1. DEFINITIONS AND MAIN RESULTS

We will study extremal mappings in the sense of Lempert function and in the sense of Kobayashi-Royden pseudometric, so let us recall the objects we will deal with in this paper. Let $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$. Let $D \subset \mathbb{C}^n$ be a domain and let $z, w \in D$, and $v \in \mathbb{C}^n$. In this paper we consider two objects:

$$k_D(z,w) := \inf\{\text{hyp dist}\,(\zeta,\xi) : \exists f \in \mathcal{O}(\mathbb{D}, D) \text{ such that } f(\zeta) = z, f(\xi) = w\}, \tag{1}$$

and

$$\kappa_D(z,v) := \sup\{\lambda > 0 : \exists f \in \mathcal{O}(\mathbb{D}, D) : f(0) = z, f'(0) = \lambda v\}. \tag{2}$$

Using appropriate automorphisms of the unit disk, we can always assume that in (1) $\zeta = 0$. First one we call *Lempert function* and second - $Kobayashi - Royden$ *pseudeometric*. We call $f : \mathbb{D} \to D$ a k_D-extremal (resp. κ_D-extremal) if for f in (1) (resp. (2)) the 'inf' is attained for some $z, w \in D$, $z \neq w$ (resp. the 'sup' is attained for some $z \in D$, $X \in \mathbb{C}^n \setminus \{0\}$). In general k is not a pseudodistance - consider a domain $D_\alpha := \{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1, |zw| < \alpha\}$, then for all $\alpha \in (0,1)$ the triangle inequality does not hold for k_{D_α}. More examples the reader may find in [4].

To overcome the difficulty connected with the triangle inequality we modify the function k_D in such a way that the new function becomes a pseudodistance. For $z, w \in D$ we put

$$k'_D(z,w) := \inf\{\sum_{j=1}^{N} k_D(z_{j-1}, z_j) : N \in \mathbb{N}, z_0 = z, z_1, ..., z_N \in D, z_N = w\}.$$

The function k'_D is called the *Kobayashi pseudodistance* for D.

However, if D is strictly linearly convex, k_D will be a distance. This is because of

Theorem 1.1. *Let $D \subset \mathbb{C}^n$ be a strictly linearly convex domain with $\mathcal{C}^k$ boundary ($k = \infty$ or $k = \omega$). Then $k_D = k'_D = c_D$, where for $z, w \in D$ we define*

$$c_D(z,w) = \sup\{\text{hyp dist}\,(F(z), F(w)) : F \in \mathcal{O}(D, \mathbb{D})\}. \tag{3}$$

Function (3) is called a *Carathéodory distance*.

Our main goal is to describe extremals in the sense of Lempert function and in the sense of Kobayashi-Royden pseudometric, in the case when D is strictly linearly convex domain with $\mathcal{C}^k$ boundary (in this paper we always assume that $k = \infty$ or $k = \omega$). We say that

Definition 1.2 (See [1]). *Let $D \subset \mathbb{C}^n$ be a bounded domain. D is called* linearly convex *if trough any boundary point $z \in \partial D$ there goes an $(n-1)$-dimensional complex hyperplane that is disjoint from D. D is called* strictly linearly convex *if*

(1) D has $\mathcal{C}^2$-smooth boundary,

(2) the defining function r of D satisfies the inequality

$$\sum_{j,k} r_{z_j \bar{z}_k}(a) w_j \overline{w}_k > \left| \sum_{j,k} r_{z_j z_k}(a) w_k w_k \right|,$$

where $a \in \partial D$, $w = (w_1, ..., w_n) \in (\mathbb{C}^n)_*$ with $\sum_j r_{z_j}(a) w_j = 0$.

We remark here that in the following sections D will always denote a strictly linearly convex domain which is, for the sake of simplicity, bounded by a real analytic hypersurface.

Remark 1.3. *If D is strictly linearly convex domain, then each complex tangent plane intersect the boundary ∂D in precisely one point.*

In addition, we shall use the following notations: $\mathcal{C}^k(K)$, where K is compact subset of $\mathbb{C}^n$, denotes the spaces of all mappings that are $[k]$-times differentiable in the interior of K, and in the case when k is an integer the derivatives up to order k extend continuously to K, in other case, i.e. $k-[k]:=c>0$, the derivatives up to order $[k]$ are c-Hölder continuous; $\mathcal{C}^\omega(K)$ denotes the set of functions that extend analytically to a neighborhood of K. Generally, if A is an arbitrary set in $\mathbb{C}^n$, then $\mathcal{C}^k(A)=\bigcap\{\mathcal{C}^k(K): K$ compact and $K\subset A\}$. $|\cdot|$ denotes the euclidean norm in $\mathbb{C}^n$. For $(z_1,...,z_n)\in\mathbb{C}^n$ we define $\widehat{z}:=(z_2,...,z_n)$, and similarly, if $f=(f_1,...,f_n)$ is a mapping into $\mathbb{C}^n$, then by $\widehat{f}$ we define a mapping $(f_2,...,f_n)$ into $\mathbb{C}^{n-1}$. Finally: for $z=(z_1,...,z_n)$, $w=(w_1,...,w_n)\in\mathbb{C}^n$ $z\cdot w:=\sum_j z_jw_j$.

Before we formulate the main result of this paper we need another definition.

Definition 1.4. *Let $D\in\mathbb{C}^n$ be a domain. We call a holomorphic mapping $f:\mathbb{D}\to D$ an $E-$*mapping, *if*

(1) f extends to a $\mathcal{C}^k$ function on $\overline{\mathbb{D}}$ (to be denoted by the same letter f);

(2) $f(\partial\mathbb{D})\subset\partial D$;

(3) there exist a positive $\mathcal{C}^k$ function $\rho:\partial\mathbb{D}\to\mathbb{R}$ such that the mapping $\partial\mathbb{D}\ni\zeta\mapsto\zeta\rho(\zeta)\overline{\nu(f(\zeta))}\in\mathbb{C}^n$ extends to a $\mathcal{C}^k$ mapping $\widetilde{f}:\overline{\mathbb{D}}\to\mathbb{C}^n$, holomorphic in $\mathbb{D}$ (here $\nu(z)$ denotes the outward unit normal vector to ∂D in z);

(4) the winding number of the function $\varphi(\zeta):=\overline{\nu(f(\zeta))}\cdot(z-f(\zeta))$ on $\partial\mathbb{D}$ is zero for all $z\in\mathbb{D}$.

Furthermore, we shall call a holomorphic mapping $f:\mathbb{D}\to D$ weakly-$E-$mapping *if it possesses the above properties (1)-(4) with $k=1/2$.*

Soon we shall see that there is no difference between $E-$mappings and weakly-$E-$mappings. $f(\mathbb{D})$ will be called a (weak) $E-$disk, if f will be a (weak) $E-$mapping.

From definition we have to compare f with all other $g\in\mathcal{O}(\mathbb{D},D)$ to check that f is extremal. Next theorem shows how to describe extremal mappings, by checking certain properties of f alone.

Theorem 1.5. *Let D be a strictly linearly convex domain with a $\mathcal{C}^k$ boundary ($k=\infty$ or $k=\omega$). Then a holomorphic mapping $f:\mathbb{D}\to D$ is extremal in the sense of Lempert function (resp. in the sense of Kobayashi-Royden pseudometric) with respect to the points $(f(0),f(\xi))$ (resp. with respect to $(f(0),f'(0))$), if and only if f is an E-mapping.*

Theorem above is the main result of this paper. The idea of its proof is the following: for any $z,w\in D$ (resp. any $z\in D$ and $v\in\mathbb{C}^n$) we prove that there is unique (weak) $E-$mapping, which is extremal for (z,w) (resp. (z,v)). Using standard tool, i.e. explicit function theorem, Arzela-Ascoli theorem we shall prove that trough any given pair of points there goes a $E-$disk. This will then establish Theorem 1.5.

2. Extremal mappings and E-mappings

Proposition 2.1. *Let $f:\mathbb{D}\to D$ be an E-mapping. Then there exists a continuous mapping $F:\overline{D}\setminus f(\partial\mathbb{D})\to\mathbb{D}$, holomorphic on D and such that $F\circ f=id_{\mathbb{D}}$.*

Proof. Set $A := \overline{D} \setminus f(\partial\mathbb{D})$ and let φ_z denote the function from the condition (4) from the definition of $E-$mapping. Since D is strictly linearly convex, φ_z does not vanish in $\partial\mathbb{D}$ for any $z \in A$, so by the continuity argument the condition (4) holds for every z in some open neighbourhood W of the set A. Consider the function $G : W \times \overline{\mathbb{D}} \to \mathbb{C}$ given by

$$G(z,\zeta) := \widetilde{f}(\zeta) \cdot (z - f(\zeta)).$$

We claim that for given $z \in W$ the equation $G(z,\zeta) = 0$ has exactly one solution $\zeta \in \mathbb{D}$. Fix $z \in W$ and let ρ be as in the condition (3). We have

$$G(z,\zeta) = \zeta\rho(\zeta)\varphi_z(\zeta)$$

for $\zeta \in \partial\mathbb{D}$, so the winding number of the function $G(z,\cdot)$ on $\partial\mathbb{D}$ is equal to 1. Since this function is holomorphic on $\mathbb{D}$, it has exactly one simple root $F(z) \in \mathbb{D}$. Therefore $G(z, F(z)) = 0$ and $\frac{\partial G}{\partial \zeta}(z, F(z)) \neq 0$. In virtue of the implicit mapping theorem, the function F is holomorphic on W. □

Let us note that for given $E-$mapping f the mapping F satisfies the equation

$$\widetilde{f}(F(z)) \cdot (z - f(F(z))) = 0 \tag{4}$$

at every point $z \in \overline{D} \setminus f(\partial\mathbb{D})$.

