Centre symmetry sets in four space

Będlewo 2011

By Graham Reeve

In this talk I shall continue a series of studies (see below) of singular varieties associated in a natural way to a pair of submanifolds in an affine space.

[4] M.V.Berry *Semi-classical mechanics in phase space: A study of Wigner's function*, Philos. Trans. R. Soc. Lond., A 287(1977), 237-271.

[6] W.Domitrz and P.de M.Rios, *Singularities of global centre symmetry sets of Lagrangian sub- manifolds*, preprint.

[8] P.J.Giblin and P.A.Holtom, *The centre symmetry set*, Geometry and Topology of Caustics, Banach Center Publications, Vol 50, ed. S.Janeczko and V.M.Zakalyukin, Warsaw, 1999, pp.91-105.

[9] P.J.Giblin, & S.Janeczko, *Geometry of curves and surfaces through the contact map*, Preprint.

[10] P.J.Giblin, J.P.Warder, & V.M.Zakalyukin, *Bifurcations of affine equidistants*, Proceedings of the Steklov Institute of Mathematics 267 (2009), 57-75.

[11] P.J.Giblin, & V.M.Zakalyukin, *Singularities of centre symmetry sets*, London Math Soc. (3) 90 (2005) 132-166.

[12] P.J.Giblin, & V.M.Zakalyukin, *Recognition of centre symmetry set singularities*, Geom. Dedicata (2007) 130:43-58.

[15] S.Janeczko, *Bifurcation of the center of symmetry*, Geom. Dedicata 60 (1996) 9-16.

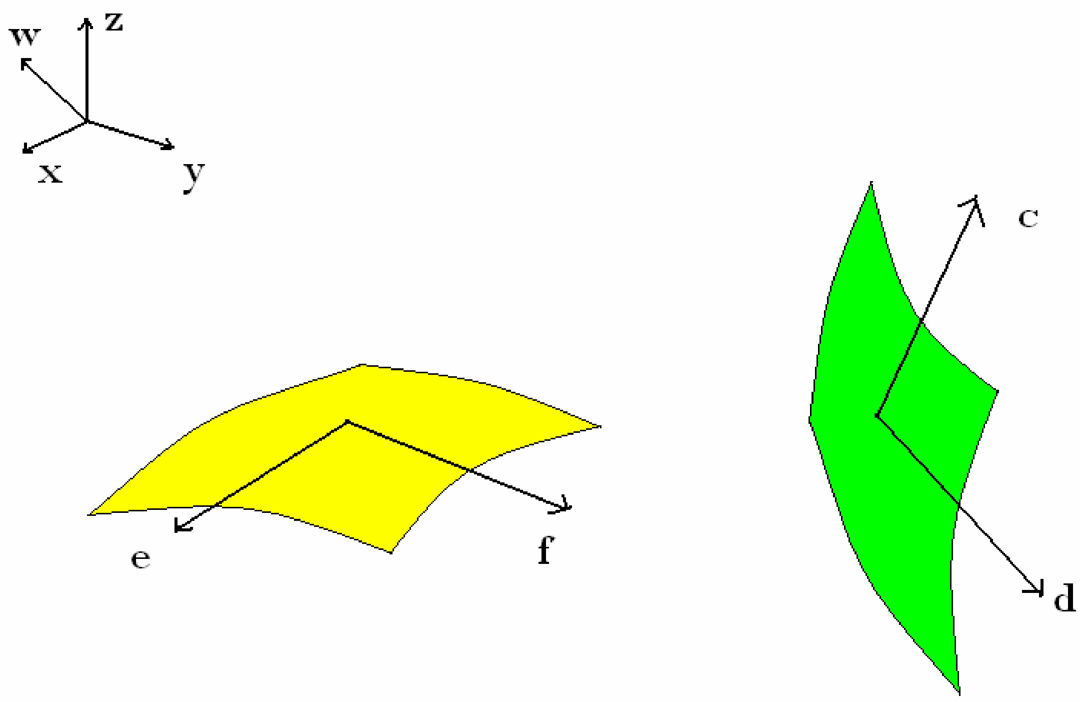
[17] G.M.Reeve, V.M.Zakalyukin, *Singularities of the Minkowski Set and affine equidistants for a curve and a surface*, Preprint, University of Liverpool, 2010.

[18] J.P.Warder, *Symmetries of curves and surfaces*, PhD Dissertation, Liverpool (2009), Available on <http://www.liv.ac.uk/~pjgiblin>

Given two 2-surfaces M and N in 4-dimensional affine space we consider pairs of points $a \in M$ and $b \in N$ such that the tangent planes T_aM and T_bN are not in general position.

We say that two 2-planes in $\mathbb{R}^4$ are in *general position* if being shifted to the origin they span $\mathbb{R}^4$ as a linear space $T_aM \bigoplus T_bN = \mathbb{R}^4$.

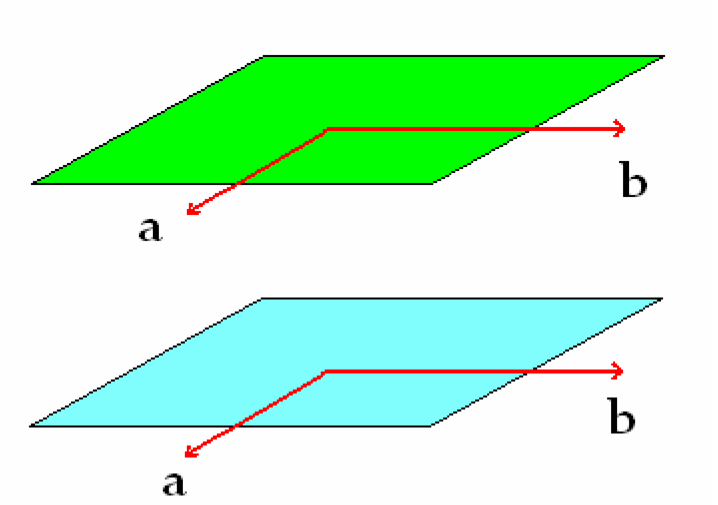
Here the planes are considered as linear subspaces in linear $\mathbb{R}^4$.



3 – space

In 3-space two generic 2-planes intersect along a line.

In 3-space if two planes share two common directions they are said to be parallel.



4 – space

In 4-space two generic 2-planes intersect at point.

In 4-space if two planes share two common directions they are said to be parallel.

In 4-space if two planes share just one common direction they are called ***weakly parallel***.

Note: In 3 space all planes are at least weakly parallel.

If $T_aM \bigoplus T_bN \neq \mathbb{R}^4$ the pair of points a, b a *pseudo parallel pair.*

That is the two tangent planes are weakly parallel.

A straight line that contains a pseudo parallel pair is called a *chord.*

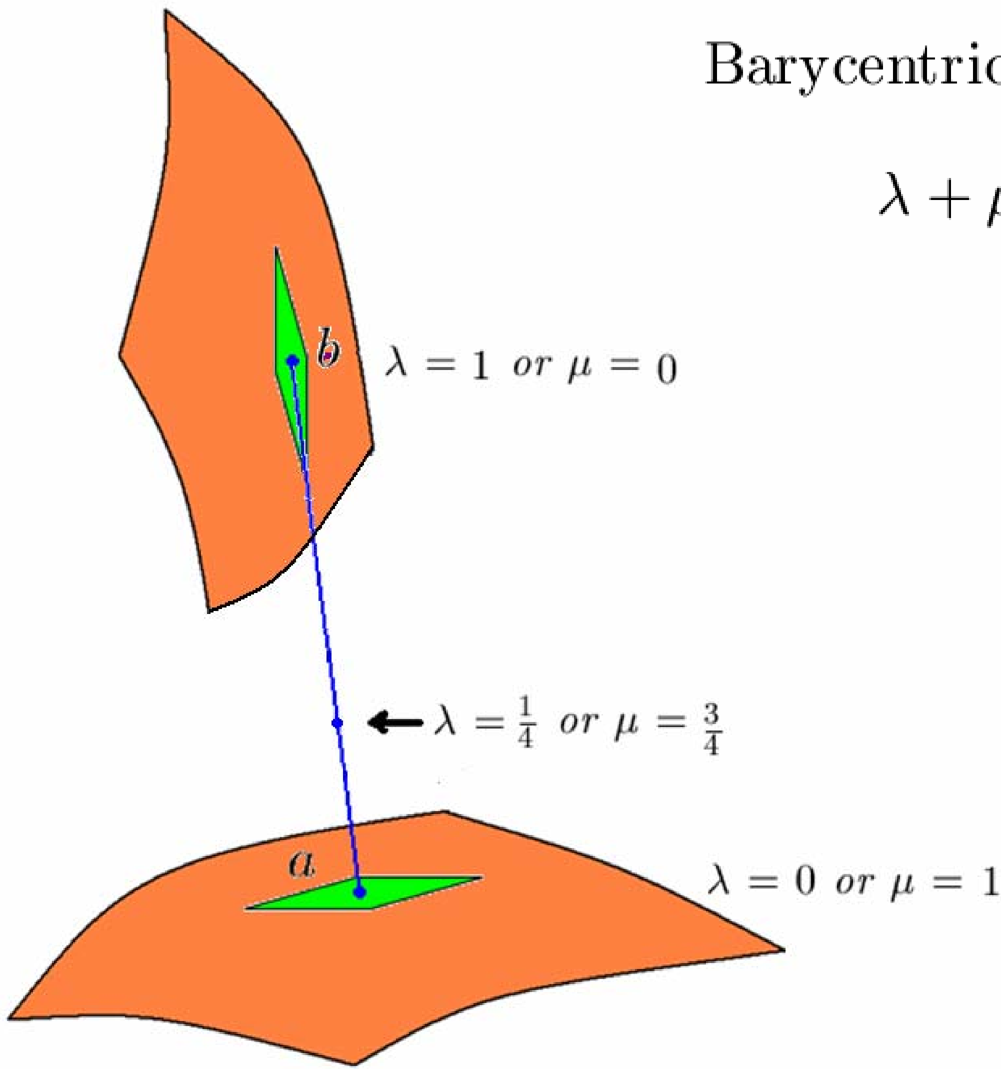
The envelope of these chords for 2 generic 2-surfaces is called the *Minkowski set* and is our main object.

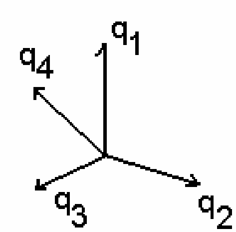
Parametrise the chord $l(a,b)$ passing through a and b by a parameter λ

$$l(a,b) = \{q \in \mathbb{R}^4 \mid q = \lambda a + \mu b, \lambda \in \mathbb{R}, \mu \in \mathbb{R}, \lambda + \mu = 1\}.$$

