Harnack inequalities for subordinate Brownian motions

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joint work with Panki Kim

$S = (S_t)_{t \geq 0}$
subordinator
(i.e. an increasing
Lévy process in $\mathbb{R}$)

$B = (B_t)_{t \geq 0}$
Brownian motion in $\mathbb{R}^d$
independent of S

$X_t = B_{S_t}$
subordinate
Brownian motion

Harmonic function

$u\colon \mathbb{R}^d \to [0, \infty)$ is harmonic in $D \subset \mathbb{R}^d$ open and bounded (w.r.t X) if for any open $B \subset \overline{B} \subset D$

$$u(x) = \mathbb{E}_x[u(X_{\tau_B})] \ \ \forall \ x \in B\,.$$

$\tau_B = \inf\{t > 0\colon X_t \notin B\}$

Harnack Inequality 1

Harnack inequality

Harnack inequality holds for X if there is a constant $C > 0$ such that for any $r \in (0,1)$ and any $u\colon \mathbb{R}^d \to [0,\infty)$ which is harmonic in $B(0,r)$

$$u(x) \leq C\, u(y) \quad \forall\, x, y \in B(0, \tfrac{r}{2})\,.$$

Some applications:

- boundary Harnack principle
- Green function estimates ($\to$ talk of P. Kim)
- regularity estimates of harmonic functions

Known cases of SBM when HI holds:

- rotationally invariant α-stable processes $(0 < \alpha < 2)$:

$$\mathbb{E}[e^{i\xi\cdot X_t}] = e^{-t|\xi|^\alpha}$$

- or, more generally, when $0 < \alpha < 2$ and

$$\mathbb{E}[e^{i\xi\cdot X_t}] = e^{-t\phi(|\xi|^2)}$$

  with

$$\lim_{\lambda\to\infty} \frac{\phi(\lambda x)}{\phi(\lambda)} = x^{\alpha/2} \qquad \forall\ x > 0$$

Our motivation:

- geometric β-stable process $(0 < \beta \le 2)$:

$$\mathbb{E}[e^{i\xi\cdot X_t}] = e^{-t\log(1+|\xi|^\beta)}$$

$'\alpha = 0'$

  - $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$
  - weaker form of HI was known before (Šikić, Song, Vondraček, PTRF '06):

$$u(x) \le C(r)\, u(y) \quad \forall\ x, y \in B(0, \tfrac{r}{2})$$

    with $\lim_{r\to 0+} C(r) = \infty$

$S = (S_t)_{t\geq 0}$ subordinator

Laplace transform

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \lambda > 0$$

Laplace exponent

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt)$$

- $b \geq 0$
- μ Lévy measure

$$\int_{(0,\infty)} (1 \wedge t)\mu(dt) < \infty$$

Potential measure

$$U(A) = \mathbb{E}\left[\int_0^\infty 1_{\{S_t \in A\}}\, dt\right]$$

$B = (B_t, \mathbb{P}_x)$ Brownian motion in $\mathbb{R}^d \perp\!\!\!\perp S$

Subordinate Brownian motion $X = (X_t, \mathbb{P}_x)$

$$X_t := B_{S_t}, \; t \geq 0$$

Properties of SBM

- X is a Lévy process
- Characteristic exponent

$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t-x)}\right]=e^{-t\Phi(\xi)} \qquad \Phi(\xi)=\int\limits_{\mathbb{R}^d}\left(1-e^{i\xi\cdot x}+i\xi\cdot x1_{\{|x|<1\}}\right)\Pi(dx)$$

Lévy measure Π is of the form

$$\Pi(dx)=j(|x|)\,dx \qquad j(r)=(4\pi)^{-d/2}\int\limits_0^\infty t^{-d/2}e^{-\frac{r^2}{4t}}\mu(dt)$$

- If X is transient, the Green function is given by

$$G(x,y)=(4\pi)^{-d/2}\int\limits_0^\infty t^{-d/2}e^{-\frac{|x-y|^2}{4t}}U(dt), \quad x,y\in\mathbb{R}^d,\, x\neq y$$

- The Green function of an open set $D\subset\mathbb{R}^d$ is given by

$$G_D(x,y)=G(x,y)-\mathbb{E}_x[G(X_{\tau_D},y);\tau_D<\infty], \quad x,y\in D,\, x\neq y$$

Note: $G_D(x,y)\leq G(x,y)$

Assumptions

(A-1) $\mu(0,\infty) = \infty$ and

$$\mu(dt) = \mu(t)\,dt\,, \qquad \mu\colon (0,\infty) \to (0,\infty) \quad \text{decreasing}$$

(A-2) $U(dt) = u(t)\,dt, \quad u\colon (0,\infty) \to (0,\infty)$ decreasing

(A-3) there exist $\sigma > 0$ and $\alpha \in [0,2)$ such that

$$\frac{\phi'(\lambda x)}{\phi'(\lambda)} \leq \sigma x^{\frac{\alpha}{2}-1} \qquad \forall \ \ \lambda, x \geq 1$$

Remarks.