Proposition 2.2. *An E-mapping $f : \mathbb{D} \to D$ is the unique extremal mapping with respect to the point $z = f(0)$ and direction $v = f'(0)$, and also with respect to the couple of points $z = f(0), w = f(\xi)$, with $\xi \in (0,1)$ being arbitrary.*

Proof. We carry the proof in both cases simultaneously. Let F be as in the Proposition 2.1. Suppose $g : \mathbb{D} \to D$ is a holomorphic mapping such that $g(0) = z$ and:

- $g'(0) = \lambda v$ for some $\lambda \geq 0$, in the first case,
- $g(\eta) = w$ for some $\eta \in (0,1)$, in the second case.

The function $F \circ g$ maps the unit disc to itself and satisfy $F(g(0)) = F(f(0)) = 0$. Therefore by the Schwarz' lemma we get:

- $1 \geq |(F \circ g)'(0)| = \lambda|(F \circ f)'(0)| = \lambda$, so $|f'(0)| \geq |g'(0)|$, in the first case,
- $\eta \geq |(F \circ g)(\eta)| = |F(w)| = |(F \circ f)(\xi)| = \xi$, in the second case.

Therefore f is an extremal mapping.

We show that f is the unique extremal mapping. Suppose g is extremal. Then $\lambda = 1$ (in the first case) or $\eta = \xi$ (in the second case), so there holds the equality in the above application of the Schwarz' lemma. This implies $F \circ g = id_{\mathbb{D}}$.

We claim that $\lim_{\mathbb{D}\ni\zeta\to\zeta_0} g(\zeta) = f(\zeta_0)$ for each $\zeta_0 \in \partial\mathbb{D}$. Suppose not. Then for some $\zeta_0 \in \partial\mathbb{D}$ there is a sequence $(\zeta_m)_m \subset \mathbb{D}$ convergent to ζ_0 and such that the limit $Z := \lim_{m\to\infty} g(\zeta_m) \in \overline{D}$ exists and is not equal to $f(\zeta_0)$. Putting $z = g(\zeta_m)$ in the equation (4) we get

$$0 = \widetilde{f}(F(g(\zeta_m))) \cdot (g(\zeta_m) - f(F(g(\zeta_m)))) = \widetilde{f}(\zeta_m) \cdot (g(\zeta_m) - f(\zeta_m)).$$

Passing $m \to \infty$ gives

$$0 = \widetilde{f}(\zeta_0) \cdot (Z - f(\zeta_0)) = \zeta_0 p(\zeta_0)\, \overline{\nu(f(\zeta_0))} \cdot (Z - f(\zeta_0)),$$

so the vector $Z - f(\zeta_0)$ belongs to the complex tangent space of ∂D at $f(\zeta_0)$. Hence $Z = f(\zeta_0)$, because $Z \in \overline{D}$ and D is strictly linearly convex. This is a contradiction. □

Proposition 2.3. *If $f : \mathbb{D} \to D$ is an E-mapping and a is an automorphism of $\mathbb{D}$, then $f \circ a$ is an E-mapping.*

Proof. Set $g := f \circ a$. The conditions (1) and (2) are clear. To prove the condition (4), fix a point $z \in D$ and let φ_f, φ_g be as in the condition (4). Then $\varphi_g = \varphi_f \circ a$. The winding number of $a|_{\partial\mathbb{D}}$ is 1, so the winding numbers of the mappings φ_f and φ_g are equal.

We prove the condition (3). The winding number of the function $\zeta \mapsto \frac{\zeta}{a(\zeta)}$ on $\partial\mathbb{D}$ is 0, so there exists a real-valued $\mathcal{C}^\omega(\partial\mathbb{D})$ function v such that $\frac{\zeta}{a(\zeta)} = e^{iv(\zeta)}$ on $\partial\mathbb{D}$. Hence there exists a real-valued $\mathcal{C}^\omega(\partial\mathbb{D})$ function u such that the function $\partial\mathbb{D} \ni \zeta \mapsto \frac{\zeta}{a(\zeta)} e^{u(\zeta)} \in \mathbb{C}$ extends to a nowhere-vanishing function $h : \overline{\mathbb{D}} \to \mathbb{C}$ holomorphic on $\mathbb{D}$. Moreover, u and v are of class $\mathcal{C}^\omega$ on $\partial\mathbb{D}$, so h can be extended to a mapping of class $\mathcal{C}^\omega$ on $\overline{\mathbb{D}}$. Let ρ be as in the condition (3) for f. For $\zeta \in \partial\mathbb{D}$ put $r(\zeta) := \rho(a(\zeta))e^{u(\zeta)}$. We get

$$\zeta r(\zeta)\overline{\nu(g(\zeta))} = \zeta e^{u(\zeta)}\rho(a(\zeta))\overline{\nu(f(a(\zeta)))} = a(\zeta)h(\zeta)\rho(a(\zeta))\overline{\nu(f(a(\zeta)))} = h(\zeta)\widetilde{f}(a(\zeta)),$$

and this mapping extends to a $\mathcal{C}^\omega(\overline{\mathbb{D}})$ mapping, holomorphic on $\mathbb{D}$. □

Corollary 2.4. *An E-disc $f(\mathbb{D})$ is the unique extremal disc with respect to any couple of different points $z, w \in f(\mathbb{D})$, and also with respect to any point $z = f(\zeta)$ and direction $v = f'(\zeta)$.*

Proposition 2.5. *Let f be an E-mapping. Then the function $f' \cdot \widetilde{f}$ is a positive constant.*

Proof. Since the curve $t \mapsto f(e^{it})$ is contained in ∂D, its tangent vector $ie^{it}f'(e^{it})$ belongs to the tangent space $T_{f(e^{it})}\partial D$, so is orthogonal to $\nu(f(e^{it}))$ with respect to the real scalar product. Hence for $\zeta \in \partial\mathbb{D}$ we have

$$\operatorname{Im} f'(\zeta) \cdot \widetilde{f}(\zeta) = \rho(\zeta) \operatorname{Re}\left(i\zeta f'(\zeta) \cdot \overline{\nu(f(\zeta))}\right) = 0,$$

so the holomorphic function $f' \cdot \widetilde{f}$ is a real constant C.

The curve $[0, 1) \ni t \mapsto f(t)$ lies in D and $f(1) \in \partial D$, so the tangent vector $f'(1)$ outwards from D. Hence

$$0 \leq \operatorname{Re}\left(f'(1) \cdot \overline{\nu(f(1))}\right) = \frac{1}{\rho(1)} \operatorname{Re}\left(f'(1) \cdot \widetilde{f}(1)\right) = \frac{C}{\rho(1)}.$$

This implies $C \geq 0$. For each $\zeta \in \partial\mathbb{D}$ we have

$$\frac{f(\zeta) - f(0)}{\zeta} \cdot \widetilde{f}(\zeta) = \rho(\zeta)\,\overline{\nu(f(\zeta))} \cdot (f(\zeta) - f(0)).$$

By the condition (4), the last function has the winding number equal to 0. Therefore the holomorphic function $h(\zeta) := \frac{f(\zeta)-f(0)}{\zeta} \cdot \widetilde{f}(\zeta)$ does not vanish in $\mathbb{D}$. In particular, $C = h(0) \neq 0$. □

Proposition 2.6. *Let f be an E-mapping and let $z = f(\zeta), w = f(\omega)$, where $\zeta, \omega \in \mathbb{D}$. Then*

$$c_D(z, w) = k_D(z, w) = \widetilde{k}_D(z, w) = \text{hyp dist}\,(\zeta, \omega).$$

Proof. Let F be as in the Proposition 2.1. Using the equality $F \circ f = id_{\mathbb{D}}$ we get

$$c_D(z, w) \geq \text{hyp dist}\,(F(z), F(w)) = \text{hyp dist}\,(\zeta, \omega) \geq \widetilde{k}_D(z, w) \geq k_D(z, w) \geq c_D(z, w)$$

and we are done. □

Corollary 2.7. *An E-mapping gives an embedding of $\mathbb{D}$ endowed with the hyperbolic distance into D endowed with the Kobayashi or the Carathéodory distance.*

3. Regularity

Let $M \subset \mathbb{C}^m$ be a totally real local $\mathcal{C}^\omega$ submanifold having the real dimension m. Take an arbitrary point $z \in M$. There are open subsets U, V of $\mathbb{C}^m$ and a $\mathcal{C}^\omega$-diffeomorphism $\widetilde{\Phi} : U \to V$ such that V is a neighbourhood of z, $\widetilde{\Phi}^{-1}(z) = 0$ and $V \cap M = \widetilde{\Phi}(U \cap \mathbb{R}^m)$. The mapping $\widetilde{\Phi}|_{U\cap\mathbb{R}^m}$ can be extended to a mapping Φ analytic on an open neighbourhood of the point 0. We have

$$\frac{\partial \Phi_j}{\partial z_k}(0) = \frac{\partial \Phi_j}{\partial x_k}(0) = \frac{\partial \widetilde{\Phi}_j}{\partial x_k}(0),$$

so the complex derivative $\Phi'(0)$ in an isomorphism. Therefore Φ restricted to a small neighbourhood of 0 is a biholomorphism of two open subsets of $\mathbb{C}^m$ which carries an open neighbourhood of 0 in $\mathbb{R}^m$ in an open neighbourhood of z in M.