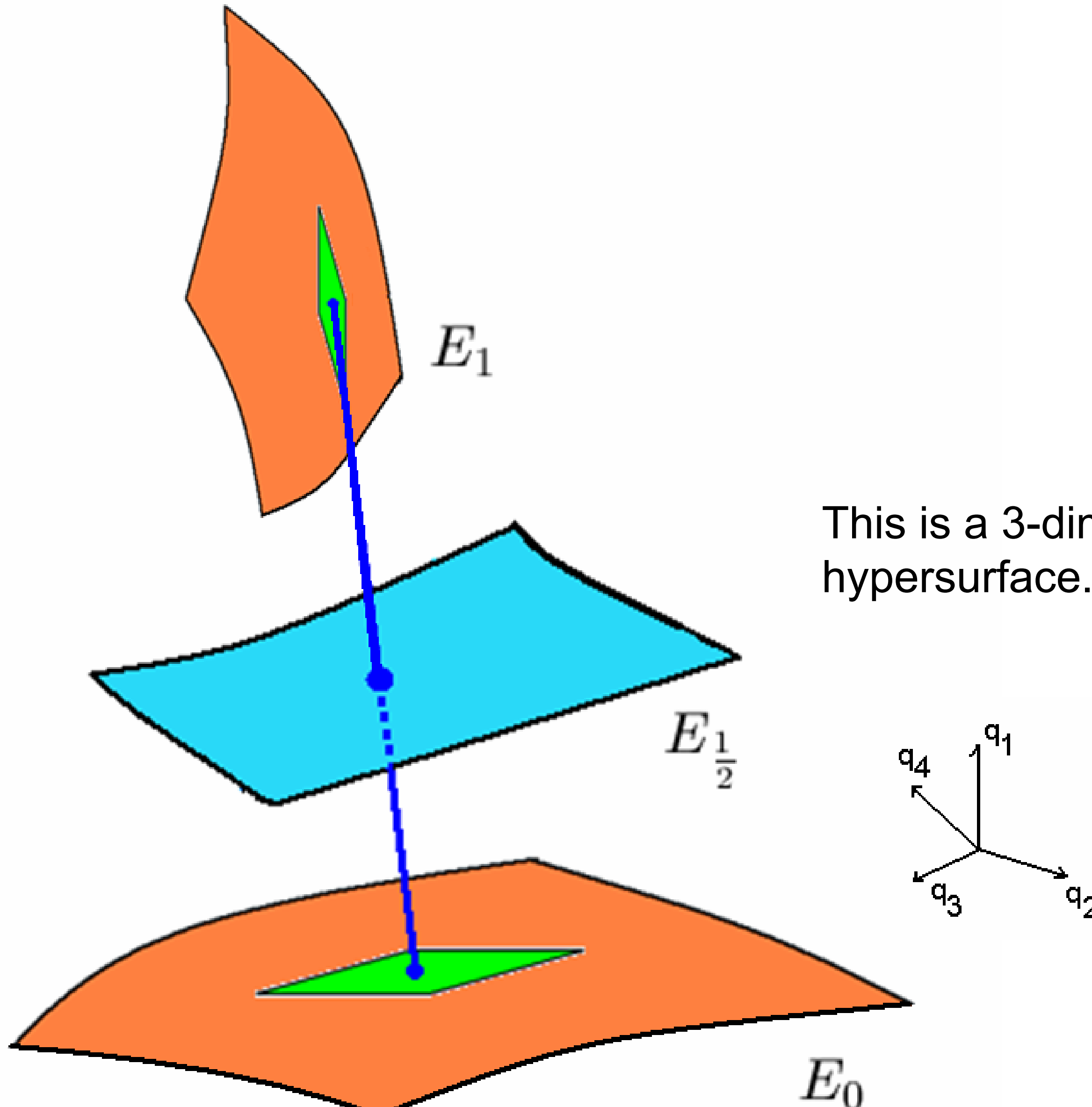
Barycentric Coordinates

$$\lambda + \mu = 1$$





The *affine* (λ, μ)*-equidistant* E_λ is the set of points $q \in \mathbb{R}^4$ such that $q = \lambda a + \mu b$ for given $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$ for all parallel pairs (a, b).



This is a 3-dimensional hypersurface.

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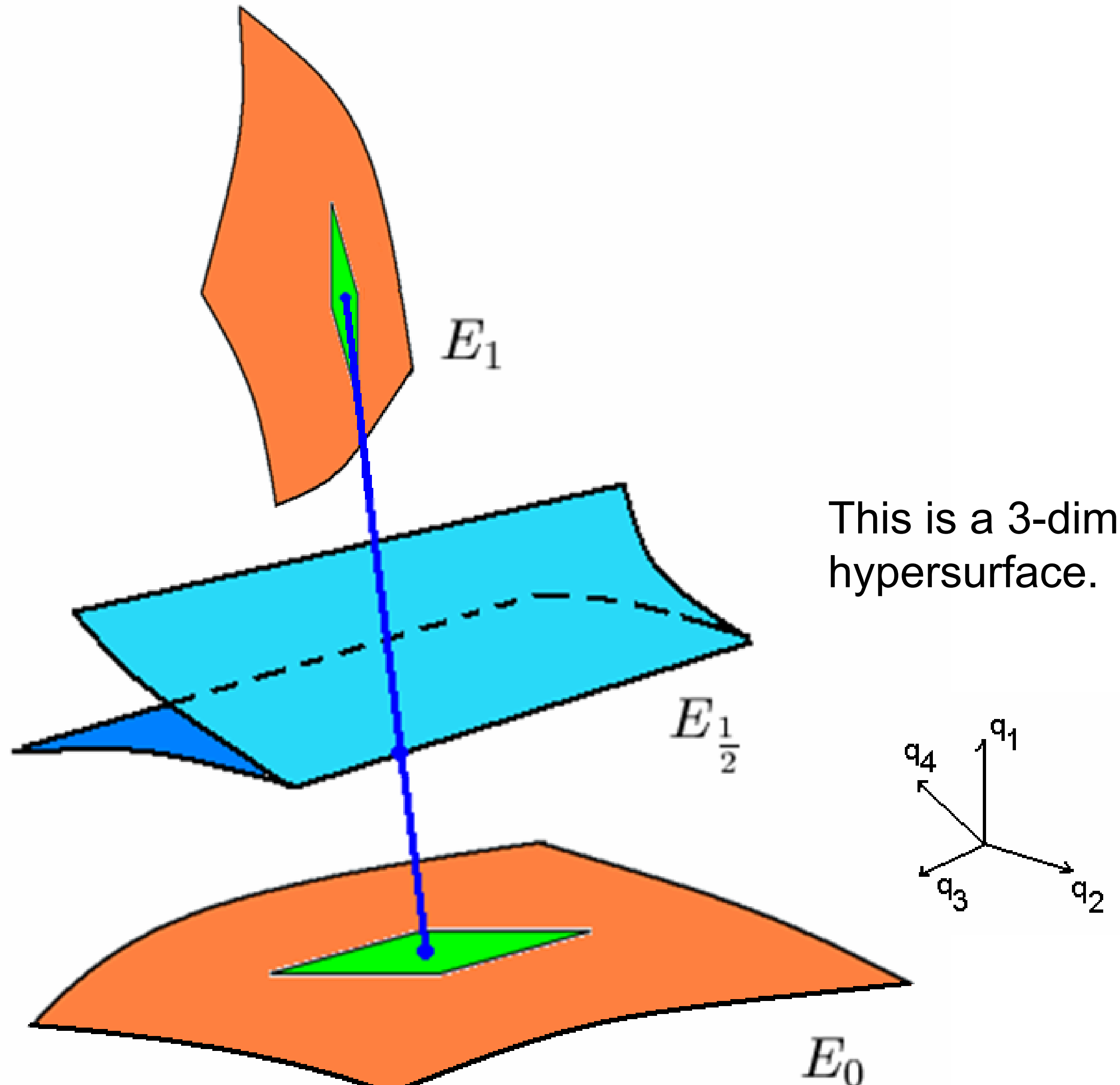
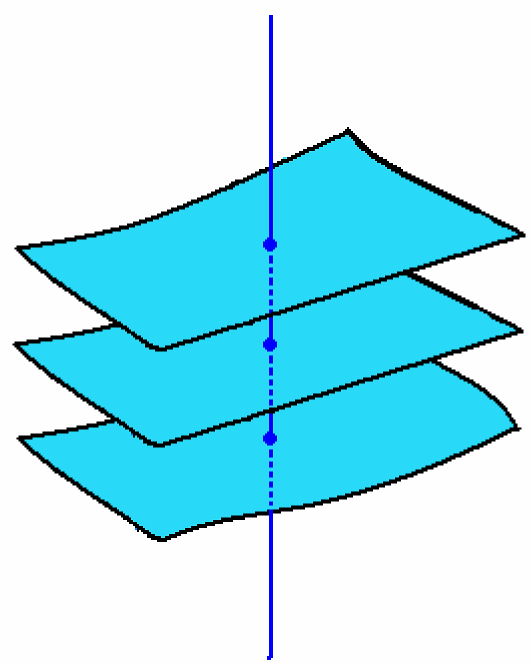


The space $\mathbb{R}^4_e = \mathbb{R} \times \mathbb{R}^3$ with coordinate $\lambda \in \mathbb{R}$ (affine time), on the first factor is called the *extended affine space.*

An *affine extended wave front* $W(M,N)$ of the pair M, N is the union of all affine equidistants each embedded into its own slice of the extended affine space:

$$W(M,N) = \{(\lambda, E_\lambda)\} \subset \mathbb{R}^4_e$$

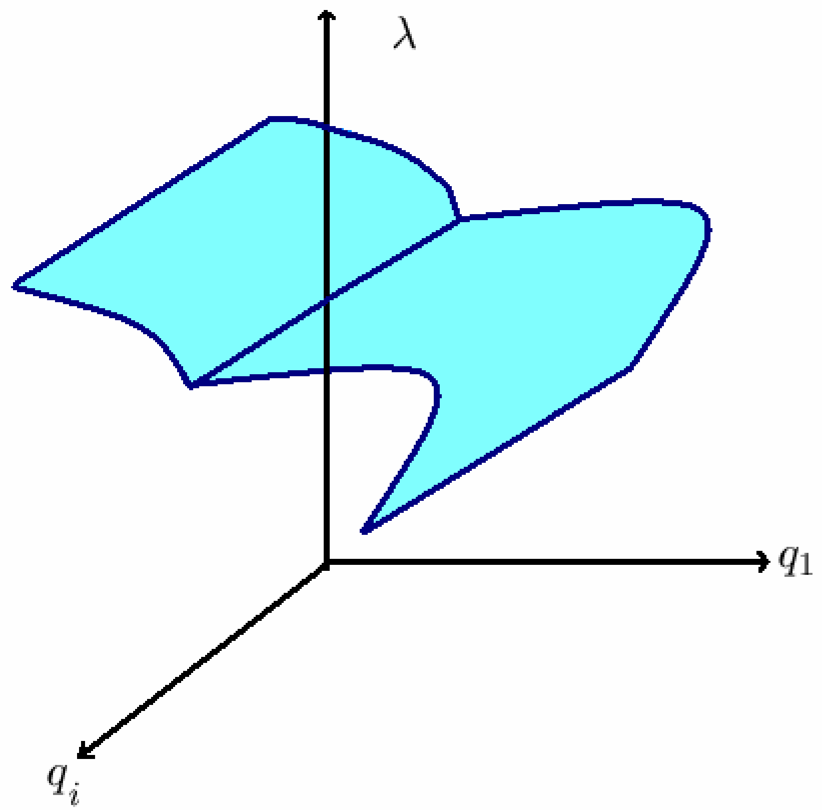


The Bifurcation or Minkowski Set. (CSS)

Consider the projection of $\mathbb{R}^4_e$

$$\pi : (\lambda, q) \mapsto q \ .$$

The Minkowski set (M, N) is the image under π of the locus of the critical points of the restriction $\pi_r = \pi|_{W(M,N)}$.