1. (A-3) holds, in particular, when ϕ varies regularly at infinity with index $\alpha/2$ with $0 < \alpha < 2$ (by the Karamata monotone density theorem), i.e.

$$\lim_{\lambda \to \infty} \frac{\phi(\lambda x)}{\phi(\lambda)} = x^{\frac{\alpha}{2}} \qquad \forall\ x > 0$$

2. (A-3) $\implies$

$$b = 0 \qquad \text{and} \qquad \frac{\phi(\lambda x)}{\phi(\lambda)} \leq \sigma' x^{\frac{\alpha}{2}} \qquad \forall \ \ \lambda, x \geq 1$$

Main result

Theorem (Harnack Inequality; Kim - M, EJP '12)

Let S be a subordinator satisfying (A-1), (A-2) and (A-3). Assume that the Lévy density $J(x) = j(|x|)$ of the corresponding subordinate Brownian motion X satisfies

$$j(r+1) \le j(r) \le c\, j(r+1) \quad \forall\ r > 1\,. \qquad (\star)$$

for some constant $c \ge 1$. Then the Harnack inequality holds for X, i.e. there exists a constant $C > 0$ such that

- *for any $r \in (0,1)$*
- *for any non-negative function u which is harmonic in $B(0,r)$*

$$u(x) \le C\, u(y) \qquad \forall\ x, y \in B(0, \tfrac{r}{2})\,.$$

Remark. New result for '$\alpha = 0$', e. g.

- geometric stable

$$\phi(\lambda) = \log(1 + \lambda^{\beta/2}),\ 0 < \beta \le 2$$

- iterated geometric stable

$$\phi(\lambda) = \log(1 + \log(1 + \lambda^{\beta/2})^{\beta/2}),\ 0 < \beta \le 2$$

Key Estimates 1

$$\Pi(dx) = j(|x|)\,dx \qquad j(r) = (4\pi)^{-d/2}\int_0^\infty t^{-d/2}e^{-\frac{r^2}{4t}}\mu(t)\,dt$$

$$G(x,y) = g(|x-y|) \qquad g(r) = (4\pi)^{-d/2}\int_0^\infty t^{-d/2}e^{-\frac{r^2}{4t}}u(t)\,dt$$

Proposition (Lévy and potential density, Green function; Kim - M, EJP '12)

$$\mu(t) \asymp t^{-2}\phi'(t^{-1}),\ t\to 0+ \qquad j(r) \asymp r^{-d-2}\phi'(r^{-2}),\ r\to 0+$$

$$u(t) \asymp t^{-2}\frac{\phi'(t^{-1})}{\phi(t^{-1})^2},\ t\to 0+ \qquad g(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2},\ r\to 0+$$

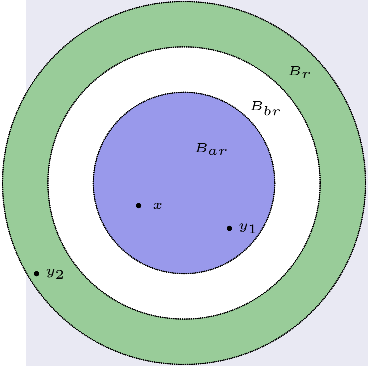
Example (Geometric stable process $\phi(\lambda) = \log(1+\lambda^{\beta/2})$ $(0<\beta\le 2)$)

$$\mu(t) \asymp \frac{1}{t} \qquad j(r) \asymp \frac{1}{r^d}$$

$$u(t) \asymp \frac{1}{t\left(\log\frac{1}{t}\right)^2} \qquad G(x,y) \asymp \frac{1}{|x-y|^d\left(\log\frac{1}{|x-y|}\right)^2}$$

Set $B_r := B(0,r)$.

Proposition (Green function of the ball; Kim-M, EJP '12)

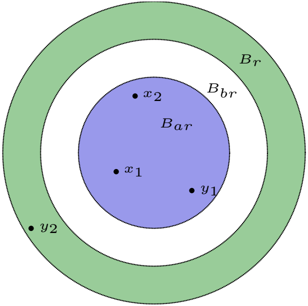


There exist $0 < a < b < 1$ *such that*

$$G_{B_r}(x,y_1) \geq c\,|x-y_1|^{-d-2}\frac{\phi'(|x-y_1|^{-2})}{\phi(|x-y_1|^{-2})^2}$$

$$G_{B_r}(x,y_2) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})}\mathbb{E}_{y_2}\tau_{B_r}$$

Let u be a nonnegative function that is harmonic in B_{2r} and $x_1, x_2 \in B_{ar}$.



$$
\begin{aligned}
u(x) &= \mathbb{E}_x[u(X_{\tau_{B_r}})] \\
&= \int\limits_{\overline{B_r}^c} \int\limits_{B_r} G_{B_r}(x,y) j(|z-y|) u(z)\, dy\, dz \\
&= \int\limits_{\overline{B_r}^c} \int\limits_{B_{br}} + \int\limits_{\overline{B_r}^c} \int\limits_{B_r \setminus B_{br}} =: u_1(x) + u_2(x)
\end{aligned}
$$

- Green function estimate in $B_r \setminus B_{br} \implies$

$$
u_2(x_1) \le c_1 \frac{r^{-d-2}\phi'(r^{-2})}{\phi(r^{-2})} \int\limits_{\overline{B_r}^c} \int\limits_{B_r \setminus B_{br}} E_{y_2}[\tau_{B_r}] j(|z-y_2|) u(z)\, dy_2\, dz \le c_2\, u_2(x_2)
$$

- $(\star)$ + fact that $G_{B_r} \le G$ + Green function lower estimate in $B_{ar} \implies$

$$
u_1(x_1) \le c_3 \cdot \underbrace{\int\limits_{B_{br}} G_{B_r}(x_1, y_1)\, dy_1}_{\le \frac{c_4}{\phi(r^{-2})} \le c_5 \int_{B_{ar}} G_{B_r}(x_2, y_1)\, dy_1} \cdot \int\limits_{\overline{B_r}^c} j(|z|) u(z)\, dz \le c_6\, u_1(x_2)
$$

Thank you for your attention !