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S-topology on the Skorokhod space Recent developments and complements

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- Skorokhod (1956) introduced 4 separable metric topologies on $\mathbb{D}$, with J_1 and M_1 the most important and finding many applications. In particular $(\mathbb{D}, J_1)$ is a Polish space.
- It must be emphasized that Skorokhod's topologies are non-linear, hence practically useless in handling e.g. minimization problems. Contrary to the case of $(\mathbb{C}, \|\cdot\|_\infty)$, there is no analysis on $(\mathbb{D}, J_1)$!

Example: Convergence of linear processes with heavy-tailed innovations

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Let

$$\alpha \in (0,2)$$

and $\{\xi_j\}_{j\in\mathbb{Z}}$ be an i.i.d sequence with marginal distribution satisfying

$$P(|\xi_j| > x) = x^{-\alpha} h(x), \; x > 0,$$

$$\lim_{x\to\infty} \frac{P(Y > x)}{P(|Y| > x)} = p \quad \text{and} \quad \lim_{x\to\infty} \frac{P(Y < -x)}{P(|Y| > x)} = q$$

for some $p \geqslant 0$ and $q \geqslant 0$ with $p + q = 1$.

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This means that under additional assumptions ($E\xi_j = 0$ if $\alpha \in (1,2)$ and if ξ_j are symmetric if $\alpha = 1$) there exists a sequence $B_n \to \infty$ such that

$$\frac{\xi_1 + \xi_2 + \ldots + \xi_n}{B_n} \underset{\mathcal{D}}{\longrightarrow} \mu,$$

where μ is a non-degenerate strictly α-stable distribution.

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Under some assumptions on $\{c_k\}$ one can prove that for each $t > 0$

$$\widetilde{X}_n(t) = \frac{1}{B_n} \sum_{i=1}^{[nt]} X_i \underset{\mathcal{D}}{\longrightarrow} AZ_t$$

where $A = \sum_{j\in\mathbb{Z}} c_j$ and $\{Z_t\}$ is a stable Lévy motion such that $Z_1 \sim \mu$.

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where $A = \sum_{j\in\mathbb{Z}} c_j$ and $\{Z_t\}$ is a stable Lévy motion such that $Z_1 \sim \mu$. In fact we have the finite dimensional convergence of X_n to Z.

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Question: is there any functional topology on $\mathbb{D}$ such that X_n's are convergent to Z in this topology without additional assumptions on the nature of $\{c_k\}$?

Definition of the S-topology

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- Let $\mathbb{V} \subset \mathbb{D}$ be the space of (regularized) functions of finite variation on $[0,1]$, equipped with the norm of total variation $\|v\| = \|v\|(1)$, where

$$\|v\|(t) = \sup \Big\{ |v(0)| + \sum_{i=1}^{m} |v(t_i) - v(t_{i-1})| \Big\},$$

and the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_m = t$.

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- Since $\mathbb{V}$ can be identified with a dual of $(\mathbb{C}, \|\cdot\|_\infty)$, we have on it the weak-$*$ topology. We shall write $v_n \Rightarrow v_0$ if for every $f \in \mathbb{C}([0, 1] : \mathbb{R}^1)$

$$\int_{[0,1]} f(t)dv_n(t) \to \int_{[0,1]} f(t)dv_0(t).$$

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We shall write $x_n \to_S x_0$ if for every $\varepsilon > 0$ one can find elements $v_{n,\varepsilon} \in \mathbb{V}$, $n = 0, 1, 2, \ldots$ which are ε-uniformly close to x_n's and weakly-$*$ convergent:

$$\|x_n - v_{n,\varepsilon}\|_\infty \leqslant \varepsilon, \qquad n = 0, 1, 2, \ldots, \tag{1}$$

$$v_{n,\varepsilon} \Rightarrow v_{0,\varepsilon}, \qquad \text{as } n \to \infty. \tag{2}$$

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This definition as well as many properties of S-topology were given in:
A. J., A non-Skorohod topology on the Skorohod space, Electron. J. Probab. **2 (1997)**, paper no 4, 1–21.