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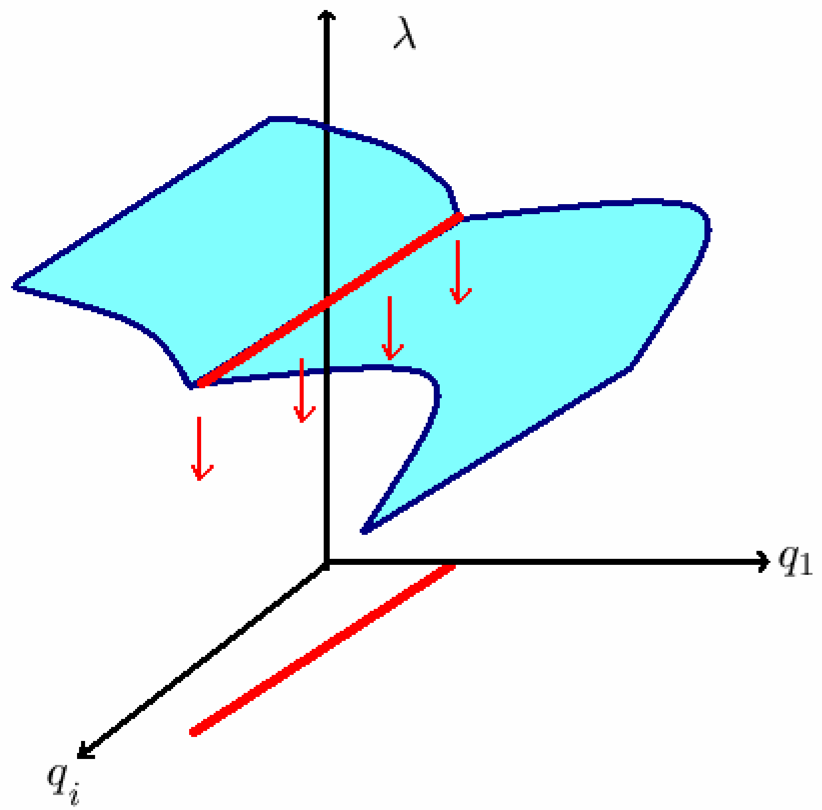

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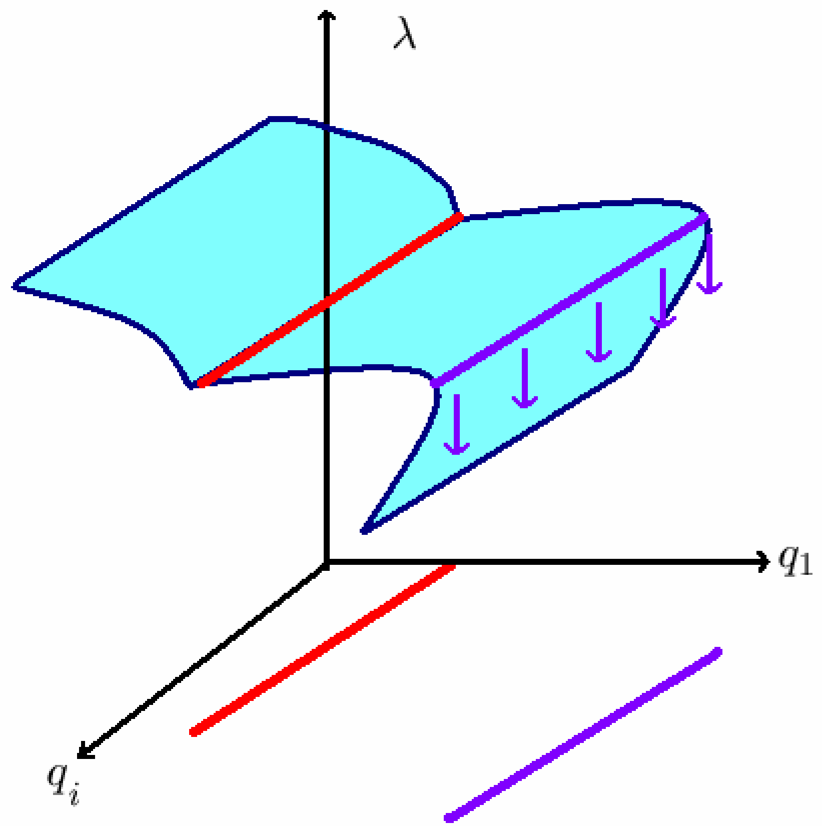
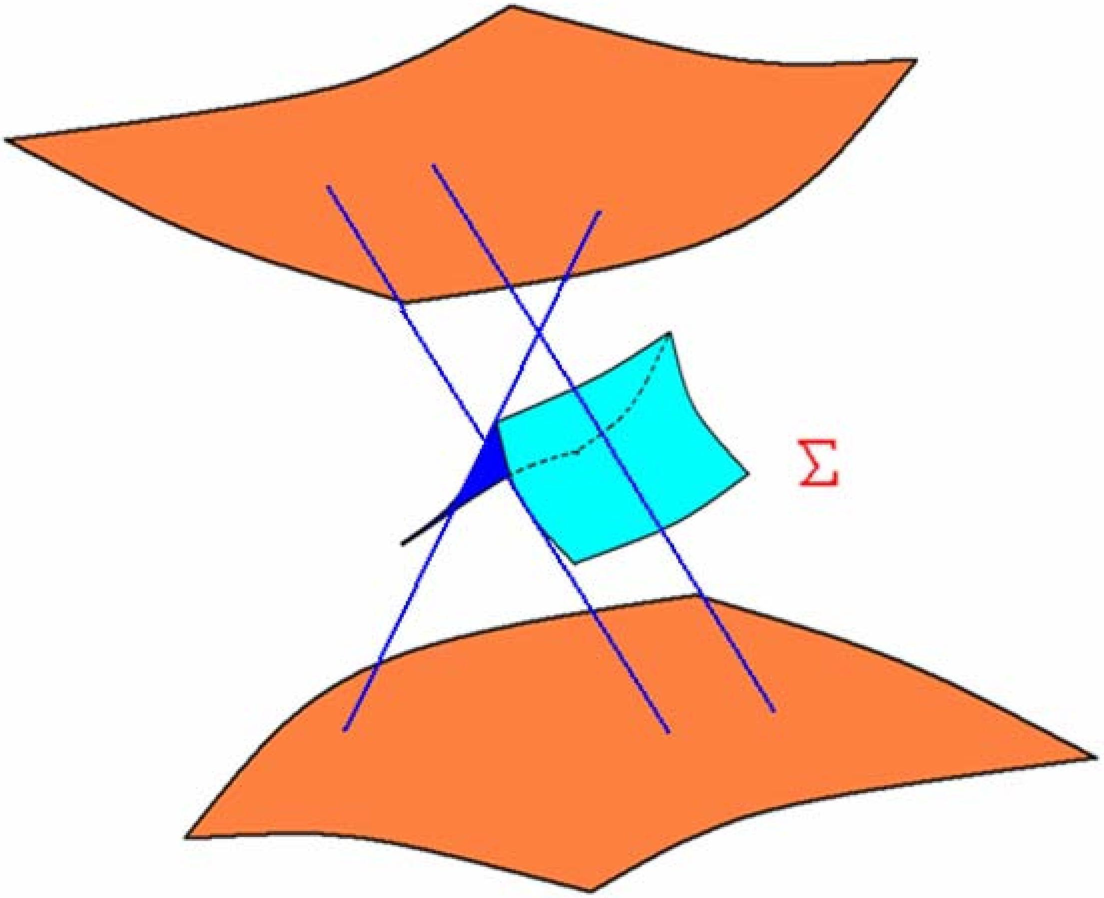
Caustic Σ

Criminant Δ





The Minkowski set is the union $\Delta \cup \Sigma$.

It coincides with the envelope of the family of chords.

The singularities of the Minkowski set happen to be singularities of Lagrange and Legendre projections determined by an appropriate generating family.

Parametrise M and N by local coordinates

$$(x, y) \in U \subset \mathbb{R}^2$$

and

$$(s, t) \in V \subset \mathbb{R}^2$$

where U and V are two charts in the plane.

So assume M and N are the images of the embeddings

$$r_2 : U \to \mathbb{R}^4$$

and

$$r_1 : V \to \mathbb{R}^4$$

respectively.

Consider the following family $\mathcal{F}$ of functions in variables $n \in (\mathbb{R}^4)^\wedge \setminus \{0\}$, (s,t) and (x,y), and parameters $(\lambda, q) \in \mathbb{R} \times \mathbb{R}^4$ of the form

$$\mathcal{F}(n, s, t, x, y, \lambda, q) = \langle \lambda r_1(s,t) + \mu r_2(x,y) - q, n \rangle$$

where $\langle , \rangle$ is the standard pairing of vectors in $\mathbb{R}^4$ with covectors n from the dual space $(\mathbb{R}^4)^\wedge$.

Recall that the wave front $W(\mathcal{F})$ of a family of functions $\mathcal{F}$ depending on parameters is the set of parameter values which correspond to the appearance of a critical point at the zero level set of the function. The wave front $W(\mathcal{F})$ is defined by the Legendre conditions:

$$\mathcal{F} = 0, \frac{\partial \mathcal{F}}{\partial n} = 0, \frac{\partial \mathcal{F}}{\partial t} = \frac{\partial \mathcal{F}}{\partial s} = \frac{\partial \mathcal{F}}{\partial x} = \frac{\partial \mathcal{F}}{\partial y} = 0.$$

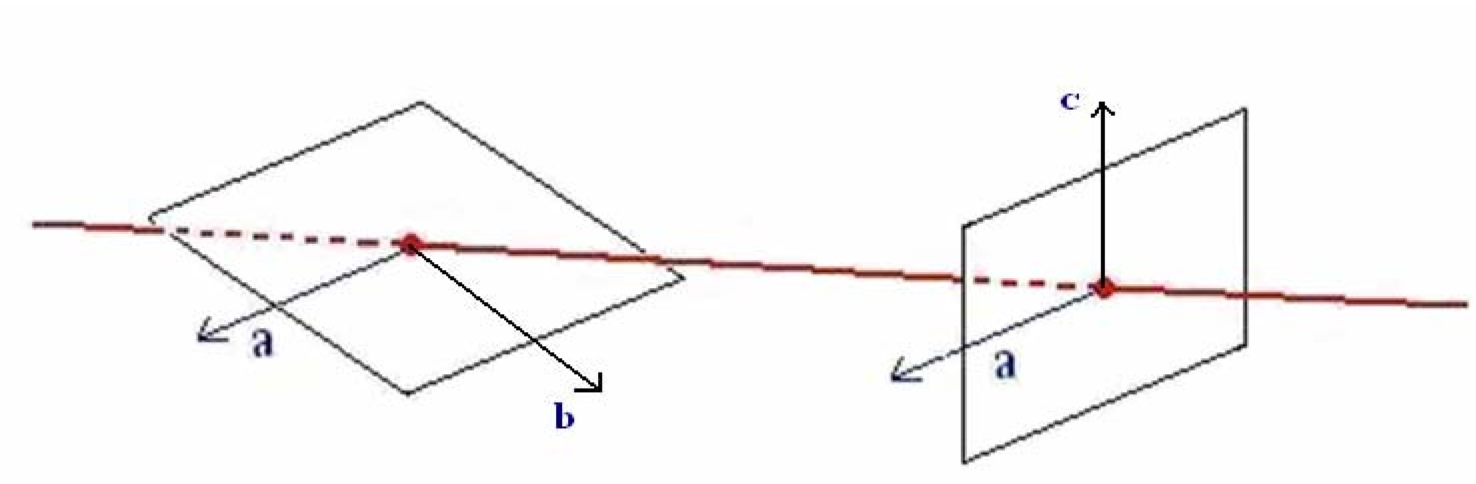
The affine group acts on the space of 1-jets of the two surface germs M and N.

The jet is determined by the pair (a, b) and the tangent planes T_aM, T_bN.

The non-generic jets (of codimension not exceeding 4) splits into the following 5 orbits:

Transversal, non-transversal, tangential, bitangential and parallel.

Transversal Case

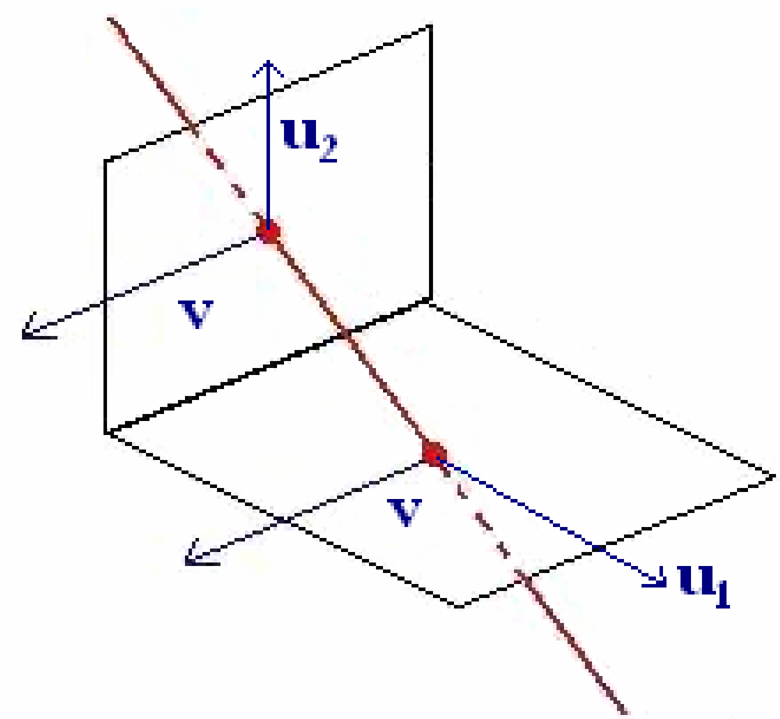


The two tangent planes share a common direction a.

