Lévy driven BSDEs: L_2-regularity and fractional smoothness

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1. Motivation

L_2 -variation of the solution (Y, Z) ← Malliavin fractional smoothness of ξ

↘

discrete-time approximation of the BSDE

$$Y_t = \xi + \int_t^T f\left(s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right) ds - \sigma \int_{(t,T]} Z_{s,0} dW_s$$
$$- \int_{(t,T]\times\mathbb{R}} Z_{s,x}\, x\tilde{N}(dt, dx), \quad 0 \le t \le T.$$

2. BSDEs driven by Lévy noise

Let L be an L_2 - Lévy process. Lévy-Itô decomposition:

$$L_t = \gamma t + \sigma W_t + \int_{(0,t]\times\mathbb{R}_0} x\tilde{N}(ds, dx)$$

- N Poisson random measure: $A \in \mathcal{B}(\mathbb{R})$

$$N([0,t]\times A) = \#\{s \in [0,t] : L_s - L_{s-} \in A\}$$

- ν Lévy measure $\nu(A) := \mathbb{E}N([0,1]\times A)$
- $\tilde{N}$ compensated Poisson random measure

$$\tilde{N}([0,t]\times A) \quad := \quad N([0,t]\times A) - t\nu(A)$$

- random measure M

$$M(ds, dx) = \begin{cases} \sigma dW_s & \text{if } x = 0 \\ x\tilde{N}(ds, dx) & \text{if } x \neq 0 \end{cases}$$

$$\mathbb{E}M([0,t]\times A)^2 = t\left(\sigma^2\delta_0(A) + \int_A x^2\nu(dx)\right) =: t\mu(A)$$

2. BSDEs driven by Lévy noise

$$
\begin{aligned}
X_t &= x_0 + \int_0^t b(X_s)ds + \int_0^t \beta(X_s)dW_s + \int_{(0,t]\times\mathbb{R}} \delta(X_{s-},x)\tilde{N}(ds,dx), \\
Y_t &= g(X_T) + \int_t^T f\left(s, X_s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right)ds - \int_{(t,T]\times\mathbb{R}} Z_{s,x}M(ds,dx), \\
&\qquad 0 \le t \le T,
\end{aligned}
$$

with
Y progressively measurable,
$Z : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$-measurable,

$$\|Y\|_{\mathcal{S}_2} + \|Z\|_{\mathcal{L}_2^\mu} < \infty$$

where

$$\|Y\|^2_{\mathcal{S}_2} := \sup_{0\le t\le T} \mathbb{E}|Y_t|^2 \quad \text{and} \quad \|Z\|^2_{\mathcal{L}_2^\mu} := \mathbb{E}\int_{(0,T]\times\mathbb{R}} |Z_{t,x}|^2 dt\mu(dx).$$

Existence and uniqueness of a solution (Y,Z) by S. Tang and X. Li (1994)

3. Discretization

- discretization scheme (Bouchard & Elie, 2005)
  Let $\pi_n = \{T = t_n > t_{n-1} > ... > t_1 = 0\}$.
- Euler approximation X^{π_n}
- Backward process intuitive idea:

$$Y_t = g(X_T) + \int_t^T f\left(s, X_s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right) ds - \int_{(t,T]\times\mathbb{R}} Z_{s,x} M(ds, dx)$$

$$Y^\pi_{t_{k-1}} \approx Y^\pi_{t_k} + f(t_{k-1}, X^\pi_{t_{k-1}}, Y^\pi_{t_{k-1}}, Z^\pi_{t_{k-1}})(t_k - t_{k-1}) - \int_{(t_{k-1},t_k]\times\mathbb{R}} Z_{s,x} M(ds, dx)$$

scheme

$$Y^\pi_T := g(X^\pi_T)$$

$$Z^\pi_{t_{k-1}} := \frac{\mathbb{E}[Y^\pi_{t_k} M((t_{k-1}, t_k] \times \mathbb{R})|\mathcal{F}_{t_{k-1}}]}{t_k - t_{k-1}} \approx \frac{\mathbb{E}[\int_{\mathbb{R}} \int_{t_{k-1}}^{t_k} Z_{s,x}\mu(dx)ds|\mathcal{F}_{t_{k-1}}]}{t_k - t_{k-1}}$$

$$Y^\pi_{t_{k-1}} := \mathbb{E}[Y^\pi_{t_k}|\mathcal{F}_{t_{k-1}}] + f(t_{k-1}, X^\pi_{t_{k-1}}, Y^\pi_{t_{k-1}}, Z^\pi_{t_{k-1}})(t_k - t_{k-1}),$$

- $\bar{Y}^\pi_s := Y^\pi_{t_{k-1}}$ constant on $[t_{k-1}, t_k)$
  $\bar{Z}^\pi_s := Z^\pi_{t_{k-1}}$ constant on $(t_{k-1}, t_k]$