Criteria of compactness

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Equivalent criteria of compactness

Let $K \subset \mathbb{D}$ and suppose that

$$\sup_{x \in K} \|x\|_\infty < +\infty. \tag{3}$$

Then the statements **(i)** and **(ii)** below are equivalent:
(i) For each $a < b$

$$\sup_{x \in K} N^{a,b}(x) < +\infty. \tag{4}$$

(ii) For each $\eta > 0$

$$\sup_{x \in K} N_\eta(x) < +\infty. \tag{5}$$

Criteria of compactness

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Equivalent criteria of compactness - continued

Moreover, either set of conditions (3)+(4) and (3)+(5) is equivalent to
(iii) For each $\varepsilon > 0$ and for each $x \in K$ there exists $v_{x,\varepsilon} \in \mathbb{V}$ such that

$$\sup_{x \in K} \|x - v_{x,\varepsilon}\|_\infty \leqslant \varepsilon, \tag{6}$$

and

$$\sup_{x \in K} \|v_{x,\varepsilon}\|(1) < +\infty. \tag{7}$$

(iv) K is relatively compact with respect to S-topology

S-topology and pointwise convergence

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Corollary

If $x_n \to_S x_0$, then in each subsequence $\{x_{n_k}\}$ one can find a further subsequence $\{x_{n_{k_l}}\}$ and a countable set $D \subset [0,1)$ such that

$$x_{n_{k_l}}(t) \to x_0(t), \quad t \in [0,1] \setminus D.$$

This gives the lower semicontinuity of many useful functionals on $\mathbb{D}$.

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Remark

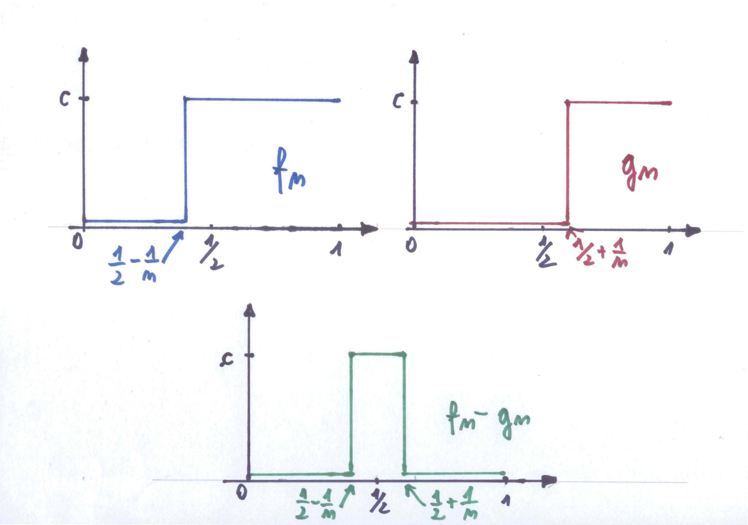
If $t \in [0, 1)$, then the evaluation at point t

$$\mathbb{D} \ni x \mapsto \pi_t(x) = x(t)$$

is nowhere S-continuous.

A typical phenomenon for S-converegnce

A typical phenomenon for *S*-converegnce



A summary on the S-topology

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This is a weak topology on $\mathbb{D}$, which is non-Skorokhod, sequential, not metrisable, but being submetric it is still good enough to build a satisfactory theory of the convergence in distribution.

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In particular:

- The σ-field $\mathcal{B}_S$ of Borel subsets for S coincides with the usual σ-field generated by projections (or evaluations) on $\mathbb{D}$: $\mathcal{B}_S = \sigma(\pi_t : t \in [0, 1])$.

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- S is a submetric topology, for there exists a countable family of S-continuous functions which separate points in $\mathbb{D}$. In particular, compact subsets of $(\mathbb{D}, S)$ are metrisable.
- S is weaker than Skorohod's J_1-topology (and even than M_1-topology).

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- $K \subset \mathbb{D}$ is S-relatively compact iff

$$\sup_{x \in K} \sup_{t \in [0,1]} |x(t)| \leqslant C_K < +\infty,$$

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- A family $\{X_\alpha\}$ of stochastic processes with trajectories in $\mathbb{D}$ is uniformly S-tight iff $\{\|X_\alpha\|_\infty\}$ is a uniformly tight family as well as for each $\eta > 0$ $\{N_\eta(X_\alpha)\}$ is uniformly tight.