The vectors a, b, c and the chord joining the two planes are independent.

This case has codimension 1.

Non-Transversal Case

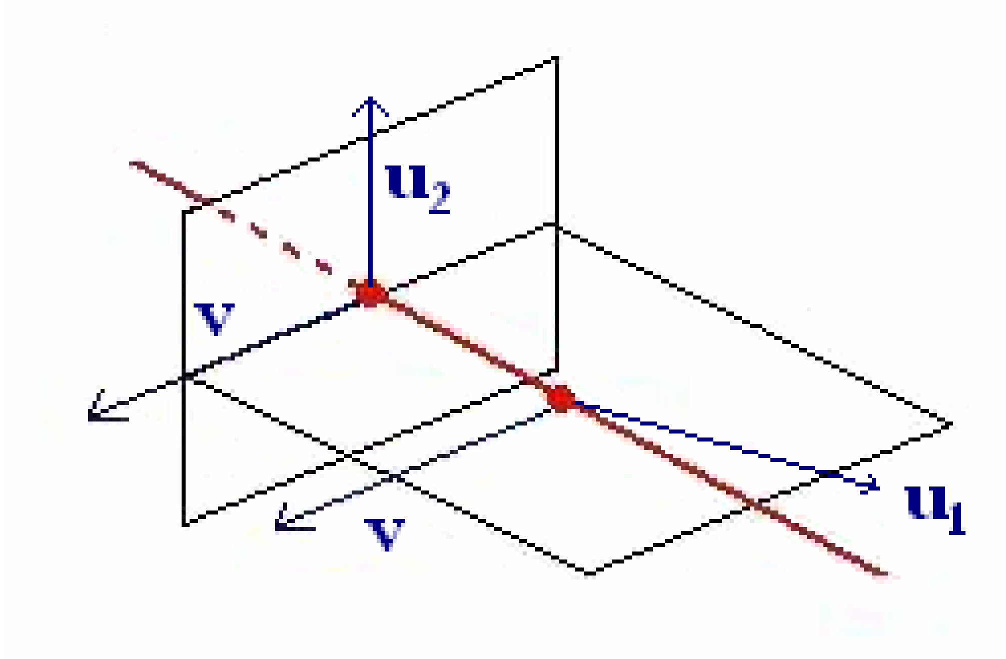


In this case the two tangent planes share a common direction and intersect along a line that does not contain either of the base points.

Equivalently we can say that direction of the chord belongs to the space $T_aM \bigoplus T_bN$.

This case has codimension 2.

Tangential Case

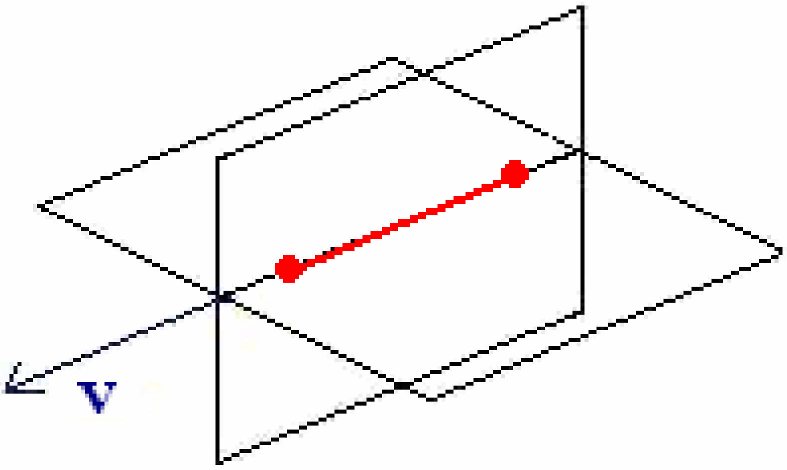


In this case the two tangent planes intersect along a common line that contains one of the points.

In other words the chord belongs to one of the tangent planes

This case has codimension 3.

Bitangential Case

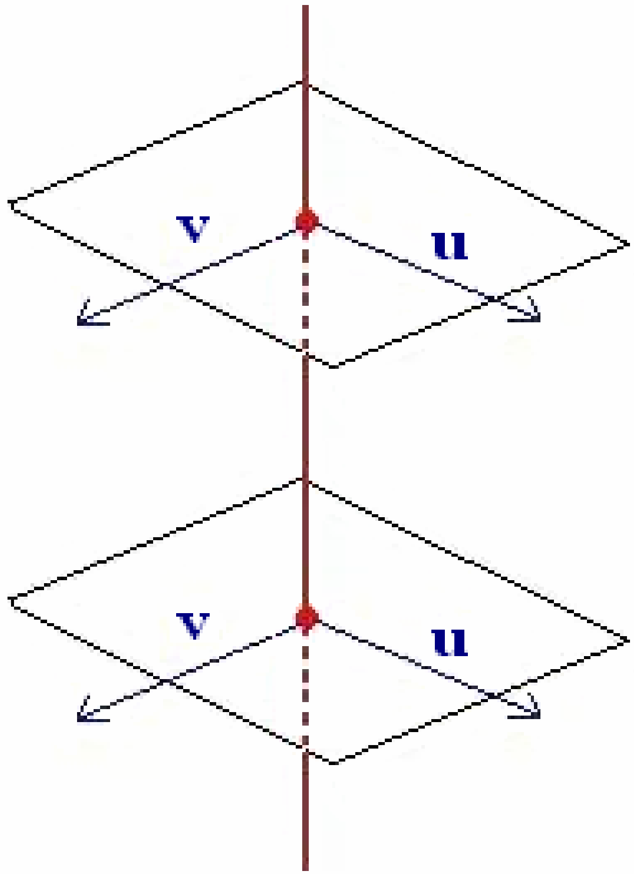


In this case the tangent planes intersect along a common line that contains both base points.

This case has codimension 4 and so occurs at isolated points.

In fact various formulae exist relating the number of bitangencies with the number of double points, elliptic points, parabolic points and the Euler number for generic immersions of closed manifolds without boundary.
See D. Dreibelbis (2007).

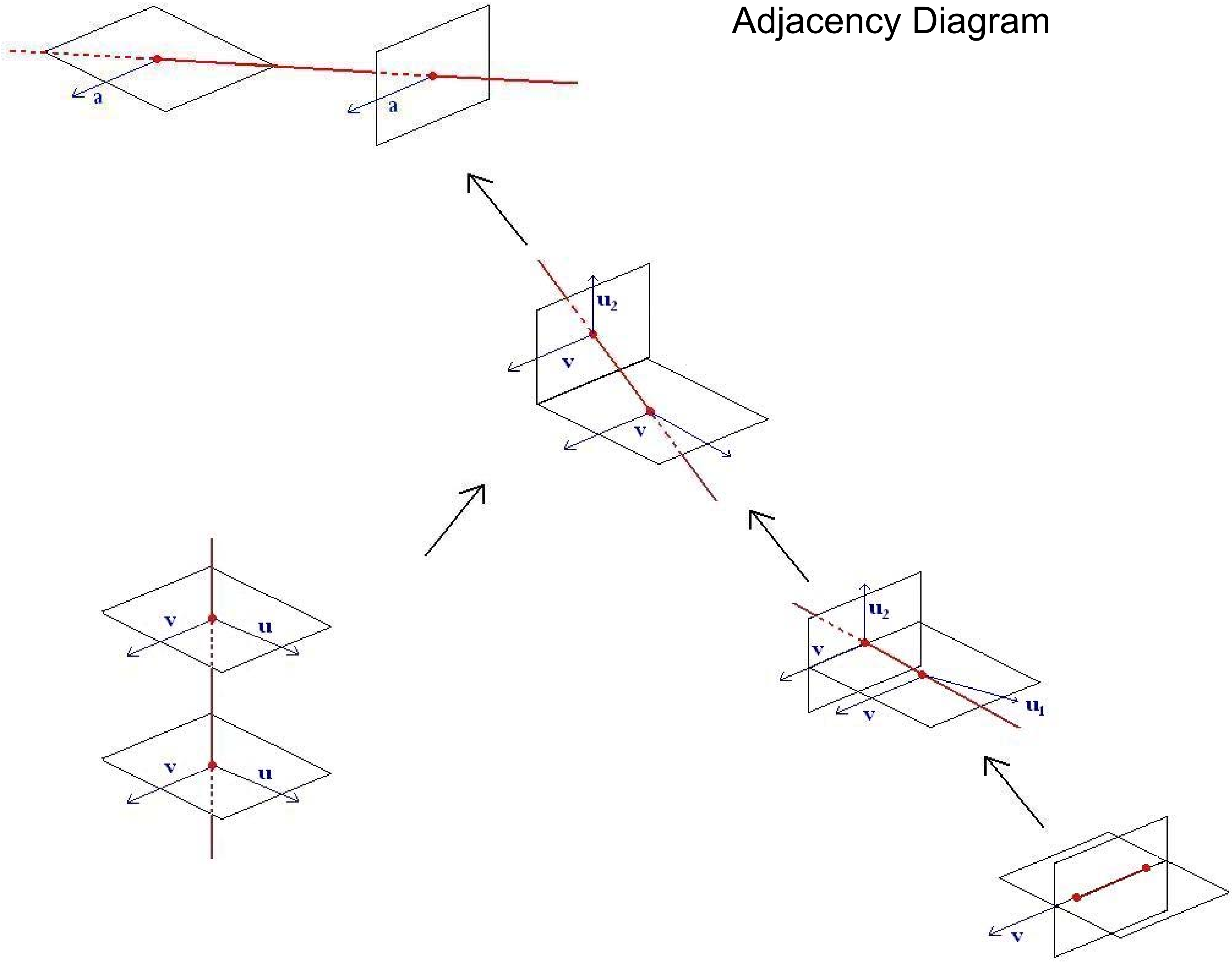
Parallel Case



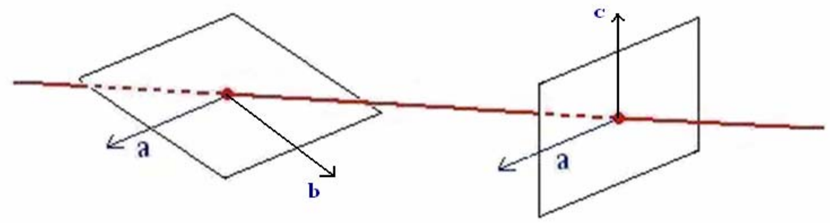
In this case the tangent planes are parallel but distinct

This case has codimension 4 so occurs at isolated points

Adjacency Diagram



Transversal Case



In this case up to an appropriate affine transformation of $\mathbb{R}^4 = (x, y, w, z)$ we can always assume that the base parallel pair a_0, b_0 coincides with the pair of points $(0,0,0,-1), (0,0,0,0)$,

the tangent plane to the surface M at a_0 is parallel to the (x, y)-coordinate plane,

and the tangent plane to the surface N at b_0 coincides with the (y, w)-plane.

In these coordinates locally

$$M = \{(x, y, f(x, y), g(x, y) - 1)\}$$

$$N = \{(r(s, t), t, s, p(s, t))\}$$

Where the functions f, g, r and p all have vanishing 1-jets.

Recall:

$$\mathcal{F}(n, s, t, x, y, \lambda, q) = \langle \lambda r_1(s, t) + \mu r_2(x, y) - q, n \rangle$$

We then reduce the number of variables using a stabilisation procedure.