Lemma 3.1 (Reflection principle)**.** *Let $M \subset \mathbb{C}^m$ be a totally real local $\mathcal{C}^\omega$ submanifold, having the real dimension m. Let $V \subset \mathbb{C}$ be an open neighbourhood of a point $\zeta_0 \in \partial\mathbb{D}$ and let $g : V \cap \overline{\mathbb{D}} \to \mathbb{C}^m$ be a continuous mapping. Suppose g is holomorphic on $V \cap \mathbb{D}$ and $g(V \cap \partial\mathbb{D}) \subset M$. Then g can be continued holomorphically past $V \cap \partial\mathbb{D}$.*

Proof. In virtue of the identity principle it is sufficient to continue g locally past an arbitrary point $\zeta_0 \in V \cap \partial\mathbb{D}$. Fix ζ_0 and take Φ is as above, for the point $g(\zeta_0) \in M$. Let $V_1 \subset V$ be an neighbourhood of ζ_0 such that $g(V_1 \cap \overline{\mathbb{D}})$ is contained in the image of Φ. The mapping $\Phi^{-1} \circ g$ is holomorphic on $V_1 \cap \mathbb{D}$ and has real values on $V_1 \cap \partial\mathbb{D}$. Hence by the ordinary reflection principle we can extend this mapping holomorphically past $V_1 \cap \partial\mathbb{D}$. Denote that extension by h. Then $\Phi \circ h$ is an extension of g in a neighbourhood of ζ_0. □

Proposition 3.2. *Every weak E-mapping is also an E-mapping.*

Proof. Let f be a weak $E-$mapping. Our goal is to prove that the mappings f and $\widetilde{f}$ are of the class $\mathcal{C}^\omega$. Write $f = (f_1, \dots, f_n)$, $\widetilde{f} = (\widetilde{f}_1, \dots, \widetilde{f}_n)$. Choose a point $\zeta_0 \in \partial\mathbb{D}$. Since $\widetilde{f}(\zeta_0) \neq 0$, we can suppose $\widetilde{f}_1(\zeta) \neq 0$ in $U \cap \overline{\mathbb{D}}$, where U is a neighbourhood of ζ_0. This implies $\nu_1(f(\zeta_0)) \neq 0$, so $\nu_1(z)$ does not vanish on some set V open in ∂D and containing the point $f(\zeta_0)$.

Define the mapping $\psi : V \to \mathbb{C}^{2n-1}$ by

$$\psi(z) = \left(z_1, \dots, z_n, \overline{\nu_2(z)} \,/\, \overline{\nu_1(z)}, \dots, \overline{\nu_n(z)} \,/\, \overline{\nu_1(z)}\right).$$

The set $M := \psi(V)$ is the graph of a $\mathcal{C}^\omega$ function defined on the local $\mathcal{C}^\omega$ submanifold V, so obviously is a local $\mathcal{C}^\omega$ submanifold of $\mathbb{C}^{2n-1}$, having the real dimension $2n - 1$. Assume for the moment that M is totally real.

Consider the mapping

$$g(\zeta) := \left(f_1(\zeta), \dots, f_n(\zeta), \widetilde{f}_2(\zeta) \,/\, \widetilde{f}_1(\zeta), \dots, \widetilde{f}_n(\zeta) \,/\, \widetilde{f}_1(\zeta)\right),$$

defined for $\zeta \in U \cap \overline{\mathbb{D}}$. If $\zeta \in U \cap \partial\mathbb{D}$, then $\widetilde{f}_j(\zeta) \,/\, \widetilde{f}_1(\zeta) = \overline{\nu_j(f(\zeta))} \,/\, \overline{\nu_1(f(\zeta))}$, so $g(\zeta) = \psi(f(\zeta))$. Therefore $g(U \cap \partial\mathbb{D}) \subset M$. The reflection principle implies that g extends analytically past $U \cap \partial\mathbb{D}$, so f is of class $\mathcal{C}^\omega$ near ζ_0. Since ζ_0 is arbitrary, f is of class $\mathcal{C}^\omega$ on $\partial\mathbb{D}$.

The mapping $\overline{\nu \circ f}|_{\partial\mathbb{D}}$ if of class $\mathcal{C}^\omega$, so it clearly extends to some mapping h holomorphic on the neighbourhood of $\partial\mathbb{D}$. For $\zeta \in U \cap \partial\mathbb{D}$ we have

$$\frac{\zeta h_1(\zeta)}{\widetilde{f}_1(\zeta)} = \frac{1}{p(\zeta)}.$$

The function on the left side is holomorphic on $U\cap\mathbb{D}$ and continuous on $U\cap\overline{\mathbb{D}}$. Since it has real values on $U\cap\partial\mathbb{D}$, the reflection principle implies that it is of class $\mathcal{C}^\omega$. Hence p, and then $\widetilde{f}$, is of class $\mathcal{C}^\omega$ near an arbitrarily chosen point ζ_0.

It remains to prove that M is totally real. Let r denote a defining function for ∂D. For every point $z\in\partial D$ the vectors $\nu(z)$ and $\operatorname{grad} r(z)=(r_{\overline{z_1}}(z),\dots,r_{\overline{z_n}}(z))$ are parallel over $\mathbb{R}$, so

$$\frac{\nu(z)}{\nu_1(z)}=\frac{1}{r_{\overline{z_1}}(z)}\operatorname{grad} r(z).$$

Consider the mapping $S=(S_1,\dots,S_n):V\times\mathbb{C}^{n-1}\to\mathbb{R}\times\mathbb{C}^{n-1}$ given by

$$S(z,w):=(r(z),r_{z_2}(z)-w_1r_{z_1}(z),\dots,r_{z_n}(z)-w_{n-1}r_{z_1}(z)).$$

Clearly $M=S^{-1}(\{0\})$. This implies $T_{(z,w)}M\subset\ker S'(z,w)$ for any $(z,w)\in M$.

Fix a point $(z,w)\in M$. Our goal is to prove that $T^{\mathbb{C}}_{(z,w)}M=\{0\}$. Take an arbitrary vector $(X,Y)=(X_1,\dots,X_n,Y_1,\dots,Y_{n-1})\in T^{\mathbb{C}}_{(z,w)}M$. Then $\sum_k r_{z_k}(z)X_k=0$, because $X\in T^{\mathbb{C}}_z\partial D$. For each $k=2,\dots,n$ we have $w_{k-1}=\frac{r_{z_k}(z)}{r_{z_1}(z)}$ and

$$0=\overline{\partial}_{(X,Y)}S_k(z,w)=\sum_j r_{z_k\overline{z_j}}(z)\overline{X_j}-w_{k-1}\sum_j r_{z_1\overline{z_j}}(z)\overline{X_j},$$

so

$$r_{z_1}(z)\sum_j r_{z_k\overline{z_j}}(z)\overline{X_j}=r_{z_k}(z)\sum_j r_{z_1\overline{z_j}}(z)\overline{X_j}.$$

Note that the last equality holds also for $k=1$. Hence

$$\begin{aligned}r_{z_1}(z)\sum_{j,k} r_{z_k\overline{z_j}}(z)\overline{X_j}X_k &= \sum_k r_{z_k}(z)\sum_j r_{z_1\overline{z_j}}(z)\overline{X_j}X_k=\\ &=\left(\sum_k r_{z_k}(z)X_k\right)\left(\sum_j r_{z_1\overline{z_j}}(z)\overline{X_j}\right)=0.\end{aligned}$$

Therefore by (2) from Definition 1.2 we get $X=0$, and this directly implies $Y=0$. □

4. Hölder estimates

We will prove some uniform 1/2-Hölder estimates for E-mappings $f:\mathbb{D}\longrightarrow D$ such that $f(0)=z$. These maps we will denote as a function between marked domains $f:(\mathbb{D},0)\longrightarrow(D,z)$. We need the following

Definition 4.1. *For given $c>0$ let the family $\mathcal{D}(c)$ consists of all marked domains (D,z) satisfying*

(1) $\operatorname{dist}(z,\partial D)>\frac{1}{c}$;

(2) the diameter of D and the modulus of the normal curvature of ∂D are smaller than c;

(3) for any $x,y\in D$ there exist $m<c^2$ and balls $B_0,\dots,B_m\subset D$ of radius $\frac{1}{2c}$ such that $x\in B_0$, $y\in B_m$ and the distance between the centers of the balls B_j, B_{j+1} is smaller than $\frac{1}{4c}$ for $j=0,\dots,m-1$;

(4) for every ball $B\subset\mathbb{C}^n$ of radius not greater than $\frac{1}{c}$ there exists a holomorphic map $\Phi:\overline{D}\longrightarrow\mathbb{C}^n$ such that

(a) for any $w\in\Phi(B\cap\partial D)$ there is a ball of radius smaller than c containing $\Phi(D)$ and tangent to $\partial\Phi(D)$ at w;

(b) Φ is biholomorphic on $B\cap\overline{D}$;

(c) *the partial derivatives of the first order of* Φ *and* Φ^{-1} *(on* $\Phi(B\cap\overline{D})$*) are bounded by* c*;*

(d) $\operatorname{dist}(\Phi(z),\partial\Phi(D))>\frac{1}{c}$.

For strictly pseudoconvex domain D and point $z\in D$ there exists c such that conditions (1)-(4) are satisfied. The construction of mapping Φ amounts to the construction of peak functions (see [2]). In the case of D strictly convex and the normal curvatures of ∂D greater than $\frac{1}{c}$, one can take $\Phi=\mathrm{id}$.

Fix $c>1$. Let us prove

Proposition 4.2. *Let* $f:(\mathbb{D},0)\longrightarrow(D,z)$ *be an* E*-mapping. Then*

$$d_D(f(\zeta))\le C(1-|\zeta|),\ \zeta\in\mathbb{D}$$

with constant $C>0$ *uniform if* $(D,z)\in\mathcal{D}(c)$.

