Lower bounds for traces of heat kernels

(joint work with Richard Laugesen)

Bartłomiej Siudeja

University of Oregon

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Eigenvalues of the generator of a killed process

Solutions of $-\Delta u = \lambda u$ on domain D satisfy

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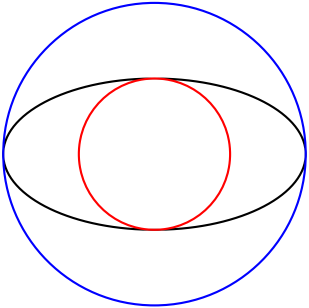
Let D^* be a ball with the same area as D.

- Isoperimetric inequality: $|\partial D| \geq |\partial D^*|$ (trace with $t \to 0$)
- Faber-Krahn inequality: $\lambda_1(D) \geq \lambda_1(D^*)$ (trace with $t \to \infty$)

Bounds for traces (Brownian motion in 2:1 ellipse E)



Easy bounds

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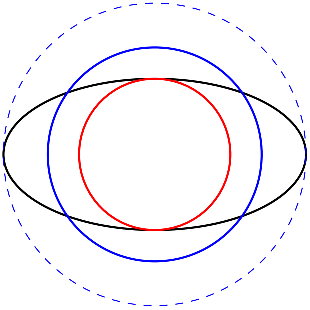

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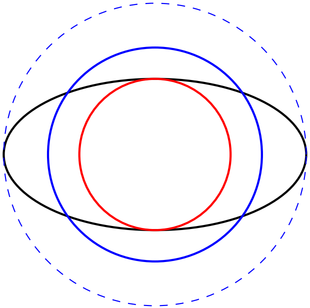

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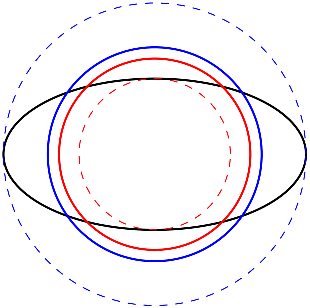

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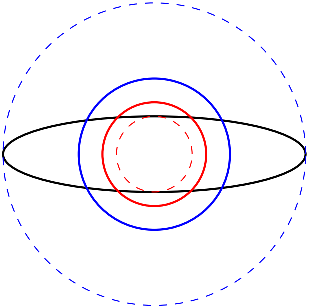

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$$Z_t(E) \geq Z_{(a^2+1)t/(2a)}(E^*)$$

For narrow ellipses exact trace should be close to our lower bound (we get an almost 1D case).

Our method for eigenvalues

Rayleigh quotient

$$R[v] = \frac{\int_D |\nabla v|^2 dx}{\int_D |v|^2 dx},$$

$$\lambda_1 + \cdots + \lambda_n = \inf \{ R[v_1] + \cdots + R[v_n] : v_i \text{ orthogonal } \}$$

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- u_i — eigenfunctions of a suspected extremizer D^*.
- U — isometry of the extremizer (isometry group irreducible)
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Averaging over isometries

$$\sum_{i=1}^{n} \lambda_i(D) \leq \quad \sum_{i=1}^{n} R[u_i \circ U \circ T]$$

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Averaging over isometries

$$\sum_{i=1}^{n} \lambda_i(D) \le \fint_U \sum_{i=1}^{n} R[u_i \circ U \circ T] = C(T) \sum_{i=1}^{n} R[u_i] = C(T) \sum_{i=1}^{n} \lambda_i(D^*)$$

Linear maps, fractional Laplacian and symmetric domains

- T - linear map
- D^* - extremizer with any irreducible isometry group (regular polygons, regular solids, ball)
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Theorem (2D statement, A — area, I — moment of inertia (2010))

Suppose that $D = T^{-1}(D^*)$.

$$\left(\lambda_1^{(\alpha)}(D) + \cdots + \lambda_n^{(\alpha)}(D)\right)^{2/\alpha} A\frac{A^2}{I} \text{ is maximal for } D^* \ (T = c\mathbb{I})$$

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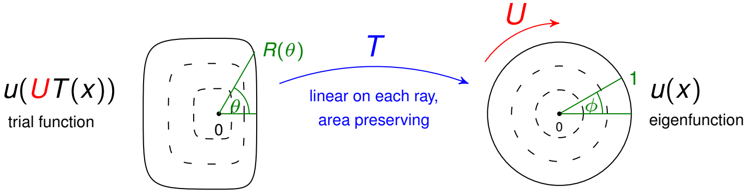
- All rectangles are extremal for $\lambda_1^{(2)}$ among parallelograms.
- Tetrahedron is the only extremizer among simplexes.
- Ball is the only extremizer among ellipsoids.