Consider the following family of functions in t with parameters λ and q

$$\Phi(t, \lambda, q) = \lambda(g(\tilde{x}, y(t)) - 1) + \mu \left[p\left(\frac{q_3 - \lambda f(\tilde{x}, y(t))}{\mu}, t \right) \right] - q_4$$

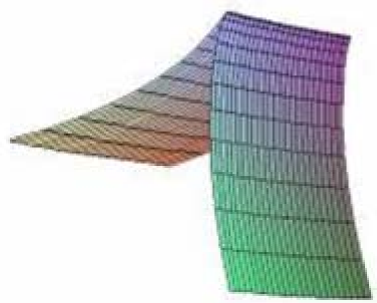
where

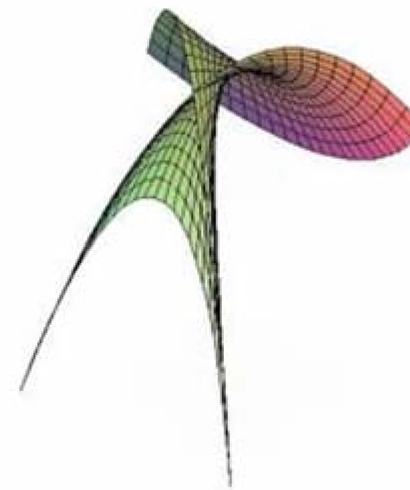
$$y(t) = \frac{q_2 - \mu t}{\lambda}$$

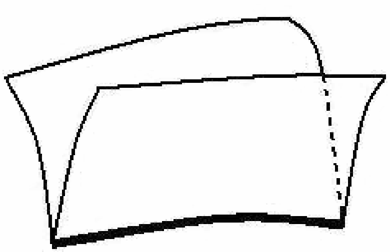
and $\tilde{x}$ and $\tilde{s}$ are solutions of the system of equations:

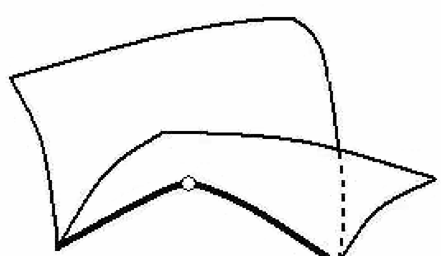
$$\tilde{x} = \frac{q_1 - \mu r(\tilde{s}, t)}{\lambda}, \tilde{s} = \frac{q_3 - \lambda f(\tilde{x}, y(t))}{\mu}.$$

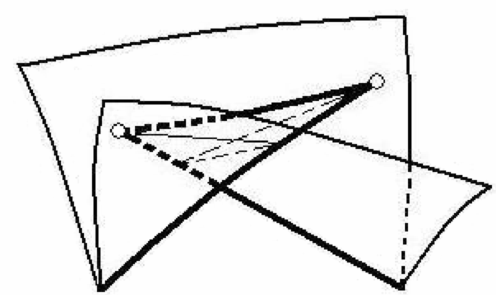
Theorem 1 *In the pseudo parallel transversal case outside M and N the germ at any point of the envelope of the family of chords for generic germs M and N is diffeomorphic to one of the standard caustics of A_k type with $k = 2, 3, 4$ or 5 (regular surface, cuspidal edge, swallowtail or butterfly)*











Non-Transversal Case

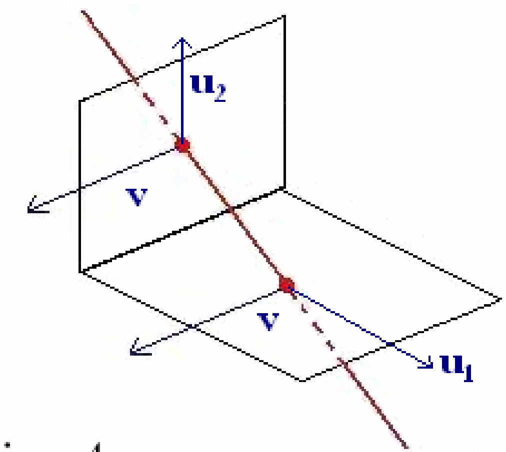
The generic space time contact stable families consist of the well known Arnold-Goryunov low dimensional fibred contact classification (which coincides with simple boundary classes)

$$B_k : \pm t^2 + \varepsilon^k + q_{k-2}\varepsilon^{k-2} + \ldots + q_4,$$

$$C_k : t^k + t\varepsilon + q_1\varepsilon + q_4 + \sum_{i=2}^{k-2} q_i t^i$$

$$C_2 \approx B_2$$

$$F_4 : t^3 + \varepsilon^2 + q_2 t\varepsilon + q_3 t + q_4.$$



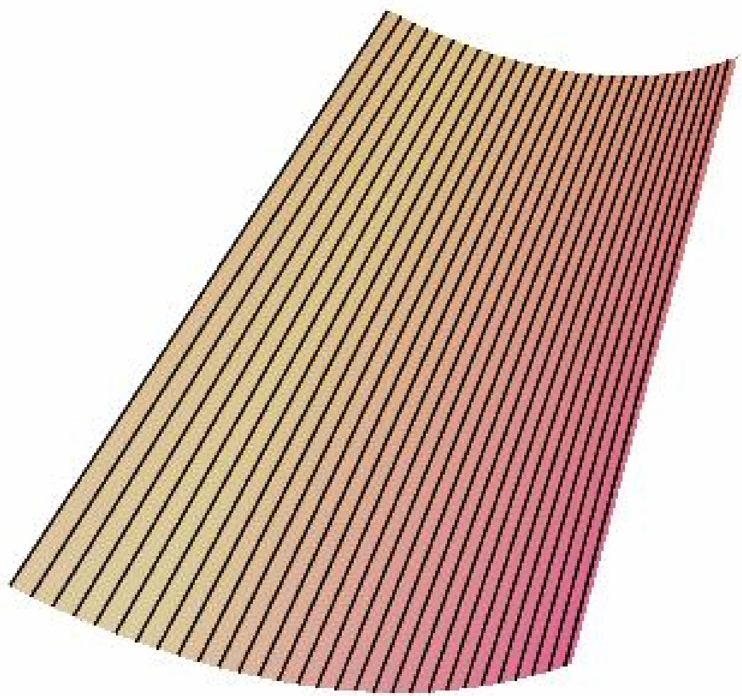
There are also two non-simple singularities of codimension 4:

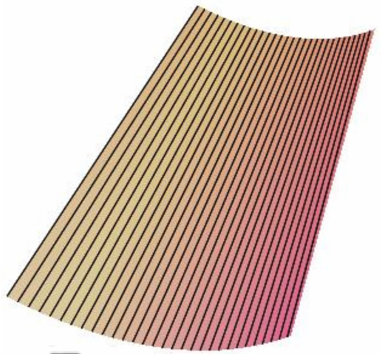
$$F_{1,0} : \mathcal{F} = t^3 + a(q_1, \ldots, q_4)t\varepsilon^2 + \varepsilon^3 + q_1 t + q_2\varepsilon + q_3\varepsilon t + q_4,$$

$$K_{4,2} : \mathcal{F} = t^4 + b(q_1, \ldots, q_4)t^2\varepsilon + \varepsilon^2 + q_1 t^2 + q_2 t\varepsilon + q_3 t + q_4,$$

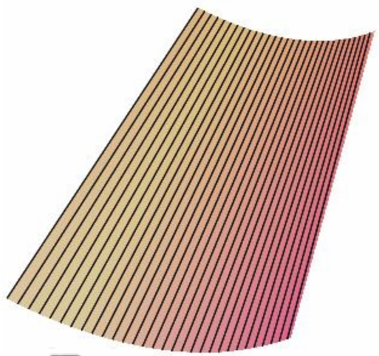
where $a(q)$ and $b(q)$ are arbitrary functions, only the conditions $4a^3(0) \neq 27$ and $b^2(0) \neq 4$ must hold.

$$\begin{tikzcd}
B_2 = C_2 & B_3 \arrow[l] & B_4 \arrow[l] & B_5 \arrow[l] \\
C_3 \arrow[u] & F_4 \arrow[l] \arrow[u] & F_{1,0} \arrow[l] & \\
C_4 \arrow[u] & K_{4,2} \arrow[l] \arrow[u] & & \\
C_5 \arrow[u] & & &
\end{tikzcd}$$