Proof. Thanks to the condition (3) there exists C_1 such that $k_D(z,w)<C_1$ if $\operatorname{dist}(w,\partial D)\ge\frac{1}{c}$. Fix $\zeta\in\mathbb{D}$ with $\operatorname{dist}(f(\zeta),\partial D)\ge\frac{1}{c}$. Then

$$k_D(f(0),f(\zeta))\le C_2-\frac{1}{2}\log(\operatorname{dist}(f(\zeta),\partial D)).$$

In the opposite case i.e. $\operatorname{dist}(f(\zeta),\partial D)<\frac{1}{c}$ let η be the nearest point to $f(\zeta)$ on ∂D. Set $w\in D$ as the center of the ball B of radius $\frac{1}{c}$ tangent to ∂D at η. By condition (2) $B\subset D$. Hence

$$k_D(f(0),f(\zeta))\le k_D(f(0),w)+k_D(w,f(\zeta))\le$$
$$\le C_1+k_B(w,f(\zeta))\le C_3-\frac{1}{2}\log(\operatorname{dist}(f(\zeta),\partial D))=C_3-\frac{1}{2}\log(\operatorname{dist}(f(\zeta),\partial D)).$$

On the other side, Proposition 2.6 used to extremal disc $f(\mathbb{D})$ through $f(0)$ and $f(\zeta)$ gives

$$k_D(f(0),f(\zeta))=\text{hyp dist}\,(0,\zeta)\ge-\frac{1}{2}\log(1-|\zeta|).$$ □

Now we are going to obtain the same Hölder estimates for an E-mapping f and associated mappings $\widetilde{f}$, ρ. Thanks to Proposition 2.5 the function $f'\widetilde{f}$ is constant, so ρ is defined up to a constant factor. We may choose ρ such that $f'\widetilde{f}\equiv1$ i.e.

$$\rho(\zeta)^{-1}=\zeta f'(\zeta)\overline{\nu(f(\zeta))},\ \zeta\in\overline{\mathbb{D}}.$$

In that way $\widetilde{f}$ and ρ are uniquely determined by f.

Proposition 4.3. *Let* $f:(\mathbb{D},0)\longrightarrow(D,z)$ *be an* E*-mapping. Then*

$$C_1<\rho(\zeta)^{-1}<C_2,\ \zeta\in\partial\mathbb{D}$$

with constants $C_1,C_2>0$ *uniform if* $(D,z)\in\mathcal{D}(c)$.

Proof. For the upper estimate choose $\zeta\in\partial\mathbb{D}$ and define $\zeta_\varepsilon:=(1-\varepsilon)\zeta$ for small $\varepsilon>0$. Set $B:=B\left(f(\zeta),\frac{1}{c}\right)$ and let $\Phi:\overline{D}\longrightarrow\mathbb{C}^n$ be chosen to the ball B as described in the condition (4). One can assume that $\Phi(f(\zeta))=0$ and the normal vector to $\partial\Phi(D)$ at 0 is $N:=(1,0,\dots,0)$. Then $\Phi(D)$ is contained in the half space $\{w\in\mathbb{C}^n:\operatorname{Re}w_1<0\}$. Putting $h:=\Phi\circ f$ we have

$$h_1(\mathbb{D})\subset\{w_1\in\mathbb{C}:\operatorname{Re}w_1<0\}.$$

In virtue of the Schwarz lemma in the half plane

$$|h_1'(\zeta_\varepsilon)|\le\frac{2\operatorname{Re}h_1(\zeta_\varepsilon)}{1-|\zeta_\varepsilon|^2}\approx\frac{\operatorname{dist}(h(\zeta_\varepsilon),\partial\Phi(D)))}{1-|\zeta_\varepsilon|}\quad\text{as }\ \varepsilon\to0,$$

since the transversality of $t \longmapsto \Phi(f(t\zeta))$ to $\partial\Phi(D)$ is equivalent to the transversality of $t \longmapsto f(t\zeta)$ to ∂D and the second transversality follows from

$$\operatorname{Re}\left(\frac{d}{dt}f(t\zeta)\bigg|_{t=1}\overline{\nu(f(\zeta))}\right) = \operatorname{Re}\left(f'(\zeta)\zeta\overline{\nu(f(\zeta))}\right) = \rho(\zeta)^{-1}.$$

Clearly

$$\frac{\operatorname{dist}(\Phi(f(\zeta_\varepsilon)), \partial\Phi(D))}{1-|\zeta_\varepsilon|} \approx \frac{\operatorname{dist}(f(\zeta_\varepsilon), \partial D)}{1-|\zeta_\varepsilon|}$$

which, by Proposition 4.2, does not exceed some constant. Now the upper estimate follows from the observation

$$|f'(\zeta)\overline{\nu(f(\zeta))}| \leq C_3|h'(\zeta)\overline{\nu(h(\zeta))}| = C_3|h_1'(\zeta)|.$$

Indeed, if ϱ is a defining function for D in the neighbourhood of $f(\zeta)$ then

$$|f'(\zeta)\overline{\nu(f(\zeta))}| = \frac{|(\varrho\circ f)'(\zeta)|}{|\nabla\varrho(f(\zeta))|}$$

and analogously

$$|h'(\zeta)\overline{\nu(h(\zeta))}| = \frac{|(\varrho\circ\Phi^{-1}\circ h)'(\zeta)|}{|\nabla(\varrho\circ\Phi^{-1})(h(\zeta))|} =$$

$$= \frac{|(\varrho\circ f)'(\zeta)|}{|\nabla\varrho(f(\zeta))\overline{(\Phi^{-1})'}(\Phi(f(\zeta)))|} \geq \frac{|(\varrho\circ f)'(\zeta)|}{|\nabla\varrho(f(\zeta))|}\frac{1}{c\sqrt{n}}.$$

The lower estimate is related to a lemma of E. Hopf. Note that for small $\varepsilon > 0$ the function

$$\varrho(w) := -\log(\varepsilon + \operatorname{dist}(w, \partial D)) + \log\varepsilon,\ \ w \in D_\varepsilon,$$

where D_ε is an ε-envelope of D i.e. the set $\{w \in \mathbb{C}^n : \operatorname{dist}(w, D) < \varepsilon\}$, is plurisubharmonic and defining for D. Indeed, we have

$$-\log(\varepsilon + \operatorname{dist}(w, \partial D)) = -\log(\operatorname{dist}(w, \partial D_\varepsilon)),\ w \in D_\varepsilon$$

and for sufficiently small ε the domain D_ε is pseudoconvex.

Let us define a non-positive subharmonic function $v := \varrho\circ f : \overline{\mathbb{D}} \longrightarrow \mathbb{R}$. Since $|f(\lambda) - z| < c$ for $\lambda \in \mathbb{D}$, we have

$$|f(\lambda) - z| < \frac{1}{2c}\ \text{ if }\ |\lambda| \leq \frac{1}{2c^2}.$$

Therefore, for fixed $\zeta \in \partial\mathbb{D}$

$$M_\zeta(x) := \max_{t\in[0,2\pi]} v(\zeta e^{x+it}) \leq -\log\left(1 + \frac{1}{2c\varepsilon}\right) =: -C_4\ \text{ if }\ x \leq -\log(2c^2).$$

Since M_ζ is convex for $x \leq 0$ and $M_\zeta(0) = 0$ we get

$$M_\zeta(x) \leq \frac{C_4 x}{\log(2c^2)}\quad\text{for}\quad -\log(2c^2) \leq x \leq 0.$$

Hence

$$\frac{C_4}{\log(2c^2)} \leq \frac{d}{dx}v(\zeta e^x)\bigg|_{x=0} = \zeta f'(\zeta)\overline{\nu(f(\zeta))}|\nabla\varrho(f(\zeta))|.$$

Easy calculations give

$$\frac{\partial\varrho}{\partial\nu}(f(\zeta)) = \frac{1}{\varepsilon}$$

thus

$$|\nabla\varrho(f(\zeta))| = \nabla\varrho(f(\zeta))\frac{\nabla\varrho(f(\zeta))}{|\nabla\varrho(f(\zeta))|} = \nabla\varrho(f(\zeta))\nu(f(\zeta)) = \frac{1}{\varepsilon}.\qquad\square$$

Proposition 4.4. *Let $f : \overline{\mathbb{D}} \to \overline{D}$ be an E-mapping. Then*
$$|f(\zeta_1) - f(\zeta_2)| \le C(|\zeta_1 - \zeta_2|)^{1/2},\ \zeta_1, \zeta_2 \in \mathbb{D},$$
where C is uniform if $(D, z) \in \mathcal{D}(c)$.

Lemma 4.5. *Let $g : \mathbb{D} \to \mathbb{B}(z_0, R)$ be a holomorphic mapping such that $|g(0) - z_0| = r$. Then*
$$|g'(0)| \le (R^2 - r^2)^{1/2}.$$

Proof. Assume that $z_0 = 0$ and $R = 1$. When $r = 0$ proof is similar to a proof of classical Schwarz Lemma. Assume $r \ne 0$ and choose an automorphism φ of $\mathbb{B}_n$ such that $\varphi(g(0)) = 0$. From the explicite formula for $\varphi'(g(0))$ we get that $|\varphi'(g(0))| \le \frac{1+\sqrt{1-r^2}}{1-r^2}$ (i.e. see [5] Theorem 2.2.2 p. 26). From thesis for $r = 0$ we get that $|(\varphi(g(0)))'| \le 1$, so $|g'(0)| \le \sqrt{1 - r^2}$.

Proof of general case, when $\mathbb{B}(z_0, R)$ is a ball with center at z_0 and radius R, is similar. □

Theorem 4.6 (Littlewood, see [3] Theorem 3 p. 397). *Let $f : \overline{\mathbb{D}} \to D$ be regular on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then for $0 < \alpha \le 1$ following properties are equivalent:*

$$|f(e^{i\theta}) - f(e^{i\theta'})| \le K|\theta - \theta'|^{\alpha} \tag{5}$$
$$|f'(\zeta)| \le M(1 - |\zeta|)^{\alpha - 1},\ \zeta \in \mathbb{D} \tag{6}$$

Theorem 4.7 (Hardy, Littlewood, see [3] Theorem 4 p. 399). *Let $f : \overline{\mathbb{D}} \to D$ be regular on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ such that $|f(e^{i\theta}) - f(e^{i\theta'})| \le K|\theta - \theta'|^{\alpha}$, $0 < \alpha \le 1$. Then*
$$|f(\zeta) - f(\zeta')| \le K|\zeta - \zeta'|^{\alpha},\ \zeta \in \mathbb{D}.$$

Proof of Proposition 4.4. Fix $\zeta_0 \in \mathbb{D}$. Let Z denote point in ∂D such that $\operatorname{dist}(f(\zeta_0), \partial D) = |f(z_0) - Z|$ and let $\mathbb{B}(Z_0, R)$ denote a smallest ball tangent to ∂D at Z containing D. Define
$$h(\zeta) = f\left(\frac{\zeta_0 - \zeta}{1 - \overline{\zeta_0}\zeta}\right).$$
Then h is holomorphic, $h(\mathbb{D}) \subset \mathbb{B}(Z_0, R)$ and $h(0) = f(\zeta_0)$. Using Lemma 4.5 we get
$$|h'(0)| \le \sqrt{|Z_0 - Z|^2 - |f(\zeta_0) - Z_0|^2} \le C_1\sqrt{|f(\zeta_0) - Z|}$$
where C_1 depends only on diameter of D.

