Let $D \subset \mathbb{C}^n$ be a linearly convex domain, $n > 1$, such that each point $z \in \partial D$ belongs to only one tangent hyperplane. Then D is $\mathbb{C}$-convex domain.

Look like theorems 1 and 2 results possible to install and in the event of domains.

Theorem 3. Let $D \subset \mathbb{C}^n$, $\Theta \in D$ be a domain,, for which all sections D by hyperplanes are connected*,* that each component of set D is $\mathbb{C}$-convex compact.

Proof similarly proof of the theorem 1. Also similarly lemma 1 possible to prove following statement.

Sentence 3. Let E be not connected linearly convex open set then for arbitrary its component D_0 there is hyperplane, tangent D_0 and $E \setminus D_0$ simultaneously.

Теорема 4 Compact $K \subset \mathbb{C}^n, \Theta \in K$ is $\mathbb{C}$- convex iff, when sets $\Gamma(z)$ are not empty and connected for arbitrary point

$$z \in \mathbb{C}^n \setminus K$$

Proof to similarly theorem 2.

Sentence 4. Let D be $\mathbb{C}$-convex domain, that sections D^* by lines are connected.

Proof. If $\gamma \cap \bar{D}$ be certain section of domain $\bar{D}$ by γ is not connected, that shall choose the pair of points *a, b* in miscellaneous components of the section $\gamma \cap \bar{D}$. Let $\{\alpha_m\}$, $\{\beta_m\} \subset D$ be two sequences of points convergent to *a, b* accordingly. Through each pair of points α_m, β_m we shall conduct line γ_m, which crosses D on aacyclic open set $D \cap \gamma_m$ because D is ℂ- convex. We shall connect inwardly set $D \cap \gamma_m$ points $\alpha_{m,} \beta_m$ by continuums σ_m. From sequence $\{\sigma_m\}$ we shall select converging subsequence $\{\sigma_j\}$ which in limit will give the continuum σ lying in $\gamma \cap \bar{D}$ and connected points *a,b.*

Sentence 5. Let K be $\mathbb{C}$- convex compact, that int K consists of $\mathbb{C}$-convex domain.

Proof. The set $D = K^*$ is $\mathbb{C}$-convex domain. From sentence 4 follows that arbitrary section $\bar{D}$ by line, including tangent, is connected. Then from theorem 1 follows that следует, что D^{**} consists of $\mathbb{C}$- convex component. But from continuity of set value mapping Φ in finite points $\Phi\left(\bar{D}\right) = \overline{\Phi(D)}$, if int $K \neq \varnothing$, $\Theta \in$ int K (last easy to obtain by linear shift) и and so

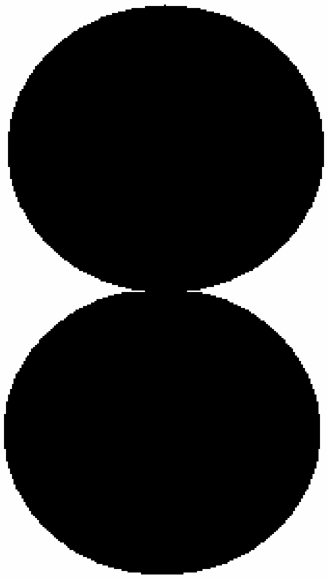
$$\bar{D} = \mathbb{C}^n \setminus \Phi\left(\bar{D}\right) = \mathbb{C}^n \setminus \overline{\Phi(D)} = \mathbb{C}^n \setminus \left(\overline{\mathbb{C}^n \setminus D^*}\right) = \mathbb{C}^n \setminus \left(\mathbb{C}^n \setminus K^{**}\right) = \mathbb{C}^n \setminus \left(\overline{\mathbb{C}^n \setminus K}\right) = \text{int } K.$$

But if int $K = \varnothing$ statement is trivial.