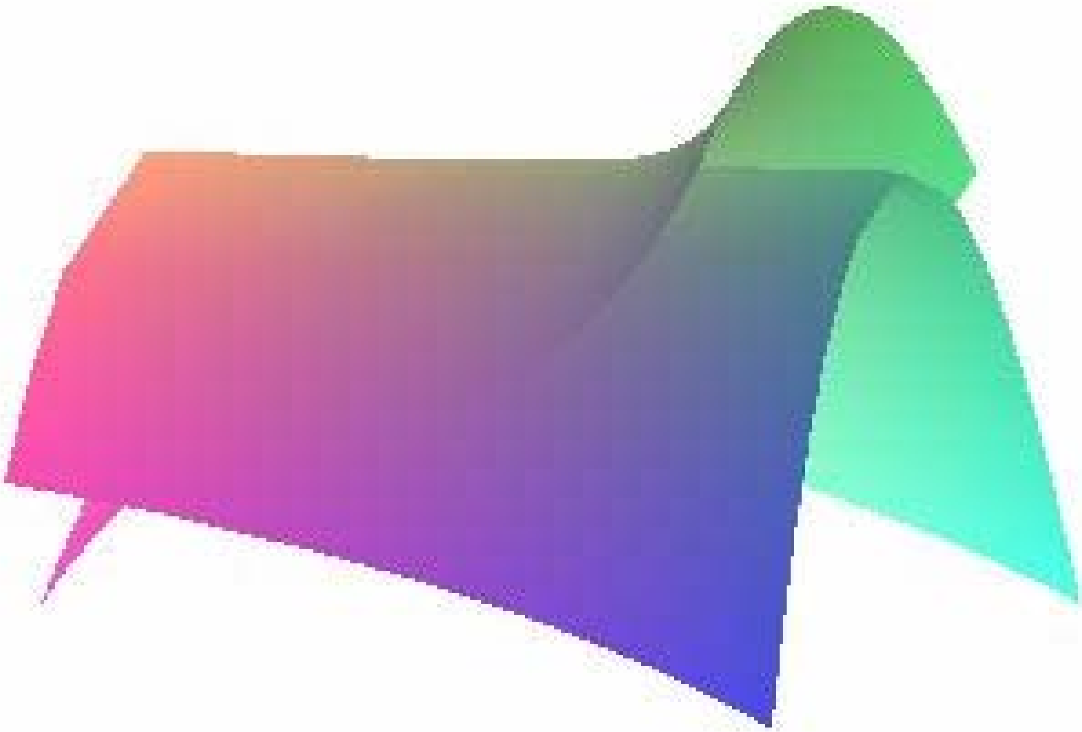


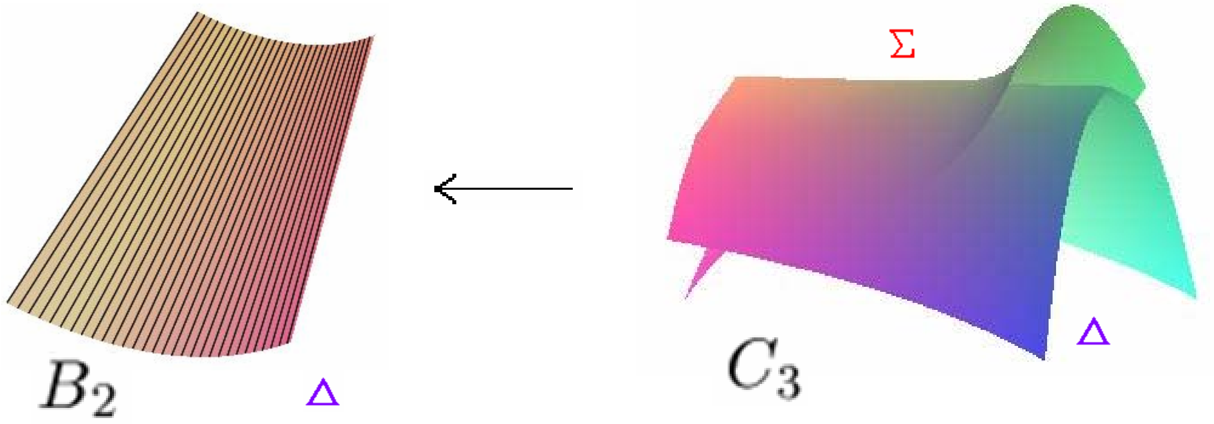


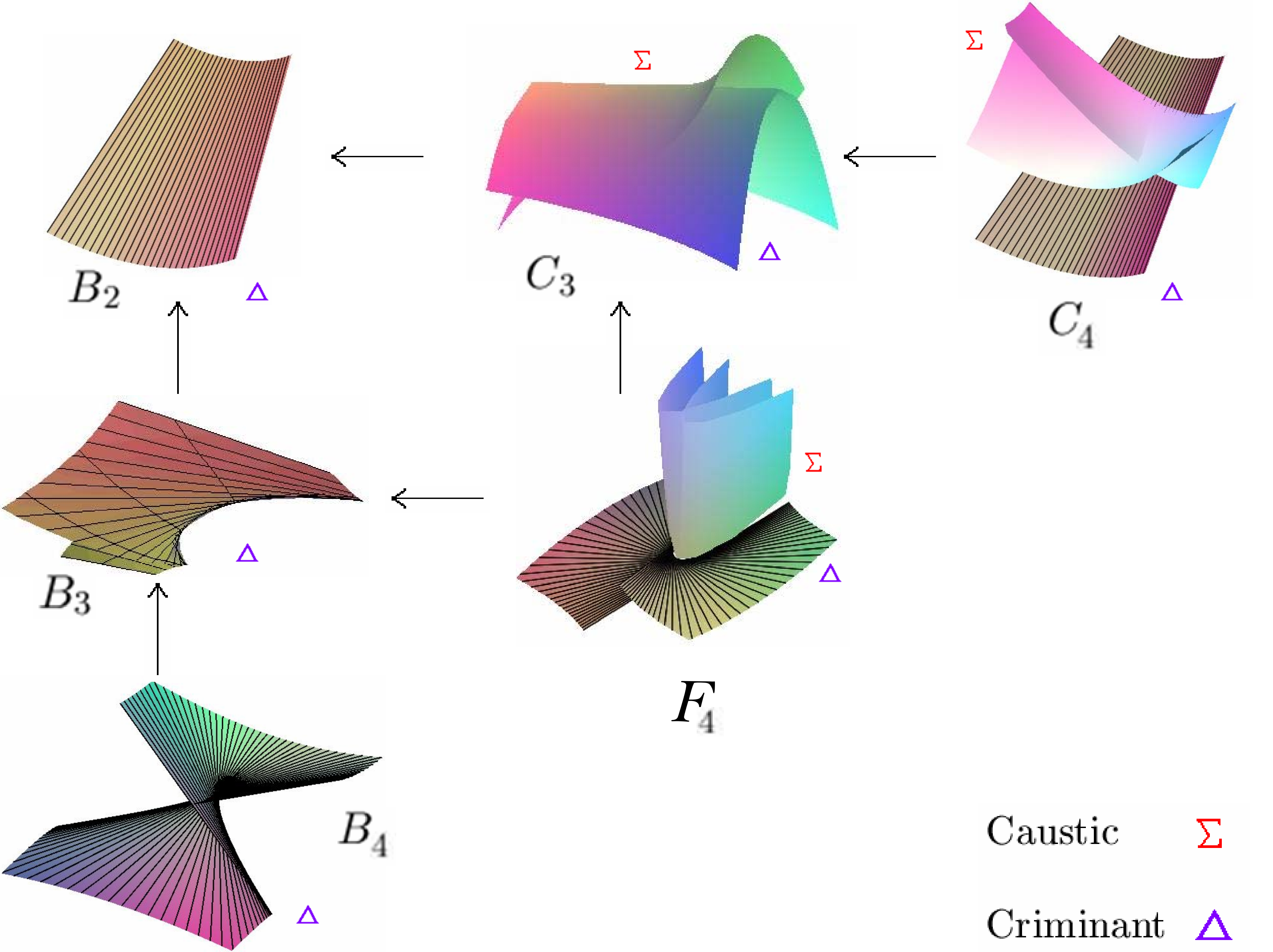
B_2 $\triangle$



B_2



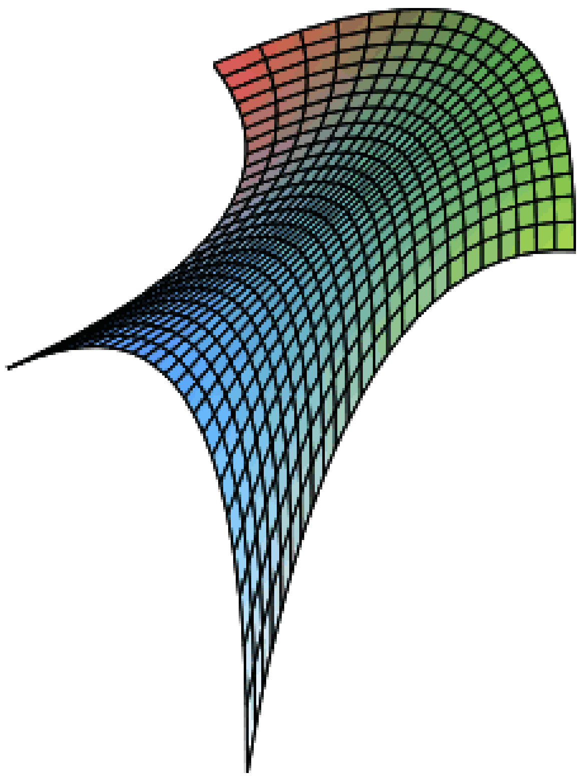

Σ
B_2
△
C_3
△



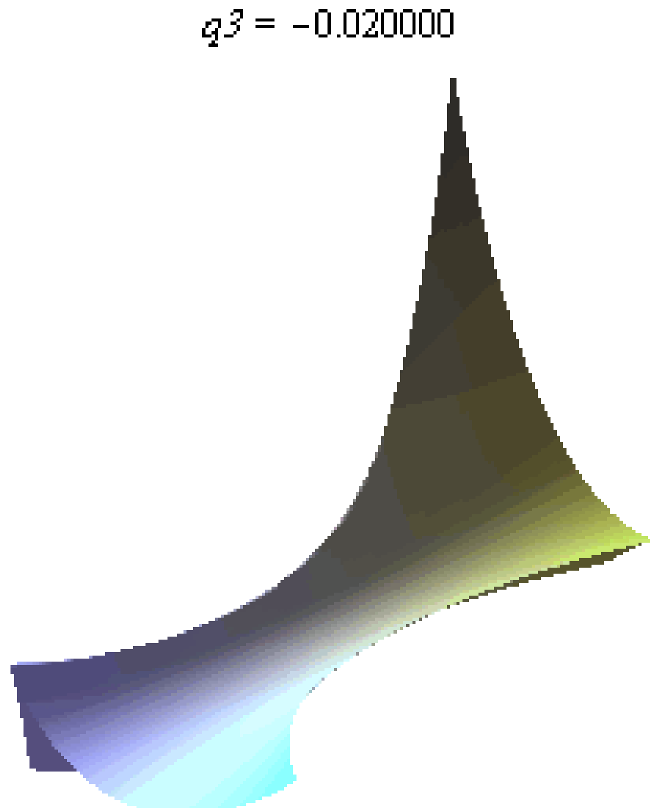
B_2
C_3
C_4
B_3
F_4
B_4
Σ
$\triangle$
Caustic Σ
Criminant $\triangle$