From the formula for $h'(\zeta)$ we get $h'(0) = f'(\zeta_0)(\zeta_0\overline{\zeta_0} - 1)$ and
$$|f'(\zeta_0)| = \frac{1}{1 - |\zeta_0|^2}|h'(0)| \le C_1\frac{\sqrt{\operatorname{dist}(f(\zeta_0), \partial D)}}{1 - |\zeta_0|^2}.$$
From Proposition 4.2
$$|f'(\zeta_0)| \le C_2\frac{\sqrt{1 - |\zeta_0|}}{1 - |\zeta_0|^2} \le C_3\frac{1}{\sqrt{1 - |\zeta_0|}}.$$

Since this inequality is true for every $\zeta \in \mathbb{D}$ we get the thesis using Theorems 4.6 and 4.7 with $\alpha = 1/2$.

□

Proposition 4.8. *Let $f : \overline{\mathbb{D}} \to \overline{D}$ be an E-mapping. Then*
$$|p(\zeta_1) - p(\zeta_2)| \le C(|\zeta_1 - \zeta_2|)^{1/2},\ \zeta_1, \zeta_2 \in \partial\mathbb{D},$$
where C is uniform if $(D, z) \in \mathcal{D}(c)$.

Proof. Assume that there exists $C_1 > 0$ such that proposition is true for $\zeta_1,\ \zeta_2 \in \partial\mathbb{D}$ such that $|\zeta_1 - \zeta_2| \le C_1$. The general case follows immediately: there exists a finite N depending only on C_1, such that for every $\zeta_1,\ \zeta_2 \in \partial\mathbb{D}$, $|\zeta_1 - \zeta_2| > C_1$ there exists $\{\eta_j\}_{j=1}^N \subset \partial\mathbb{D}$, $\eta_1 = \zeta_1$, $\eta_N = \zeta_2$, $|\eta_j - \eta_k| \le C_1$ for $j,\ k \in \{1, \ldots, N\}$. Then

$$|p(\zeta_1) - p(\zeta_2)| \le |p(\eta_1) - p(\eta_2)| + \ldots + |p(\eta_{N-1}) - p(\eta_N)|$$

$$\le C(\sqrt{|\eta_1 - \eta_2|} + \ldots + \sqrt{|\eta_{N-1} - \eta_N|}) \le CN\sqrt{C_1} < CN(|\zeta_1 - \zeta_2|)^{1/2}.$$

So it is sufficient to prove, that such C_1 really exists.

Fix $\zeta_1 \in \partial\mathbb{D}$. Without loss of generality we may assume that $\nu_1(f(\zeta_1)) = 1$. Choose C_1 such that $|\nu_1(f(\zeta)) - 1| < 1/2$ for $|\zeta - \zeta_1| \le 2C_1$. Such C_1 exists because of continuity of function $\nu \circ f$.

Construct new function $\varphi : \partial\mathbb{D} \to \mathbb{C}$ such that:

- $\varphi(\zeta) = \overline{\nu_1(f(\zeta))}$ for $|\zeta - \zeta_1| \le 2C_1$,
- $|\varphi(\zeta) - 1| < 1/2$ for all $\zeta \in \partial\mathbb{D}$,
- $\varphi \in \mathcal{C}^{1/2}(\partial\mathbb{D})$ and $\|\nu \circ f\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})} = \|\varphi\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})}$

Let $r : \partial\mathbb{D} \to \mathbb{R}$ be such that $r + i\mathrm{Im}\, \log\varphi$ extends to function holomorphic on $\mathbb{D}$. Because φ is 1/2-Hölder continuous, $\log\varphi$ has the same property and using Privaloff's theorem we can show, that r is also 1/2-Hölder continuous and it's norm is uniformly bounded. Define $q := r - \mathrm{Re}\, \log\varphi$, which from the definition is 1/2-Hölder continuous with constant C_2, depending on C_1. Function $q + \log\varphi = r + i\mathrm{Im}\,\varphi$ extends to $h : \overline{\mathbb{D}} \to \mathbb{C}$, holomorphic in $\mathbb{D}$, continuous at $\overline{\mathbb{D}}$. Because on the boundary $h = q$, which is 1/2-Hölder continuous, from Theorem 4.7 we get 1/2-Hölder continuity of h in $\mathbb{D}$.

Define functions $g(\zeta) := \widetilde{f}_1(\zeta)e^{-h(\zeta)}$ and $G(\zeta) = g(\zeta)/\zeta$. Then g is defined on $\overline{\mathbb{D}}$, holomorphic in $\mathbb{D}$, G is defined on $\overline{\mathbb{D}} \setminus \{0\}$, holomorphic in $\mathbb{D} \setminus \{0\}$. For $\zeta \in \partial\mathbb{D}$ we have that

$$g(\zeta) = \zeta p(\zeta)\overline{\nu_1(f(\zeta))}e^{-r(\zeta)}e^{i\mathrm{Im}\, \log\varphi(\zeta)}$$

which, combined with unform boundness of r and equality $\|\nu \circ f\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})} = \|\varphi\|_{\mathcal{C}^{1/2}(\partial\mathbb{D})}$, gives uniform boundness of g. Define

$$U_1 := \{\zeta \in \mathbb{C} :\ |\zeta - \zeta_1| < 2C_1\}.$$

Then G is uniformly bounded on $\overline{\mathbb{D}} \cap U_1$. Moreover, for $\zeta \in \partial\mathbb{D} \cap U_1$ we have

$$G(\zeta) = \frac{g(\zeta)}{\zeta} = p(\zeta)\overline{\nu_1(f(\zeta))}e^{-q(\zeta)}e^{-\log\varphi(\zeta)} = p(\zeta)e^{-r(\zeta)}e^{\mathrm{Re}\, \log\varphi(\zeta)} \in \mathbb{R}.$$

Because we can extend G holomorphically through $\partial\mathbb{D} \cap U_1$ to a function bounded on U_1, G is 1/2-Hölder continuous on connected components of $U_1 \cap \overline{\mathbb{D}}$, in particular for every $|\zeta_1 - \zeta_2| < C_1$

$$|G(\zeta_1) - G(\zeta_2)| \le |\zeta_1 - \zeta_2|.$$

Now, since $p(\zeta) = (G(\zeta)e^h)/(\overline{\nu_1(f(\zeta))})$ and for $|\zeta_1 - \zeta_2| < C_1$ all functions $G,\ h$ and $\nu_1 \circ f$ are 1/2-Hölder continuous, we get the thesis. □

Proposition 4.9. *Let $f : \overline{\mathbb{D}} \to \overline{D}$ be an E-mapping. Then*

$$|\widetilde{f}(\zeta_1) - \widetilde{f}(\zeta_2)| \le C(|\zeta_1 - \zeta_2|)^{1/2},\ \zeta_1, \zeta_2 \in \overline{\mathbb{D}},$$

where C is uniform if $(D, z) \in \mathcal{D}(c)$.

Proof. Using Propositions 4.4 and 4.8 we have desired inequality for $\zeta_1,\ \zeta_2 \in \partial\mathbb{D}$. Application of Theorem 4.7 with $\alpha = 1/2$ finishes the proof. □

5. Perturbation of the domain

We will describe what happens to E-mapping if the domain D is perturbed a little.

Proposition 5.1. *Let $f:\mathbb{D}\longrightarrow D$ be an E-mapping. Then there is a biholomorpihism $\Phi:\overline{D}\longrightarrow\overline{G}$ such that*

(1) $g(\zeta):=\Phi(f(\zeta))=(\zeta,0,\ldots,0)$, $\zeta\in\mathbb{D}$;
(2) $\nu(g(\zeta))=(\zeta,0,\ldots,0)$, $\zeta\in\partial\mathbb{D}$;
(3) for any $\zeta\in\partial\mathbb{D}$ the point $g(\zeta)$ is a point of strict linear convexity of ∂G, i.e. for $w\in T^{\mathbb{C}}(g(\zeta))$ near $g(\zeta)$ and positive constant c the inequality

$$\operatorname{dist}(w,G)\geq c|w-g(\zeta)|^2$$

holds.

Proof. After performing, if necessary, a linear change of coordinates one can assume that $\widetilde{f}_1$, $\widetilde{f}_2$ do not have common zeroes in $\overline{\mathbb{D}}$. Then there are holomorphic maps $h_1,h_2:\overline{\mathbb{D}}\longrightarrow\mathbb{C}$ such that $h_1\widetilde{f}_1+h_2\widetilde{f}_2\equiv 1$. Indeed, let $\widetilde{f}_j=F_jP_j$, $j=1,2$, where F_j are holomorphic and non-zero in $\overline{\mathbb{D}}$ and P_j are polynomials with all zeroes in $\overline{\mathbb{D}}$ (recall that $\widetilde{f}_j$ extend analytically through $\partial\mathbb{D}$). Then P_j are relatively prime, so there are polynomials Q_j, $j=1,2$ such that

$$Q_1P_1+Q_2P_2\equiv 1.$$

Hence

$$\frac{Q_1}{F_1}\widetilde{f}_1+\frac{Q_2}{F_2}\widetilde{f}_2\equiv 1$$

and $h_j:=Q_j/F_j$, $j=1,2$ extend analytically through $\partial\mathbb{D}$.