3. Discretization

- L_2-regularity Let $\bar{Z}_t := \int_{\mathbb{R}} Z_{t,x}\mu(dx)$. Define

$$var_2(Z;\pi_n)^2 \quad := \quad \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\bar{Z}_t - \bar{Z}_{t_{k-1}}\|_{L_2}^2 dt$$

- discretization error

$$\begin{aligned} Err_2(Y,Z;\pi_n) \quad &:= \quad \{\|Y - \bar{Y}^{\pi_n}\|_{\mathcal{S}_2}^2 + \int_0^T \|\bar{Z}_t - \bar{Z}_t^{\pi_n}\|_{L_2}^2 dt\}^{\frac{1}{2}} \\ &\leq \quad C\{\|g(X_T) - g(X_T^{\pi_n})\|_{L_2} + var_2(Z;\pi_n)\} \end{aligned}$$

- L_2-regularity estimate
  If for $\theta \in (0,1)$

$$\|\bar{Z}_t - \bar{Z}_s\|_{L_2}^2 \leq c \int_s^t (T-r)^{\theta-2} dr$$

  $\exists$ time nets π_n^θ such that

$$\limsup_n n^{\frac{1}{2}} var_2(Z;\pi_n^\theta) < \infty.$$

3. Discretization

The estimate $var_2(Z;\pi_n^\theta) \leq cn^{-\frac{1}{2}}$ holds in the Brownian motion case for:

- g is Lipschitz, π_n equidistant
  J. Zhang; B. Bouchard and N. Touzi (2004)
- generator $f = 0$, fractional smoothness of g :

$$\exists\theta \in (0,1] : \|g(X_T) - \mathbb{E}[g(X_T)|\mathcal{F}_t]\|_{L_2}^2 \leq c(T-t)^\theta$$

  for $\pi_n^\theta = 1 - (1 - \frac{k}{N})^{\frac{1}{\theta}}, \quad k = 1, ..., N.$
  C. G. and S. Geiss (2004)
- f Lipschitz, fractional smoothness of g, π_n^θ
  E. Gobet and A. Makhlouf (2010)
- $g(X_{r_1}, \ldots, X_{r_K})$ $0 < r_1 < ... < r_K = T$, fractional smoothness of g, π_n^θ, $L_p, (p \geq 2)$
  C.G., S. Geiss and E. Gobet (2012)

3. Discretization

$$var_2(Z;\pi_n^\theta) \le cn^{-\frac{1}{2}} \quad \text{in the Lévy case:}$$

$$\begin{aligned} X_t &= x_0 + \int_0^t b(X_s)ds + \int_0^t \beta(X_s)dW_s + \int_{(0,t]\times\mathbb{R}} \delta(X_{s-},x)\tilde{N}(ds,dx), \\ Y_t &= g(X_T) + \int_t^T f\left(s, X_s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right) ds - \int_{(t,T]\times\mathbb{R}} Z_{s,x} M(ds,dx), \\ & \qquad\qquad 0 \le t \le T, \end{aligned}$$

- B. Bouchard and R. Elie (2008)
  for
  - b, β, δ, f 'nice' (Lipschitz, ...)
  - L is compound Poisson + Brownian motion
  - g is Lipschitz
- Now
  - L Lévy process, square integrable
  - $X = L$
  - $g(L_{r_0}, ..., L_{r_K})$ for some $0 = r_0 < r_1 < ... < r_K = T$
    with a fractional smoothness condition

4. L_2-variation: results

Theorem (C. G. and A. Steinicke)

Assume that $\xi \in \mathbb{H}$. *Let* $k \in \{1, \dots, K\}$ *and* $0 < \theta_k \leq 1$.

(i) $\exists c_1 > 0: \quad \|Y_{r_k} - \mathbb{E}_s Y_{r_k}\|^2 \leq c_1 (r_k - s)^{\theta_k} \quad r_{k-1} < s < r_k.$

(ii) $\exists c_2 > 0: \quad \|Y_t - Y_s\|^2 \leq c_2 \int_s^t (r_k - r)^{\theta_k - 1} dr, \quad r_{k-1} < s < t < r_k.$

(iii) $\exists c_3 > 0: \quad \|Z_{s,\cdot}\|^2_{\mathrm{L}_2(\mathbb{P}\otimes\mu)} \leq c_3 (r_k - s)^{\theta_k - 1}, \quad \lambda - a.e.\ r_{k-1} < s < r_k.$

(iv) $\exists c_4 > 0:$ *for* $\lambda - a.e.\ r_{k-1} < s < t < r_k$ *it holds*

$$\|\bar{Z}_t - \bar{Z}_s\|^2 \leq c_4 \int_s^t (r_k - r)^{\theta_k - 2} dr.$$

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv).