$K^{-}_{4,2}$ Criminant

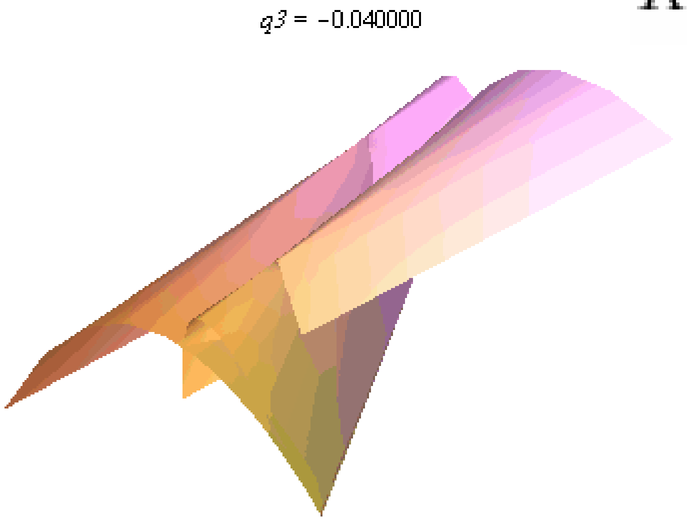
$q3 = -0.040000$



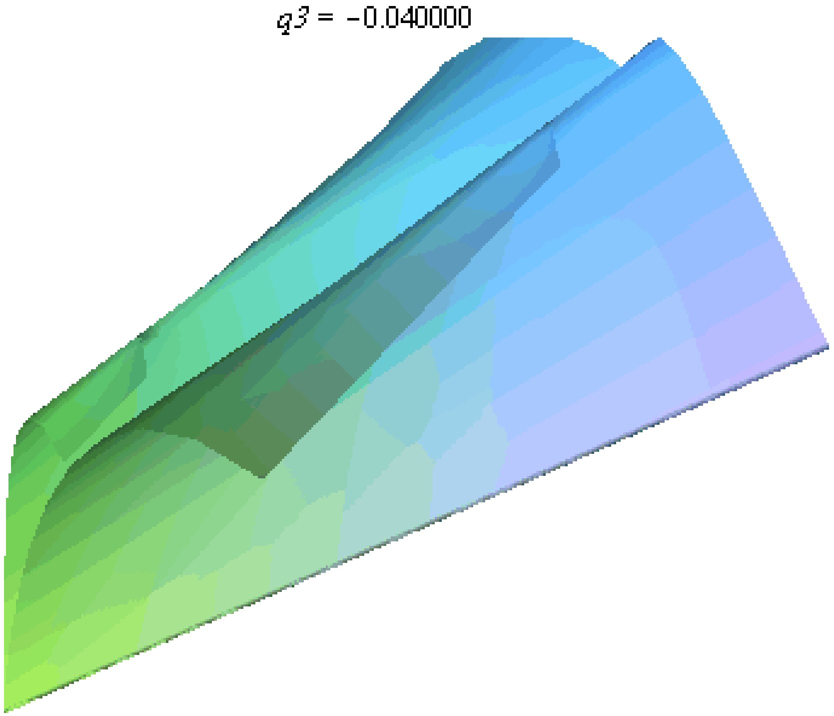
$K^{+}_{4,2}$ Criminant

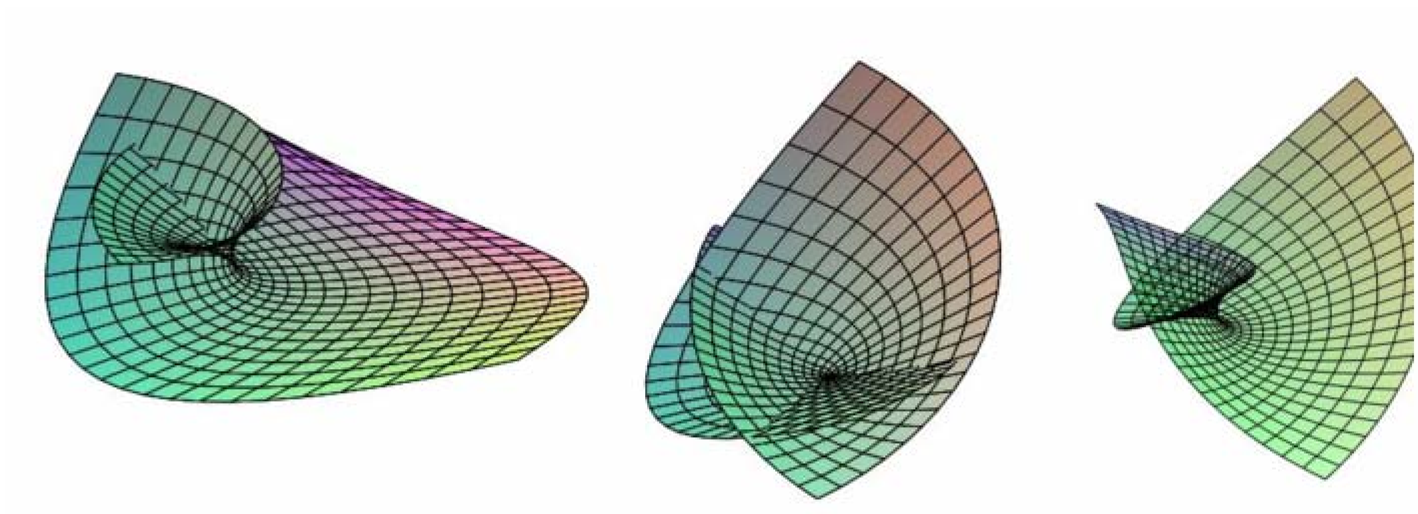


$K^{-}_{4,2}$ Caustic

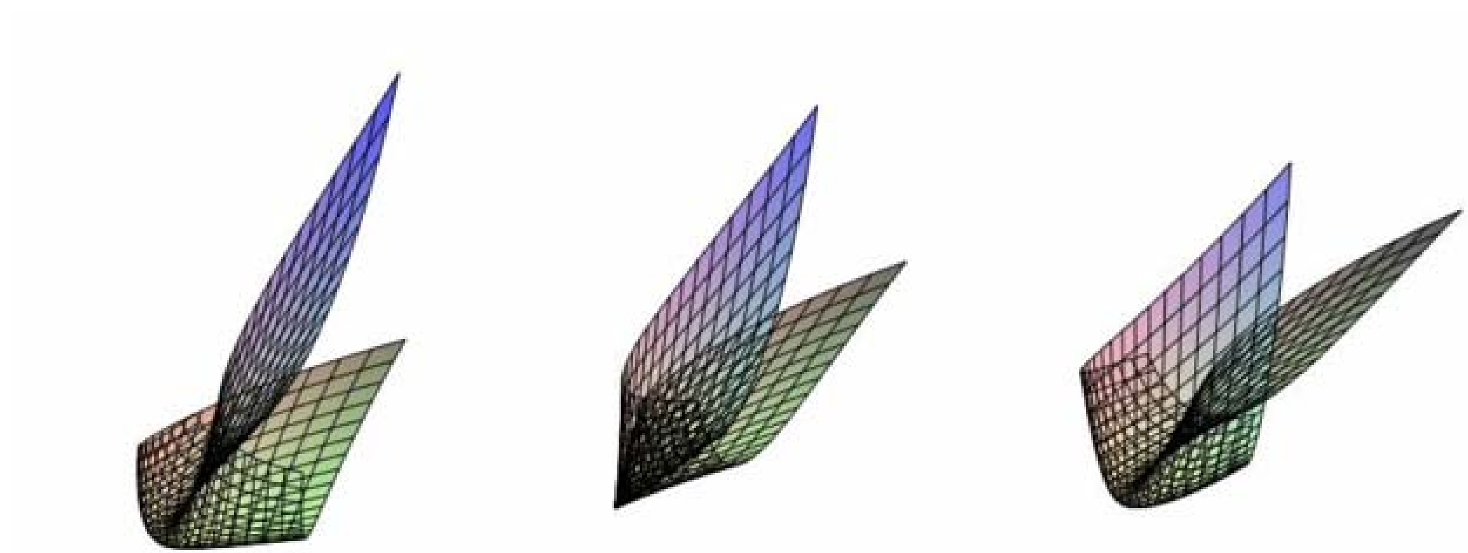


$K^{+}_{4,2}$ Caustic





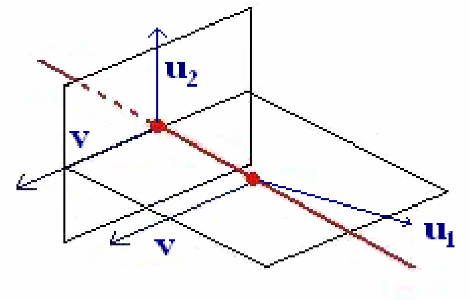
$F^{-}_{1,0}$ Criminant



$F^{+}_{1,0}$ Criminant

$F^{+}_{1,0}$ and $F^{-}_{1,0}$ Caustic

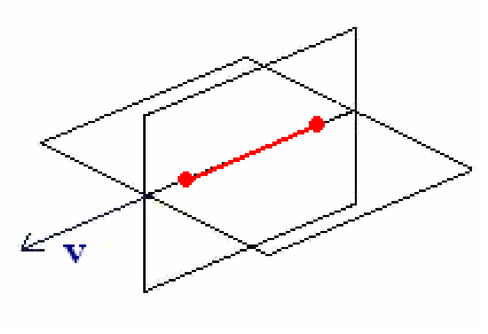
Tangential



In this case the method is very similar

One of the B_3 singularities always occurs at the point whose tangent space contains the other point.

Bitangential



In this case the method is very similar.

In this case a B_3 occurs on both surface points.

$$
\begin{tikzcd}
B_2 = C_2 & B_3 \arrow[l] & B_4 \arrow[l] & B_5 \arrow[l] \\
C_3 \arrow[u] & F_4 \arrow[l] \arrow[u] & F_{1,0} \arrow[l] & \\
C_4 \arrow[u] & K_{4,2} \arrow[l] \arrow[u] & & \\
C_5 \arrow[u] & & &
\end{tikzcd}
$$

Non-tranvsversal

$$
\begin{tikzcd}
\underline{B_2 = C_2} & \underline{B_3} \arrow[l] & \underline{B_4} \arrow[l] & B_5 \arrow[l] \\
\underline{C_3} \arrow[u] & \underline{F_4} \arrow[l] \arrow[u] & F_{1,0} \arrow[l] & \\
\underline{C_4} \arrow[u] & K_{4,2} \arrow[l] \arrow[u] & & \\
C_5 \arrow[u] & & &
\end{tikzcd}
$$

Non-tranvsversal

Tangential

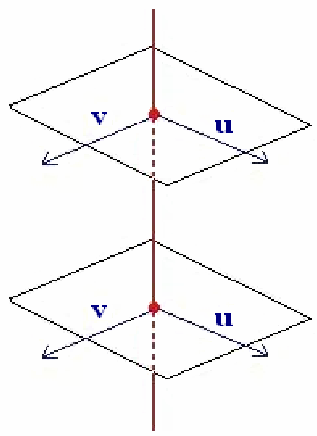
$$
\begin{tikzcd}
B_2 = C_2 & B_3 \arrow[l] & B_4 \arrow[l] & B_5 \arrow[l] \\
C_3 \arrow[u] & F_4 \arrow[l] \arrow[u] & F_{1,0} \arrow[l] & \\
C_4 \arrow[u] & K_{4,2} \arrow[l] \arrow[u] & & \\
C_5 \arrow[u] & & &
\end{tikzcd}
$$

Non-tranvsversal

Tangential

Bitangential

Parallel Case



After stabilisation the generating family is of the form:

$$\mathcal{F} = nA(s,t,\lambda,q) + B(s,t,\lambda,q)$$

for some functions A and B.

Now consider the following mapping (complete intersection) for linear generating families:

$$G_{\mathcal{F}} : \mathbb{R}^2 \times \Pi \to \mathbb{R}^2, \qquad G_{\mathcal{F}} : (s,t,\lambda,q) \mapsto \begin{pmatrix} A \\ B \end{pmatrix}$$

where Π is the parameter space $(\lambda, q) \in \mathbb{R} \times \mathbb{R}^4$.

Lemma

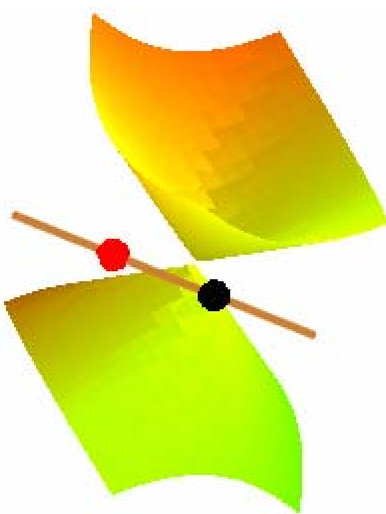
The extended wavefront of $\mathcal{F}$ in Π coincides with the bifurcation diagram of the contact class of the map $G_{\mathcal{F}}$.

Codimension 0 classes

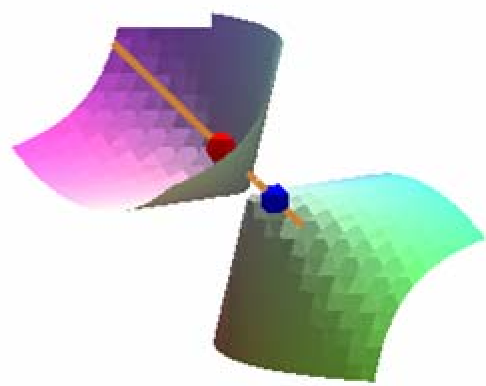
In the codimension 0 case the quadratic form $J^2(A(q))$ is a non-degenerate quadratic form and is independent of $B|_{\varepsilon=0}$.

Lemma 10.2 *In the codimension 0 case the normal form of the mapping $G_{\mathcal{F}}$ is*

$$\begin{cases} A_* = \; s^2 \pm t^2 + q_1 s + q_2 t + q_3 \\ B_* = st - \varepsilon \end{cases}$$