Consider the mapping $\Psi:\mathbb{D}\times\mathbb{C}^{n-1}\longrightarrow\mathbb{C}^n$ defined as

$$\Psi_1(Z):=f_1(Z_1)-Z_2\widetilde{f}_2(Z_1)-h_1(Z_1)\sum_{j=3}^{n}Z_j\widetilde{f}_j(Z_1), \tag{7}$$

$$\Psi_2(Z):=f_2(Z_1)+Z_2\widetilde{f}_1(Z_1)-h_2(Z_1)\sum_{j=3}^{n}Z_j\widetilde{f}_j(Z_1), \tag{8}$$

$$\Psi_j(Z):=f_j(Z_1)+Z_j,\ j=3,\ldots,n. \tag{9}$$

We claim that Ψ is biholomorphic on $G:=\Psi^{-1}(D)$. It suffices to show that if $\Psi(Z)=\Psi(W)=z\in D$ then $Z=W$.

By direct computation both $\zeta=Z_1$ and $\zeta=W_1$ solve the equation

$$\widetilde{f}(\zeta)(z-f(\zeta))=0.$$

It was demonstrated in the proof of Proposition 2.1 that it has exactly one solution. Hence $Z_1=W_1$. By (9) we have $Z_j=W_j$ for $j=3,\ldots,n$. Finally $Z_2=W_2$ follows from one of the equations (7), (8).

It is clear that Ψ extends to a neighbourhood of $\overline{\mathbb{D}}\times\mathbb{C}^{n-1}$ and Ψ is biholomorphic also on a neighbourhood of $\Psi^{-1}(\overline{D})$. The map $\Phi:=\Psi^{-1}$ has desired properties. □

Proposition 5.2. *Let $W:\partial\mathbb{D}\to GL(m,\mathbb{C})$ be a matrix valued $\mathcal{C}^\omega$ mapping such that $W(\zeta)$ is self-adjoint for every $\zeta\in\partial\mathbb{D}$. Then there exists a holomorphic mapping $H:\overline{\mathbb{D}}\to GL(m,\mathbb{C})$ such that $HH^*=W$ on $\partial\mathbb{D}$.*

Let $D_0 \subset \mathbb{C}^n$ be a strictly linearly convex domain with a real analytic boundary. Then there exists an open neighbourhood V_0 of ∂D and a real analytic defining function $r_0 : V_0 \to \mathbb{R}$ such that $dr_0 \neq 0$ and $D_0 \cap V_0 = \{z \in V_0 : r_0(z) < 0\}$. It is sraightforward that r_0 extends to a holomorphic function on an open neighbourhood $V \subset \mathbb{C}^n$ of V_0 in the complexification of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. Without losing generality we may assume that r_0 is bounded on V. Let

$$X := \{r \in \mathcal{O}(V) \text{ s.t. } r(V_0) \subset \mathbb{R} \text{ and } r \text{ is bounded}\},$$

which equipped with the sup−norm is a Banach space. If $r \in X$ is near to r_0 (w.r.t. the sup−norm), then $\{z \in V_0 : r(z) = 0\}$ is a compact real analytic hypersurface which bounds a bounded domain, say D^r.

Definition 5.3. We say that a domain D *is near to* D_0 if its defining function r can be taken from X, near to r_0.

Proposition 5.4. *Let $f_0 : \mathbb{D} \to D_0$ be an $E-$mapping. Then there exist an open neighbourhood U of the point $(r_0, f_0'(0))$ in the space $X \times \mathbb{C}^n$ and a real analytic mapping $\Gamma : U \to \mathcal{C}^{1/2}(\overline{\mathbb{D}})$ such that $\Gamma(r_0, f_0'(0)) = f_0$ and for any $(r, v) \in U$ the mapping $f := \Gamma(r, v)$ is an $E-$mapping into D^r such that $f(0) = f_0(0)$ and $f_0'(0) = \lambda v, \lambda > 0$. Furthermore, let $\xi \in (0, 1)$. Then there exist an open neighbourhood W of $(r_0, f_0(\xi))$ in $X \times D_0$ and two real analytic mappings $\Lambda : W \to \mathcal{C}^{1/2}(\overline{\mathbb{D}})$, $\Omega : W \to (0, 1)$ such that $\Lambda(r_0, f_0(\xi)) = f_0, \Omega(r_0, f_0(\xi)) = \xi$, and for any $(r, v) \in W$ the mapping $f := \Lambda(r, v)$ is an $E-$mapping into D^r satisfying $f(0) = f_0(0)$ and $f(\Omega(r, v)) = v$.*

Proof. We shall prove the first statement. The proof of the second one is similar.

Consider the Sobolev space $W^{2,2}(\mathbb{T})$ of functions on $\mathbb{T}$ whose first two derivatives are in $L^2(\mathbb{T})$. It is known that we have the following characterization:

$$W^{2,2}(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \sum_{k=-\infty}^{\infty} (1 + k^2 + k^4)|\widehat{f}_k|^2 < \infty\},$$

where $\widehat{f}_k$'s are the Fourier coefficients of f. Note we have $W^{2,2} \subset \mathcal{C}^{1/2} \subset \mathcal{C}$. To see the first inclusion take $z_1, z_2 \in \mathbb{T}$ and compute:

$$|\sum_{k=-\infty}^{\infty} \widehat{f}_k z_1^k - \sum_{k=-\infty}^{\infty} \widehat{f}_k z_2^k| \leq \sum_{k=-\infty}^{\infty} |\widehat{f}_k||z_1^k - z_2^k| \leq$$

$$\leq \sum_{k=-\infty}^{\infty} |\widehat{f}_k||z_1 - z_2||z_1^{k-1} + z_1^{k-2} z_2^{+} \ldots + z_2^{k-1}| \leq \sum_{k=-\infty}^{\infty} k|\widehat{f}_k||z_1 - z_2| \leq$$

$$\leq \sqrt{2} \sum_{k=-\infty}^{\infty} k|\widehat{f}_k|\sqrt{|z_1 - z_2|},$$

and it is an easy observation that the series $\sum\limits_{k=-\infty}^{\infty} k|\widehat{f}_k|$ is convergent. Moreover, both inclusions are continuous, which also implies their real analyticity. Put

$$Q := W^{2,2}(\mathbb{T}, \mathbb{R}), \quad Q_0 := Q \cap \{q : q(1) = 0\}, \quad A := W^{2,2}(\mathbb{T}, \mathbb{C}^n),$$

$$B := A \cap \{g : g \text{ extends holomorphically to } \mathbb{D} \text{ and the extension is 0 at the origin}\},$$

$$\overline{B} := \{\overline{g} : g \in B\}.$$

Introduce a bounded projection

$$\pi : A \ni \sum_{k=-\infty}^{\infty} a_k e^{ikt} \mapsto \sum_{k=-\infty}^{-1} a_k e^{ikt} \in \overline{B}.$$

Observe that $g \in A$ admits a holomorphic extension to $\mathbb{D}$ if and only if $\pi(g) = 0$.

Using Proposition 5.1 we may assume without losing generality that $f_0(\zeta) = (\zeta, 0, \ldots, 0)$, $\overline{\nu(f_0(\zeta))} = r_{0z}(f_0(\zeta)) = (\overline{\zeta}, 0, \ldots, 0)$ and that for any $\zeta \in \mathbb{T}$, $f_0(\zeta)$ is a point of strict linear convexity of D_0. Observe that the latter means that for any nonzero $v \in \mathbb{C}^{n-1}$ there is

$$\sum_{i,j=2}^{n} r_{0_{z_i \overline{z_j}}}(f_0(\zeta)) v_i \overline{v_j} > |\sum_{i,j=2}^{n} r_{0_{z_i z_j}}(f_0(\zeta)) v_i v_j|. \tag{10}$$

Consider the mapping $\Phi : X \times \mathbb{C}^n \times B \times Q_0 \times \mathbb{R} \to Q \times \overline{B} \times \mathbb{C}^n$ defined by

$$\Phi(r, v, f, q, \lambda) := (r \circ f, \pi((1+q)\zeta(r_z \circ f)), f'(0) - \lambda v),$$

where ζ is just the identity function on the unit circle. From now on we shall identify $f \in B$ with its extension to $\mathbb{D}$[1].

We shall show that there exist an open neighbourhood U of $(r_0, f_0'(0))$ in $X \times \mathbb{C}^n$ and a real analytic mapping $\Psi : U \to B \times Q_0 \times \mathbb{R}$ such that for any $(r, v) \in U$ there is $\Phi(r, v, \Psi(r, v)) = 0$, which will finish the proof.

Indeed, suppose we have such U and Ψ. Observe first that for $(r, f'(0))$ sufficiently close to $(r_0, f_0'(0))$, f is an $E-$mapping into D^r such that $f(0) = 0$ and $f'(0) = \lambda v$ iff there exists a $q \in Q_0$ such that $q > -1$ and $\Phi(r, v, f, q, \lambda) = 0$. The only problem here is to prove the fourth condition from the definition of an $E-$mapping in the backward implication. This fourth condition follows from the fact that for $(r, f'(0))$ near to $(r_0, f_0'(0))$, f and f_0 are uniformly close and then the respective winding numbers are equal.

In this situation taking Γ as the composition of Ψ with the projection $\pi_B : B \times Q_0 \times \mathbb{R} \to B$ and the inclusion $W^{2,2} \subset \mathcal{C}^{1/2}$ does the job.