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- How to deal with convergence in distribution with respect to the S-topology?

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- How to deal with convergence in distribution with respect to the S-topology?

Theorem, A.J. '97

Suppose $\{X_n\}$ is uniformly S-tight. Then in each subsequence $\{X_{n_k}\}_{k \in \mathbb{N}}$ one can find a further subsequence $\{X_{n_{k_l}}\}_{l \in \mathbb{N}}$ admitting the usual a.s. Skorohod representation on $([0,1], \mathcal{B}_{[0,1]})$!

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- verifying the criteria of compactness,
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- by passing to the limit deducing the existence of a stochastic process with desired properties and trajectories in $\mathbb{D}$.

A standard procedure when using any weak topology!

S-topology in existence problems

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Selected recent publications including such type of reasoning:

- K. Bahlali, A. Elouaflin, E. Pardoux, Homogenization of semilinear PDEs with discontinuous averaged coefficients, Electron. J. Probab., **14 (2009)**, 477-–499.
- R. Rhodes, Stochastic Homogenization of Reflected Stochastic Differential Equations, Electron. J. Probab., **15 (2010)**, 989–1023.
- É. Pardoux, A.B. Sow, Homogenization of a periodic degenerate semilinear elliptic PDE, Stoch. Dyn. **11 (2011)**, 475–493.
- H-W. Kang, T. Kurtz, Separation of time-scales and model reduction for stochastic reaction networks, to appear in: Ann. Appl. Probab. (2012+)

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S-topology was used in a different manner.

In this paper the predictable compensator in the Doob-Meyer decomposition is obtained as a limit of

$$\widetilde{A}_t^N = \frac{1}{N}\sum_{n=1}^{N} A_t^n.$$

which cannot be, in general, convergent in any of Skorokhod's topologies.

Other types of applications

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Another crucial step consists in obtaining the convergence of integrals

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where C_t is the predictable compensator of the process $I(\tau \leqslant t)$, for some totally inaccessible stopping time.

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Notice that C_t is continuous, but not necessarily absolutely continuous.

Hence almost sure convergence with respect to the Lebesgue measure is useless

Genesis of S-topology - Meyer-Zheng topology

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- This topology is not suitable for the "functional convergence" of stochastic processes, for it is easy to find a sequence $\{x^n\} \subset \mathbb{D}$ which converges in measure to $x^0 \equiv 0$ and is such that $x^n(q) \to 1$ for each rational $q \in [0, 1]$.

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- However, as a weak topology it can be and it is used in existence problems, due to easy-to-check compactness criteria.

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Meyer-Zheng topology and S-topology

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- First they imply criteria of compactness for S-topology (but are much stronger).
- Second, it can be shown that S-relative compactness implies convergence in $L^p(\mu)$ for each $p \in (0, \infty)$ and each atomless measure μ on $[0, 1]$
- The problem what the criteria of compactness considered by Meyer and Zheng really imply was investigated by T. Kurtz (1991).

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Theorem

There exists a locally convex linear topology $\tilde{S}$ on $\mathbb{D}$ such that the sequential topology generated by $\tilde{S}$ coincides with S.

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Theorem

There exists a locally convex linear topology $\tilde{S}$ on $\mathbb{D}$ such that the sequential topology generated by $\tilde{S}$ coincides with S.

Conjecture

The topology $\tilde{S}$ is the finest among all locally convex topologies on $\mathbb{D}$, which are coarser than the J_1 topology.

The idea of the proof

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We define the topology $\tilde{S}$ by a convex basis consisting of all sets of the form

$$\alpha U + \bigcup_{n=1}^{\infty} V_1^* \cap V + V_2^* \cap 2V + \ldots + V_n^* \cap nV,$$

where

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where

- $U = \{x \in \mathbb{D}\,;\, \|x\|_\infty \leqslant 1\} \subset \mathbb{D}$.

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We define the topology $\tilde{S}$ by a convex basis consisting of all sets of the form

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The idea goes back to Wiweger (1959), but it is worth to emphasize that among dozens of topologies considered in sixties and seventieth there is no scheme corresponding to our space $(\mathbb{D}, \tilde{S})$)