Area-preserving maps, Laplacian and star-like domains



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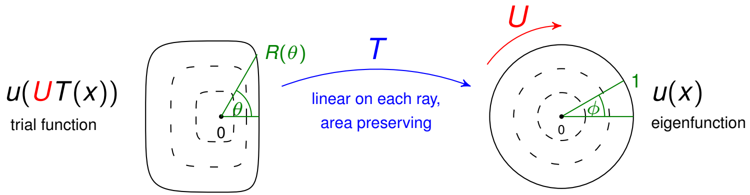

Area-preserving means $R(\theta)^2 \, d\theta = d\phi$.

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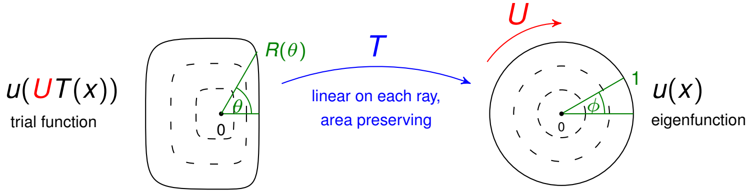

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Averaging is very challenging here. We get 2 geometric factors:

$$G_0 = \frac{1}{2\pi}\int_{\partial\Omega} \frac{1}{x\cdot N(x)}\, ds(x) \geq 1, \qquad G_1 = \frac{2\pi I}{A^2} \geq 1.$$

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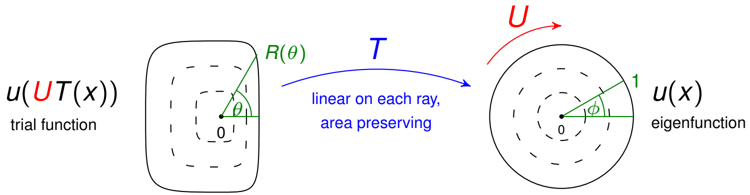

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Theorem (Laugesen-S. 2012)

Among starlike plane domains D

$$\lambda_1 A/G_0 \qquad AND \qquad (\lambda_1 + \cdots + \lambda_n)A/\max\{G_0, G_1\}$$

are maximal for centered balls.

From eigenvalues to traces

Theorem (Majorization: Hardy, Littlewood, Pólya)

If $a_1 \le a_2 \le a_3 \le \cdots$ *and* $b_1 \le b_2 \le b_3 \le \cdots$ *and*

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$$\Phi(a_1) + \cdots + \Phi(a_n) \le \Phi(b_1) + \cdots + \Phi(b_n) \qquad \forall n \ge 1$$

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Used on eigenvalues gives the following bounds:

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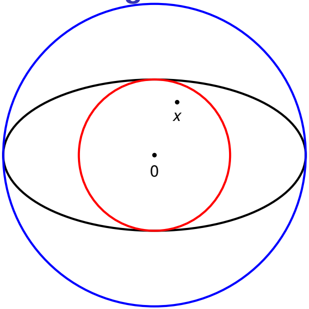
- lower for spectral zeta function: $\Phi(x) = -1/x^s$ with $s > 0$,
- upper for products: $\Phi(x) = \ln x$,
- upper for sloshing in cylinders: $\Phi(x) = \sqrt{x}\tanh(c\sqrt{x})$.

Our scaling factors and expected exit time



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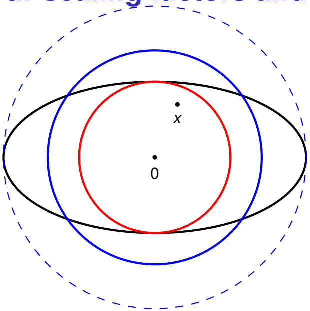

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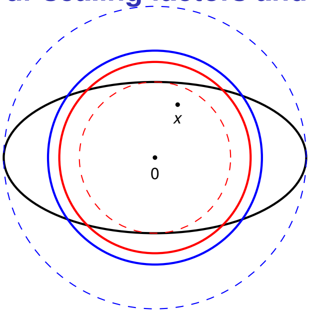

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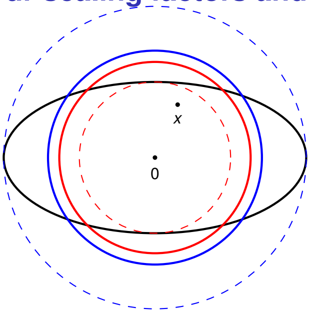

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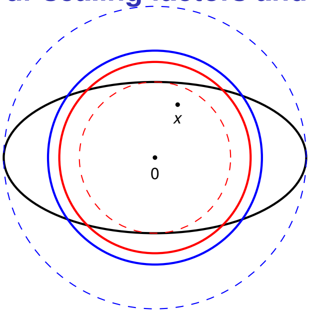

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