To this end observe that Φ is real analytic, hence the existence of such U, Ψ would be a direct consequence of the implicit function theorem if only the partial derivative

$$\Phi_{(f,q,\lambda)}(r_0, f_0'(0), f_0, 0, 1) : B \times Q_0 \times \mathbb{R} \to Q \times \overline{B} \times \mathbb{C}^n$$ [2]

is invertible. It is an easy computation to show that for a fixed $(\widetilde{f}, \widetilde{q}, \widetilde{\lambda}) \in B \times Q_0 \times \mathbb{R}$ the following equality holds:

$$\Phi_{(f,q,\lambda)}(r_0, f_0'(0), f_0, 0, 1)(\widetilde{f}, \widetilde{q}, \widetilde{\lambda}) := \frac{d}{dt}\Big|_{t=0} \Phi(r_0, f_0'(0), f_0 + t\widetilde{f}, t\widetilde{q}, 1 + t\widetilde{\lambda}) =$$
$$= ((r_{0z} \circ f_0)\widetilde{f} + (r_{0\overline{z}} \circ f_0)\overline{\widetilde{f}}, \pi(\widetilde{q}\zeta r_{0z} \circ f_0 + \zeta(r_{0zz} \circ f_0)\widetilde{f} + \zeta(r_{0z\overline{z}} \circ f_0)\overline{\widetilde{f}}), \widetilde{f}'(0) - \widetilde{\lambda} f_0'(0)).$$

From now on we will consider $r_{0z}, r_{0\overline{z}}$ as row vectors, $\widetilde{f}, \overline{\widetilde{f}}$ as column vectors and $r_{0zz} = (\frac{\partial^2 r_0}{\partial z_i \partial z_j}), r_{0z\overline{z}} = (\frac{\partial^2 r_0}{\partial z_i \partial \overline{z_j}})$ as $n \times n$ matrices.

We have to show that for fixed $\eta \in Q, \varphi \in \overline{B}, v \in \mathbb{C}^n$ there exist exactly one element $(\widetilde{f}, \widetilde{q}, \widetilde{\lambda}) \in B \times Q_0 \times \mathbb{R}$ satisfying

$$(r_{0z} \circ f_0)\widetilde{f} + (r_{0\overline{z}} \circ f_0)\overline{\widetilde{f}} = \eta, \tag{11}$$

$$\pi(\widetilde{q}\zeta r_{0z} \circ f_0 + \zeta(r_{0zz} \circ f_0)\widetilde{f} + \zeta(r_{0z\overline{z}} \circ f_0)\overline{\widetilde{f}}) = \varphi, \tag{12}$$

[1]Hence we are able to consider $f(0)$ and $f'(0)$.

[2]Observe that we have $q_0 = 0, \lambda_0 = 1$.

(13) $$\widetilde{f}'(0) - \widetilde{\lambda} f_0'(0) = v.$$

Observe that in view of our assumption (11) turns out to be

$$\overline{\zeta}\widetilde{f_1} + \zeta\overline{\widetilde{f_1}} = \eta$$

or

(14) $$\mathrm{Re}(\widetilde{\mathrm{f}_1}/\zeta) = \eta/2.$$

Since $\widetilde{f_1}(0) = 0$, the function $\widetilde{f_1}/\zeta$ is holomorphic and then (14) determines $\widetilde{f_1}/\zeta \in W^{2,2}(\mathbb{T})$ up to an imaginary additive constant, which may be computed using (13).

Indeed, let $\widetilde{f_1}/\zeta = r/2 + is(r/2) + ic$, where $s(r/2)$ is an adjoint function to $r/2$ and c is our imaginary additive constant we have to compute. Observe that $r/2(0) + is(r/2)(0) + ic = \widetilde{f_1}'(0)$ and

$$r/2(0) + is(r/2)(0) + ic - \widetilde{\lambda}\mathrm{Re f_0}_1'(0) - \mathrm{iIm f_0}_1'(0) = \mathrm{Re v_1} + \mathrm{iIm v_1},$$

$$r/2(0) - \widetilde{\lambda}\mathrm{Re f_0}_1'(0) = \mathrm{Re v_1},$$

which yields $\widetilde{\lambda}$ and then c. Observe that having $\widetilde{\lambda}$, once again using (13), we can easily find $\widetilde{f_2}'(0), \ldots, \widetilde{f_n}'(0)$.

Consider (12), which in fact is a system of n equations with unknowns $\widetilde{q}, \widetilde{f_2}, \ldots, \widetilde{f_n}$. Observe that $\widetilde{q}$ appears only in the first of the equations and the remaining $n-1$ equations mean exactly that the mapping

(15) $$\zeta(r_{0\widehat{z}\widehat{z}} \circ f_0)\widehat{\widetilde{f}} + \zeta(r_{0\widehat{z}\widehat{\overline{z}}} \circ f_0)\widehat{\overline{\widetilde{f}}} - \psi^3$$

extends to a holomorphic mapping from $\mathbb{D}$ into $\mathbb{C}^{n-1}$, where $\psi \in W^{2,2}(\mathbb{T}, \mathbb{C}^{n-1})$ may be obtained from φ and $\widetilde{f_1}$.

Indeed, to see this, write (12) in the form:

$$\pi(M_1 + \zeta M_2 + \zeta M_3) = M_4,$$

where M_1 is a column vector having $\widetilde{q}$ on the first place and zero's on the remaining $n-1$ places, $M_2 = (A_i)_{i=1}^n$ is a column vector such that $A_i = \sum_{j=1}^n (\frac{\partial^2 r_0}{\partial z_i \partial z_j} \circ f_0)\widetilde{f_j}$, $M_3 = (B_i)_{i=1}^n$ is a column vector such that $B_i = \sum_{j=1}^n (\frac{\partial^2 r_0}{\partial z_i \partial \overline{z_j}} \circ f_0)\overline{\widetilde{f_j}}$, and M_4 is a column vector with φ_i on $i-$th place. This implies as follows:

$$\widetilde{q} + \zeta A_1 + \zeta B_1 - \varphi_1$$

admits a holomorphic extension to $\mathbb{D}$ and for $i = 2, \ldots, n$,

$$\zeta A_i + \zeta B_i - \varphi_i$$

extends holomorphically to $\mathbb{D}$ and

$$\psi = ((\frac{\partial^2 r_0}{\partial z_2 \partial z_1} \circ f_0)\widetilde{f_1} + (\frac{\partial^2 r_0}{\partial z_2 \partial \overline{z_1}} \circ f_0)\overline{\widetilde{f_1}} - \varphi_2, \ldots, (\frac{\partial^2 r_0}{\partial z_n \partial z_1} \circ f_0)\widetilde{f_1} + (\frac{\partial^2 r_0}{\partial z_n \partial \overline{z_1}} \circ f_0)\overline{\widetilde{f_1}} - \varphi_n),$$

which derives (15). Put

$$g(\zeta) := \widehat{\widetilde{f}}(\zeta)/\zeta, \quad \alpha(\zeta) := \zeta^2 r_{0\widehat{z}\widehat{z}}(f_0(\zeta)), \quad \beta(\zeta) := r_{0\widehat{z}\widehat{\overline{z}}}(f_0(\zeta)).$$

Observe that α, β are $(n-1)\times(n-1)$ matrices depending analytically on ζ and g is a column vector in $\mathbb{C}^{n-1}$. This allows us to reduce our task to the following: we need to find a $g \in W^{2,2}(\mathbb{T},\mathbb{C}^{n-1})$ such that g extends holomorphically to $\mathbb{D}$ and

$$\alpha g + \beta\overline{g} - \psi \quad \text{extends holomorphically to } \mathbb{D}. \tag{16}$$

Observe that we necessarily have $g(0) = \widehat{\widetilde{f}}'(0)$. Moreover, in view of (10) it is an easy observation that for any $z \in \mathbb{C}^{n-1}\setminus\{0\}$ there is

$$|z^T\alpha z| < z^T\beta\overline{z}. \tag{17}$$

Note that $\beta(\zeta)$ is self$-$adjoint, hence using the Proposition 5.2 we get the existence of a holomorphic mapping $H:\mathbb{D}\to GL(n-1,\mathbb{C})$ satisfying $HH^\star = \beta$. In this situation, (16) is equivalent to

$$H^{-1}\alpha g + H^\star\overline{g} - H^{-1}\psi \quad \text{extends holomorphically to } \mathbb{D}, \tag{18}$$

or, if we denote $h := H^Tg, \gamma := H^{-1}\alpha(H^T)^{-1}$,

$$\gamma h + \overline{h} - H^{-1}\psi \quad \text{extends holomorphically to } \mathbb{D}. \tag{19}$$

Using (17) and the results of [6] we get for any $\zeta\in\mathbb{T}$ the norm of the symmetric matrix $\gamma(\zeta)$ is less than 1. In fact, take a $z\in\mathbb{C}^{n-1}: ||z||=1$. Then

$$|z^T\gamma z| = |z^TH^{-1}\alpha(H^T)^{-1}z| < z^TH^{-1}\beta\overline{(H^T)^{-1}z} = z^TH^{-1}HH^\star\overline{(H^T)^{-1}}\overline{z} = ||z||^2.$$

We have to prove that there is an unique solution $h\in W^{2,2}(\mathbb{T})$ of (19), holomorphic on $\mathbb{D}$ and such that $h(0)=a$ with certain a.