5. Terminal condition: assumptions via chaos expansions

Itô's chaos expansion

- for any $F \in L_2 := L_2(\Omega, \mathcal{F}_T^L, \mathbb{P})$ exists the chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n), \ a.s.$$

- $I_n(f_n)$ multiple integrals w.r.t. M
- for example:

$$\begin{aligned}
I_0(f_0) &= \mathbb{E}F, \\
I_1(f_1) &= \int_{[0,T]\times\mathbb{R}} f_1(s,x) M(ds,dx) \\
I_2(f_2) &= 2\int_{[0,T]\times\mathbb{R}} \left(\int_{[0,s)\times\mathbb{R}} f_2((u,y),(s,x)) M(du,dy) \right) M(ds,dx)
\end{aligned}$$

if $f_2((u,y),(s,x)) = f_2((s,x),(u,y))$

5. Terminal condition: assumptions via chaos expansions

$$Y_t = \xi + \int_t^T f\left(s, L_s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right) ds - \int_{(t,T]\times\mathbb{R}} Z_{s,x} M(ds, dx), \qquad 0 \le t \le T,$$

- example $\xi = g(L_T)$: chaos expansion

$$g(L_T) = \sum_{n=0}^{\infty} I_n(f_n)$$

$$\begin{aligned} f_n((t_1, x_1), \ldots, (t_n, x_n)) &= g_n(x_1, \ldots, x_n) \mathbb{1}_{(0,T]}^{\otimes n}(t_1, \ldots, t_n) \\ &= g_n(\mathbf{x}) \mathbb{1}_{(0,T]^n}(\mathbf{t}) \end{aligned}$$

5. Terminal condition: assumptions via chaos expansions

- path-dependence: $\xi = g(L_{r_K} - L_{r_{K-1}}, ..., L_{r_1} - L_{r_0}) \in L_2$

  Let
  - $0 = r_0 < ... < r_K = T$
  - $\Lambda_k := (r_{k-1}, r_k]$ for $k = 1, ..., K$
  - $A_n := \{1, ..., K\}^n$
  - $\Lambda_\alpha := \Lambda_{\alpha_1} \times \cdots \times \Lambda_{\alpha_n}$ for $\alpha \in A_n$ then

$$f_n((t_1, x_1), \ldots, (t_n, x_n)) = \sum_{\alpha \in A_n} g_{n,\alpha}(\mathbf{x}) \mathbb{1}_{\Lambda_\alpha}(\mathbf{t}).$$

- general definition

$$\mathbb{H} := \Big\{ \xi = \sum_{n=0}^{\infty} I_n(f_n) \in L_2 \colon f_n \text{ is symmetric and } f_n((t_1, x_1), \ldots, (t_n, x_n)) = \sum_{\alpha \in A_n} g_{n,\alpha}(\mathbf{x}) \mathbb{1}_{\Lambda_\alpha}(\mathbf{t}) \Big\}.$$

6. Ideas of the proof $(i) \implies (iii)$

Theorem (C. G. and A. Steinicke)

Assume that $\xi \in \mathbb{H}$. *Let* $k \in \{1, \dots, K\}$ *and* $0 < \theta_k \leq 1$.

(i) $\exists c_1 > 0 : \quad \|Y_{r_k} - \mathbb{E}_s Y_{r_k}\|^2 \leq c_1 (r_k - s)^{\theta_k} \quad r_{k-1} < s < r_k.$

(ii) $\exists c_2 > 0 : \quad \|Y_t - Y_s\|^2 \leq c_2 \int_s^t (r_k - r)^{\theta_k - 1} dr, \quad r_{k-1} < s < t < r_k.$

(iii) $\exists c_3 > 0 : \quad \|Z_{s,\cdot}\|^2_{\mathrm{L}_2(\mathbb{P}\otimes\mu)} \leq c_3 (r_k - s)^{\theta_k - 1}, \quad \lambda - a.e.\ r_{k-1} < s < r_k.$

(iv) $\exists c_4 > 0$: *for* $\lambda - a.e.\ r_{k-1} < s < t < r_k$ *and for all* $h \in L_2(\mu)$ *it holds*

$$\left\| \int_{\mathbb{R}} (Z_{t,x} - Z_{s,x})\, h(x) \mu(dx) \right\|^2 \leq \|h\|^2_{L_2(\mu)} c_4 \int_s^t (r_k - r)^{\theta_k - 2} dr.$$

Then (i) $\Leftrightarrow$(iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (iv).