Example 1. Let K be union of two circles

$$K = \left\{ z \middle| (|z-i| \le 1) \vee (|z+i| \le 1) \right\} \subset \mathbb{C}$$

Obviously that K is $\mathbb{C}$- convex compact. Except this int K is not conntcted.

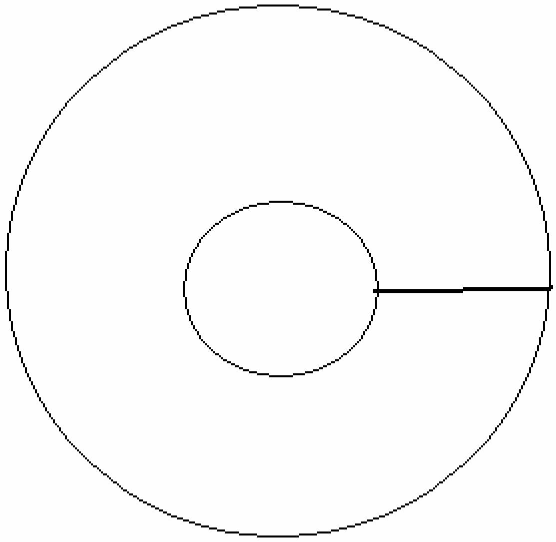


Example 2. Let $A=\left\{z=(z_1,z_2)\middle\|z\right|\le 1,\operatorname{Im} z_2\ge 0\}$ be hemiball, but $B=\left\{z=(z_1,z_2)\middle\|z_2-i\right|\ge 1\}$ be an open unlimited cylinder in $\mathbb{C}^2$. We shall consider compact $K=A\backslash B$. Any section of compact K by line, different from $z_2=const$, , is of the form of intersection two ensembles: 1) hemiline $\operatorname{Im} z_2\ge 0$ with thrown away by ball $|z_2-i|<1$ and 2) of the ball of the radius not more 1; moreover if ball completely lies in hemiline $\operatorname{Im} z_2\ge 0$, that its radius is less then 1. So K is $\mathbb{C}$- convex compact. Obviously that int K consists of two component.

Remark. We shall notice that equality $\overline{D}^* = \operatorname{int} D^*$ used in proof of the sentence 5 is true for any bounded (not only $\mathbb{C}$-convex) domain, but for unbounded domain it can be broken.

Example 3. Let $D = D_1 \times \mathbb{C}^{n-1}, n > 1,$ where D_1 is a flat domain. Then $D^* \approx \overset{0}{\mathbb{C}} \setminus D_1 \subset \mathbb{C},$ but $(\overline{D})^* \approx \overset{0}{\mathbb{C}} \setminus \overline{D}_1 \subset \mathbb{C},$ and, consequently,, $(\overline{D})^* \neq \operatorname{int} D^* = \varnothing$.

Example 4. Let $D=\left\{z=(z_1,z_2,...,z_n)\in\mathbb{C}^n \left| 1<|z_1|<2, z_1\notin[1,2]\right.\right\}$. Easy to check that D $\mathbb{C}$-convex domain, but $\overline{D}$ is already not $\mathbb{C}$-convex compact.



Problem 1. Unknown, always closure of $\mathbb{C}$-convex domain will be a linearly convex set.

The next two problems are related to compactyfication of sets. At first: can $\mathbb{C}$-convex domain have the internal boundaries, and at second can $\mathbb{C}$-convex compact have point of the boundary that does not belong to the closure of his inside?

Problem 2. Let $K \subset \mathbb{C}^n$ be $\mathbb{C}$-convex compact with nonempty inside $\operatorname{int} K \neq \varnothing$. Is this true that $\overline{\operatorname{int} K} = K$ for $n > 1$?

Problem 3. Let $D \subset \mathbb{C}^n$ be bounded $\mathbb{C}$-convex domain. Is this true that $\operatorname{int} \overline{D} = D$ for $n > 1$?

For unbounded domains, and for $n = 1$ that is not true.

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