Define the operator

$$P: W^{2,2}(\mathbb{T},\mathbb{C}^{n-1}) \ni \sum_{k=-\infty}^{\infty} a_ke^{ikt} \mapsto \overline{\sum_{k=-\infty}^{-1} a_ke^{ikt}} \in W^{2,2}(\mathbb{T},\mathbb{C}^{n-1}).$$

We shall show that a mapping $h\in W^{2,2}(\mathbb{T},\mathbb{C}^{n-1})\cap\mathcal{O}(\mathbb{D},\mathbb{C}^{n-1})$ satisfies (19) and is such that $h(0)=a$ if and only if it is a fixed point of the mapping

$$K: W^{2,2}(\mathbb{T},\mathbb{C}^{n-1}) \ni h \mapsto P(H^{-1}\psi - \gamma h) + a \in W^{2,2}(\mathbb{T},\mathbb{C}^{n-1}).$$

Indeed, take an $h\in W^{2,2}(\mathbb{T},\mathbb{C}^{n-1})\cap\mathcal{O}(\mathbb{D},\mathbb{C}^{n-1})$ and suppose $h(0)=a$ and $\gamma h+\overline{h}-H^{-1}\psi$ extends holomorphically to $\mathbb{D}$. We then have

$$h = a + \sum_{k=1}^{\infty} a_ke^{ikt}, \quad \overline{h} = \overline{a} + \sum_{k=1}^{\infty} a_ke^{-ikt} = \sum_{k=-\infty}^{-1} a_{-k}e^{ikt} + \overline{a},$$

$$P(h) = 0, \quad P(\overline{h}) = \sum_{k=1}^{\infty} a_ke^{ikt} = h - a$$

and

$$P(\gamma h + \overline{h} - H^{-1}\psi) = 0,$$

which implies

$$P(H^{-1}\psi - \gamma h) = h - a$$

and finally $K(h)=h$. Conversely, suppose $K(h)=h$. Then

$$P(H^{-1}\psi - \gamma h) = h - a = \sum_{k=1}^{\infty} a_ke^{ikt} + a_1 + a, \quad P(h) = 0$$

and

$$P(\overline{h}) = \sum_{k=-\infty}^{-1} a_{-k} e^{ikt} = \sum_{k=1}^{\infty} a_k e^{ikt} = h - a_1,$$

from which follows

$$P(\gamma h + \overline{h} - H^{-1}\psi) = P(\overline{h}) - P(H^{-1}\psi - \gamma h) = a - a_1$$

and

$$P(\gamma h + \overline{h} - H^{-1}\psi) = 0 \quad \text{if only } a = a_1.$$

Observe that $K(h)(0) = h(0) = P(H^{-1}\psi - \gamma h)(0) + a = a$ and we are done.

Thus it is enough to use the Banach fixed poit theorem. Yet, we have first to show that K is a contraction. To do this, consider in $W^{2,2}(\mathbb{T})$ the following norm

$$||h||_\varepsilon = ||h||_{L^2(\mathbb{T})} + \varepsilon ||h'||_{L^2(\mathbb{T})} + \varepsilon^2 ||h''||_{L^2(\mathbb{T})},$$

with positive ε. We shall show that for ε sufficiently small, K is a contraction with respect to the norm $|| \cdot ||_\varepsilon$.

Indeed, for each pair $h_1, h_2 \in W^{2,2}(\mathbb{T})$ there is

$$(20) \quad ||K(h_1) - k(h_2)||_{L^2(\mathbb{T})} = ||P(\gamma(h_2 - h_1))||_{L^2(\mathbb{T})} \leq \\ \leq ||\gamma(h_2 - h_1)||_{L^2(\mathbb{T})} < \Lambda ||h_2 - h_1||_{L^2(\mathbb{T})}$$

with $\Lambda := ||\gamma|| < 1$. Moreover,

$$(21) \quad ||K(h_1)' - K(h_2)'||_{L^2(\mathbb{T})} = ||P(\gamma h_2)' - P(\gamma h_1)'||_{L^2(\mathbb{T})} \leq \\ \leq ||(\gamma h_2)' - (\gamma h_1)'||_{L^2(\mathbb{T})} = ||\gamma'(h_2 - h_1) + \gamma(h_2' - h_1')||_{L^2(\mathbb{T})},$$

in view of the equality $P(h)' = P(h')$ Furthermore,

$$(22) \quad ||K(h_1)'' = K(h_2)''||_{L^2(\mathbb{T})} \leq ||\gamma''(h_2 - h_1) + 2\gamma'(h_2' - h_1') + \gamma(h_1'' - h_2'')||_{L^2(\mathbb{T})},$$

because of the formula $P(h'') = -P(h)''$. Using now the finiteness of the norms $||\gamma'||, ||\gamma''||$, the fact that $||P|| \leq 1$ and piecing together (20), (21), (22), we see there exists an $\varepsilon > 0$ such that K is a contraction with respect to the norm $|| \cdot ||_\varepsilon$.

So far we have found $\widetilde{f}$ and $\widetilde{\lambda}$ such that (11),(13) and the last $n-1$ equations from (12) are satisfied. To the end, it has to be shown that there exists an unique $\widetilde{q} \in Q_0$ such that

$$\widetilde{q} + \zeta A_1 + \zeta B_1 - \varphi_1$$

admits a holomorphic extension to $\mathbb{D}$.

It is not hard to see that if

$$\pi(\zeta A_1 + \zeta B_1 - \varphi_1) = \sum_{k=-\infty}^{-1} a_k e^{ikt},$$

then $\widetilde{q}$ should be taken as follows:

$$\widetilde{q} = -\sum_{k=-\infty}^{-1} a_k e^{ikt} - \sum_{k=0}^{\infty} a_k e^{ikt},$$

with a real b_0 and $b_k = \overline{a_k}$ for $k = 1, 2, \ldots$ □

6. Proof of Theorem 1.5.

Here we prove the main result, i.e. Theorem 1.5. First, from Proposition 2.2, we know that $E-$mappings are (unique) extremals. So it is sufficient to prove that if we have $z, w \in D$ (resp $z \in D$ and $v \in \mathbb{C}^n$) then there is a $E-$mapping $f : \mathbb{D} \to D$ such that $f(0) = z$, $f(\zeta) = w$, with $1 > \zeta > 0$ (resp. $f(0) = z$ and $f'(0) = \lambda v$ for $\lambda > 0$). Then any extremal mapping for (z, w) (resp. for (z, v)) must be equal to f (since they are unique extremal).

First we consider the case, when D is strictly convex and we prove that there is a $E-$mapping for $(z, w) \in D \times D$ (resp. for $(z, v) \in D \times \mathbb{C}^n$). Without loos of generality we may assume that $D \subset \mathbb{B}_n$, where $\mathbb{B}_n$ is a open unit ball in $\mathbb{C}^n$. For $t \in [0, 1]$ consider the domains $D_t := tD + (1 - t)\mathbb{B}_n$. It is easy to see that D_t are strictly convex (since D and $\mathbb{B}_n$ are) and $D \subset D_t$ for all $t \in [0, 1]$. Let T be a subset of $[0, 1]$ such that for all $t \in T$ there is a $E-$mapping $f_t : \mathbb{D} \to D_t$ for $(z, v) \in D \times \mathbb{C}^n$ (resp. for $(z, w) \in D \times D$). It is easy to see that $0 \in T$. To prove that $1 \in T$, we need to know that $(D_t, z) \in \mathcal{D}(c)$ for some $c > 0$ independent of t. Since for $t = 0$ there is $E-$mapping, then from Proposition **??** there is a neighborhood T_0 of 0 in $[0, 1]$ and

- there are an $E-$mappings $f_t : \mathbb{D} \to D_t$ and $\xi_t \in (0, 1)$ such that for all $t \in T_0$ we have $f_t : \mathbb{D} \to D_t$, $f_t(0) = f(0)$, and $f_t(\xi_t) = f(\xi)$ (in the case of Lempert function);
- there are an $E-$mappings $f_t : \mathbb{D} \to D_t$ and $\lambda_t > 0$ such that for all $t \in T_0$ we have $f_t(0) = f(0)$, and $f_t'(0) = \lambda_t v$ (in the case of Kobayashi-Royden pseudometric).

It means that T is open in $[0, 1]$.

Now we prove that T is closed in $[0, 1]$. Let us take $(t_n) \subset T$ such that $t_n \to t$. We prove that $t \in T$. From Proposition 4.4, 4.8 and 4.9 we have that f_{t_n} and $\widetilde{f_{t_n}}$ are equicontinuous in $\mathcal{C}^{1/2}(\overline{\mathbb{D}})$. From Arzela-Ascoli theorem there is a subsequence $(s_n) \subset (t_n)$ such that $f_{s_n} \to g$ and $\widetilde{f_{s_n}} \to G$ uniformly. It is easy to see that $g : \mathbb{D} \to D_t$ is an $E-$mapping. So T is closed subset of $[0, 1]$. This ends the proof in the case strict convexity of D.

Let us back to general situation. Let D be a strictly linearly convex domain and let $(z, w) \in D \times D$ (resp. $(z, v) \in D \times \mathbb{C}^n$). Take $\mu \in \partial D$ such that $\text{dist}(z, \partial D) = ||z - \mu||$. Since D is strictly linearly convex then μ is a point of strict convexity. There exist a neighborhood V_0 of μ in $\mathbb{C}^n$ such that $V_0 \cap D$ is strictly convex. From previous part of proof there exist an $E-$mapping $g : \mathbb{D} \to V_0 \cap D$ for $(g(0), g(\xi))$ (resp. for $(g(0), g'(0))$) such that $g(\partial\mathbb{D}) \subset V_0 \cap \partial D$, so g is an $E-$mapping in D. Let $Z := g(0)$ and $W := g(\xi)$ (resp. $Z := g(0)$ and $V := g'(0)$). If $Z = z$ and $W = w$ (resp. $Z = z$ and $V = v$) then we are done. In opposite situation we take a curves $z_t : [0.1] \to D$, $w_t : [0, 1] \to D$ (resp. $z_t : [0.1] \to D$, $v_t : [0.1] \to \mathbb{C}^n$) which joint z and Z, w and W (resp. z and Z, v and V). Again let T be a subset of $[0, 1]$ such that for $t \in T$ there is an $E-$mapping g_t in D such that $(g_t(0), g_t(\xi_t)) = (z_t, w_t)$ for some $\xi_t \in (0, 1)$ (resp. $(g_t(0), g_t'(0)) = (z_t, \lambda_t v_t)$ for some $\lambda_t > 0$). The same argumentation as above leads to $T = [0, 1]$. In particular there exist an $E-$mapping $f : \mathbb{D} \to D$ for (z, w) (resp. (z, v)).

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