or



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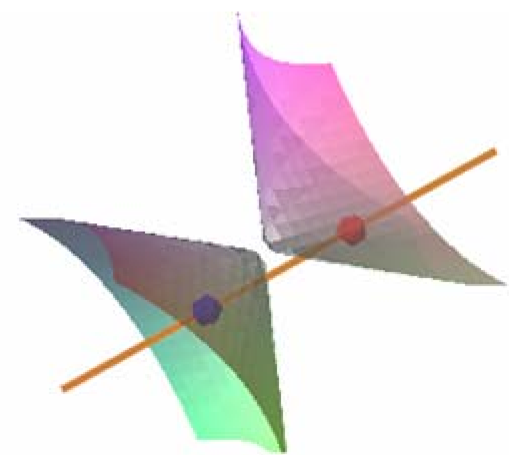
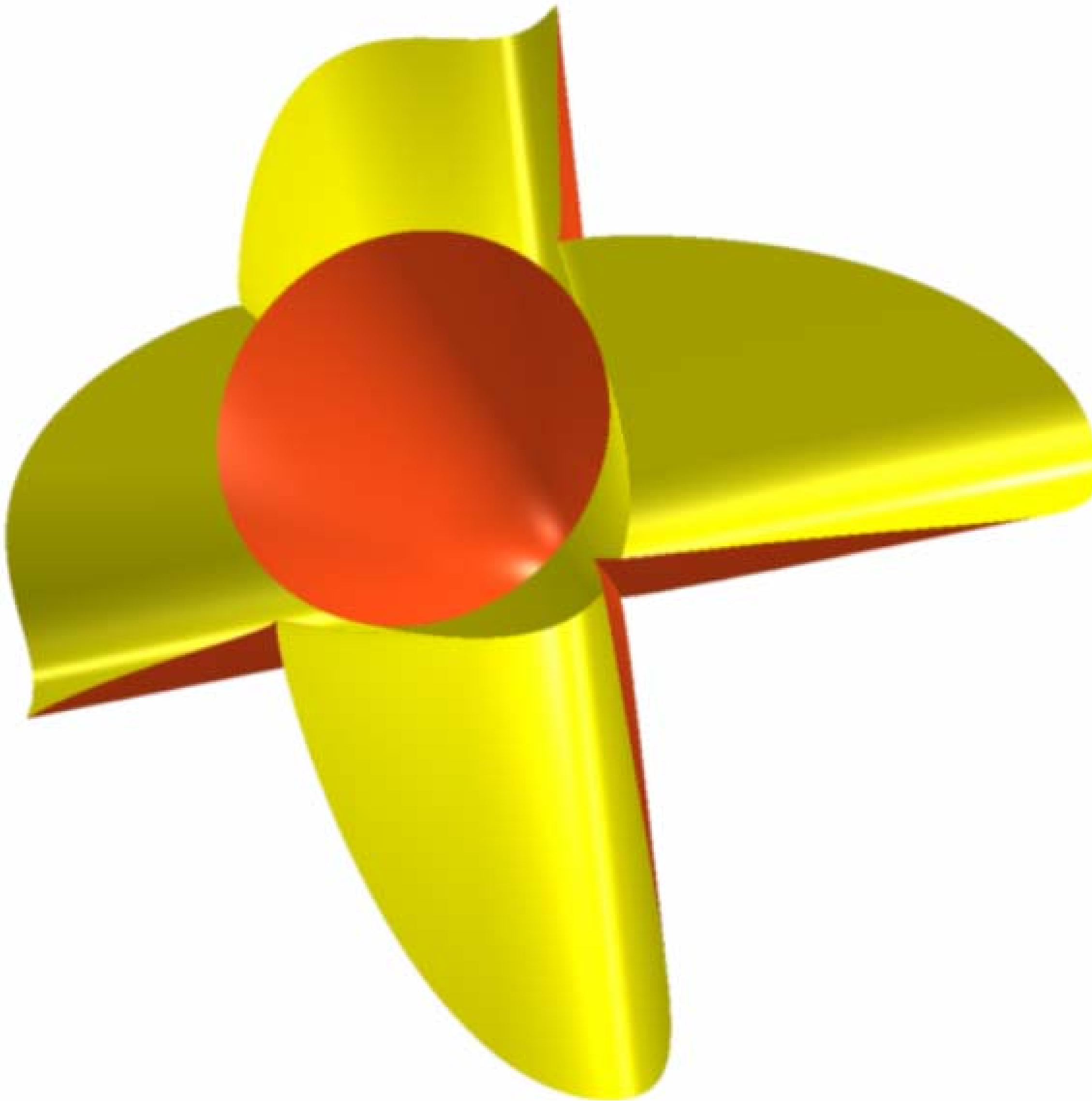
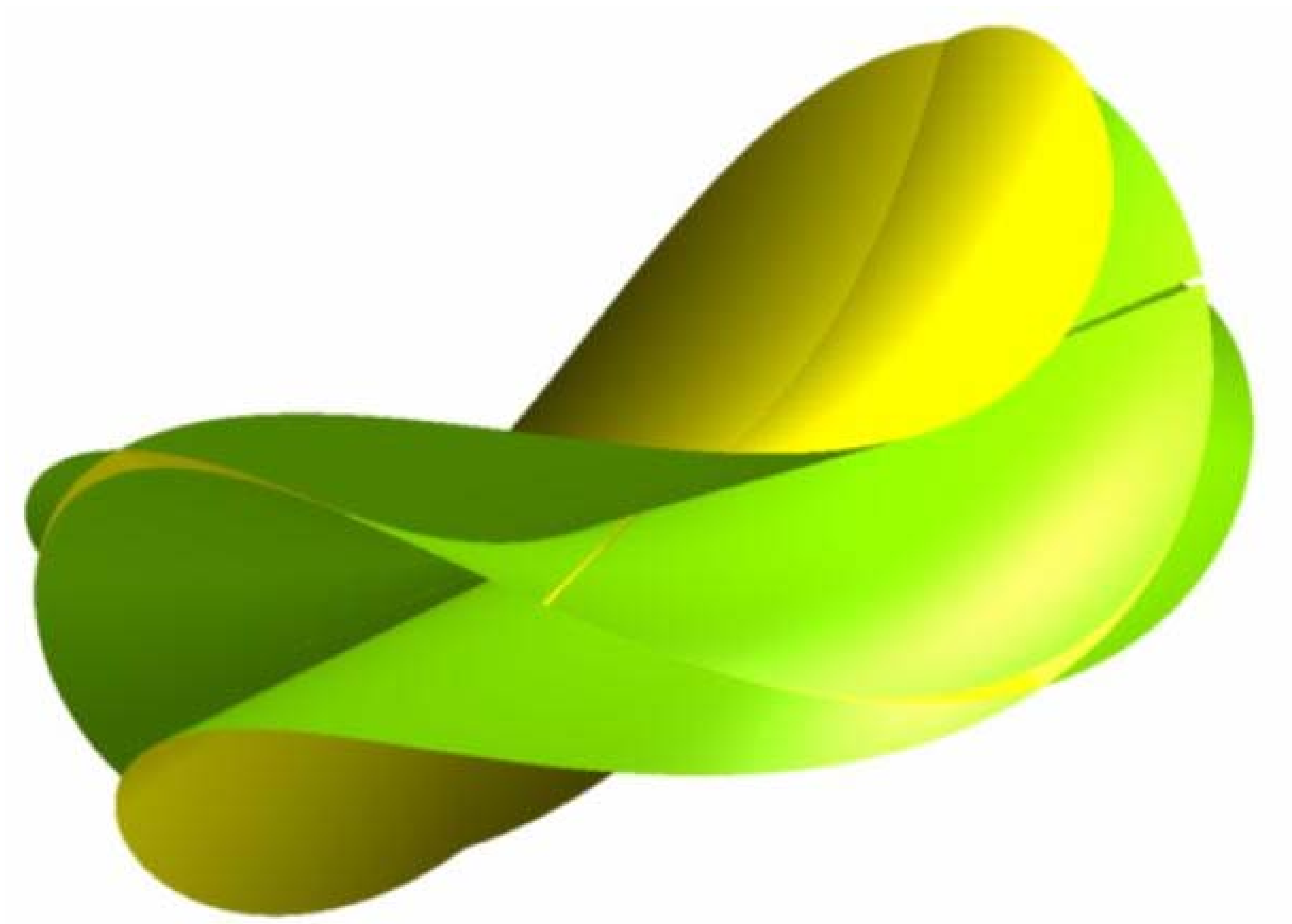




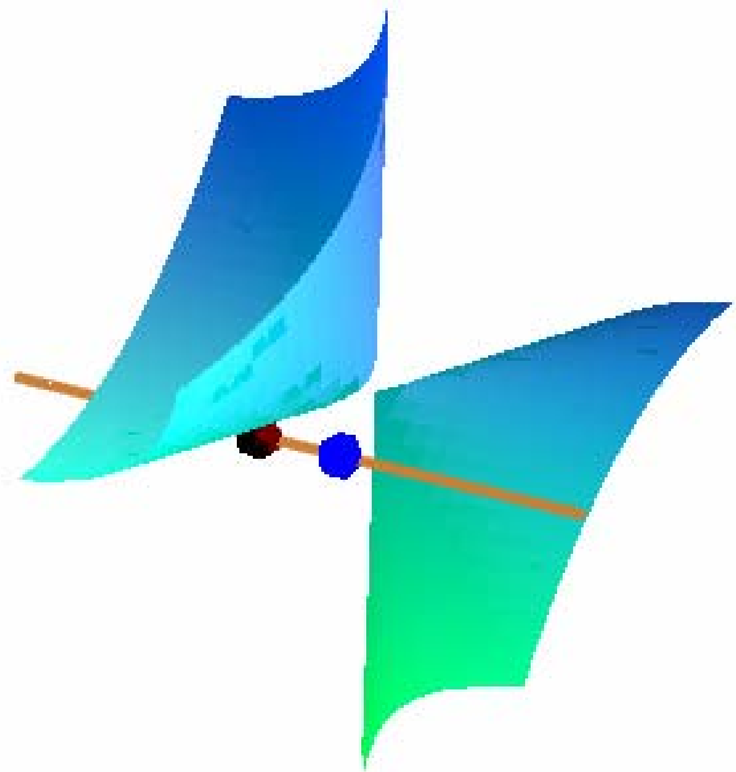
$$\begin{cases} A_* = \ s^2 + t^2 + q_1 s + q_2 t + q_3 \\ B_* = st - \varepsilon \end{cases}$$

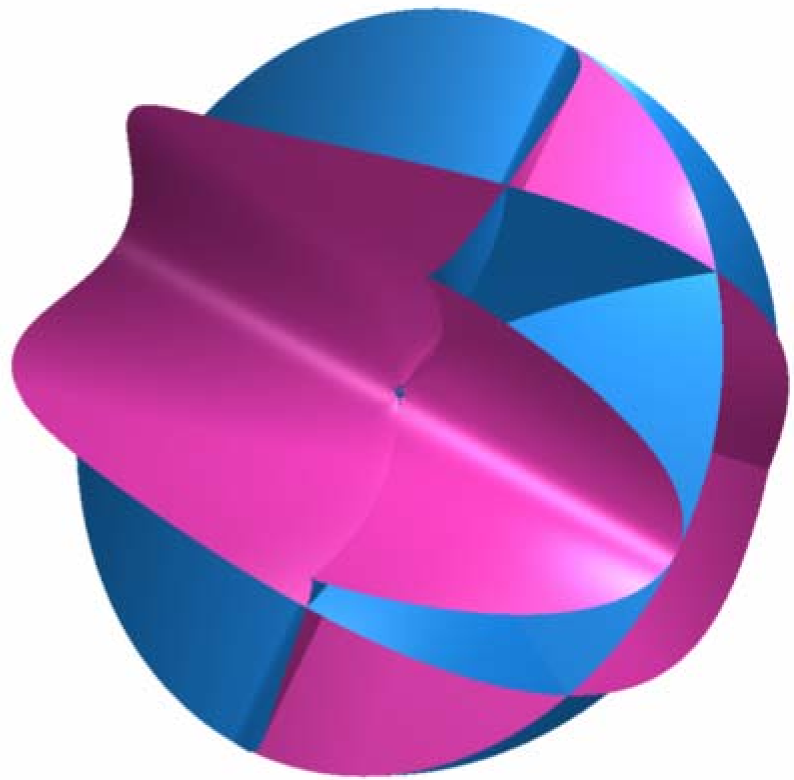
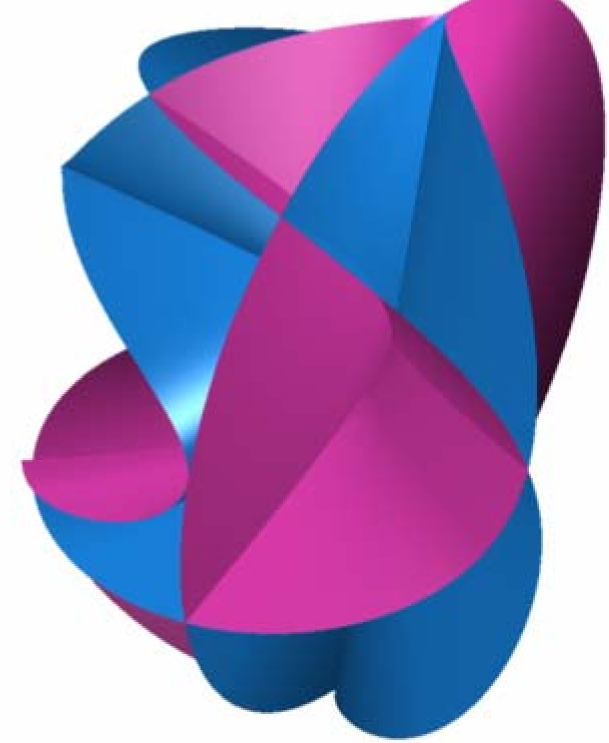


$$\begin{cases} A_* = & s^2 - t^2 + q_1 s + q_2 t + q_3 \\ B_* = & st - \varepsilon \end{cases}$$

Lemma 4.5 *For some special values of λ the pencil of quadratic forms can be tangent to the parabolic cone the normal form is given by:*

$$\begin{cases} A_* = st - s^2 + t^3 + q_4 t^2 + q_1 s + q_2 t + q_3 \\ B_* = st - \varepsilon \end{cases}$$



$$\begin{cases} A_* = st - s^2 + t^3 + q_4 t^2 + q_1 s + q_2 t + q_3 \\ B_* = st - \varepsilon \end{cases}$$

Thanks for listening