7. Ideas of the proof $(i) \implies (iii)$

$$Y_t = \xi + \int_t^T f\left(s, L_s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right) ds - \int_{(t,T]\times\mathbb{R}} Z_{s,x} M(ds, dx),$$
$$0 \le t \le T,$$

- $\exists$ stability results $\implies$ choose ξ and f smooth enough
- $Y_t = \mathbb{E}_t\left[\xi + \int_t^T f\left(s, L_s, Y_s, \int_{\mathbb{R}} Z_{s,x}\mu(dx)\right) ds\right]$

$$Z_{t,x} = \lim_{u \downarrow t} D_{t,x} Y_u$$

- $\xi \in \mathbb{H} \cap \mathbb{D}_{1,2}$: for $r_{k-1} < s < r_k$

$$\|\mathbb{E}_s D_{s,\cdot}\xi\|^2_{\mathrm{L}_2(\mathbb{P}\otimes\mu)} \le \frac{\|\mathbb{E}_{r_k}\xi - \mathbb{E}_s\xi\|^2}{(r_k - s)} \le \frac{c_1 (r_k - s)^{\theta_k}}{(r_k - s)}.$$

- Y and Z inherit the $\mathbb{H}$-structure from ξ :

$$\eta \in \mathbb{H}, g(\eta) \in L_2 \implies g(\eta) \in \mathbb{H}$$

8. Fractional smoothness of $\xi \implies L_2$-variation of (Y, Z)

- fractional smoothness of ξ - the 'decoupling condition':
  $\tilde{L}$ independent copy of L. For $0 \le t \le r \le T$ let

$$L_s^{t,r} := \int_{(0,s]} \mathbb{I}_{(0,T]\setminus(t,r]}(u) dL_u + \int_{(0,s]} \mathbb{I}_{(t,r]}(u) d\tilde{L}_u$$

$$\implies M^{t,r}(B) = M\left(B \setminus ((t,r] \times \mathbb{R})\right) + \widetilde{M}\left(B \cap ((t,r] \times \mathbb{R})\right).$$

$$\implies I_n^{t,r}(f_n) \implies \xi^{t,r} := \sum_{n=0}^{\infty} I_n^{t,r}(f_n)$$

Theorem (C. G. and A. Steinicke)

Assume $\xi \in \mathbb{H}$ *and* $\exists \Theta = (\theta_1, ..., \theta_K) \in (0,1]^K$ *and* $c > 0$ *such that*

$$\|\xi - \xi^{t,r_k}\|_{L_2}^2 \le c(r_k - t)^{\theta_k}, \; t \in (r_{k-1}, r_k], \; k = 1, ..., K \quad (A_\Theta)$$

Then for all $k = 1, ..., K$
$\exists c_1 > 0 : \quad \|Y_{r_k} - \mathbb{E}_s Y_{r_k}\|^2 \le c_1 (r_k - s)^{\theta_k} \quad r_{k-1} < s < r_k.$

8. Fractional smoothness of $\xi \implies L_2$-variation of (Y, Z)

Interpretation of (A_Θ)

- For $\xi = \sum_{n=0}^{\infty} I_n(f_n) \in \mathbb{H}$ the case $\Theta = (1, 1, ..., 1)$ corresponds to Malliavin differentiability:

$$\exists c > 0: \quad \|\xi - \xi^{t,r_k}\|^2 \leq c(r_k - t) \text{ for all } t \in (r_{k-1}, r_k],\ k = 1, ..., K$$
$$\iff \xi \in \mathbb{D}_{1,2}$$
$$\iff \sum_{n=0}^{\infty} n \|I_n(f_n)\|^2 < \infty$$

- If $K = 1$ and $\theta \in (0, 1)$ then

$$\exists c > 0: \quad \|\xi - \xi^{t,T}\|^2 \leq c(T - t)^{\theta} \text{ for all } t \in (0, T]$$
$$\iff \xi \in (L_2, \mathbb{D}_{1,2})_{\theta,\infty}$$
$$\implies \sum_{n=0}^{\infty} n^{\eta} \|I_n(f_n)\|^2 < \infty \quad \forall 0 < \eta < \theta.$$

9. The example $\mathbb{I}_{(K,\infty)}(L_1)$

For $\delta > 0$ we let

$$\psi(\delta) := \sup_{\lambda \in \mathbb{R}} \mathbb{P}(|X_1 - \lambda| \le \delta).$$

σ	ψ	additional assumption on ν	smoothness of $\mathbb{I}_{(K,\infty)}(L_1)$
$\sigma = 0$	arbitrary	$\int_{\lvert x\rvert \le 1} \nu(dx) < \infty$	$\mathbb{D}_{1,2}$
$\sigma = 0$	$\psi(\delta) \le c\delta$	$\int_{\lvert x\rvert \le 1} \lvert x\rvert \nu(dx) < \infty$	$\mathbb{D}_{1,2}$
arbitrary	$\psi(\delta) \le c\delta$		$(L_2, \mathbb{D}_{1,2})_{\frac{1}{2},\infty}$

C.G., S. Geiss and E. Laukkarinen

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