Nonlocal porous medium equation

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joint work with

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FRACTIONAL LAPLACIAN

$$u_t + (-\Delta)^{\alpha/2} u = 0 \qquad x \in \mathbb{R}^d$$

$$\alpha \in (0, 2]$$

Fundamental solution

$$P_\alpha(x,t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\xi} e^{-t|\xi|^\alpha} d\xi$$

Terminology:

- α-stable distribution
- fundamental solution
- source solution
- Gauss-Weierstrass kernel for $\alpha = 2$.

PROPERTIES of $p_\alpha(x,t)$

1° probability distributions for each $t \geq 0$

2° **SCALING** (self-similar solution)

$$p_\alpha(x,t) = t^{-\frac{d}{\alpha}} \, p_\alpha\left(\frac{x}{t^{1/\alpha}}, 1\right)$$

3° **ASYMPTOTICS**

$$p_\alpha(x,1) \sim \frac{C}{|x|^{\alpha+d}}, \quad |x| \to \infty$$

4° **CAUCHY PROBLEM**

$$\begin{cases} u_t + (-\Delta)^{\alpha/2} u = 0 \\ u(x,0) = u_0(x) \end{cases}$$

solution $u(x,t) = p_\alpha(\cdot,t) * u_0(x)$

$u_0 \in L^1(\mathbb{R}^d)$

SELF-SIMILAR ASYMPTOTICS

Let
$$u(x,t) = p_\alpha(\cdot,t) * u_0(x)$$
then
$$\Lambda^d u(\Lambda x, \Lambda^\alpha t) \to p_\alpha(x,t) \cdot \int_{\mathbb{R}^d} u_0(x)\,dx$$
as $\Lambda \to \infty$

in any reasonable sense.

DIFFUSION EQUATION

Let $\Omega \subset \mathbb{R}^d$ be arbitrary and bounded with a smooth boundary $\partial\Omega$

Conservation law

J is a FLUX

$$\frac{d}{dt} \int_{\Omega} u(x,t)\, dx = - \int_{\partial\Omega} J \, d\sigma$$

by the divergence theorem

$$= - \int_{\Omega} \nabla \cdot J \, dx$$

Hence,

$$\frac{\partial u}{\partial t} = - \nabla \cdot J$$

EXAMPLES of FLUXES

1°. FICK LAW

FLUX is proportional to the gradient of concentration

$$J = -D \nabla u$$

Hence, from the conservation law

$$\frac{\partial}{\partial t} u = -\nabla \cdot J = D \Delta u$$

2° BOUSSINESQ LAW

$$J = -u \nabla p \qquad \text{where the pressure } p = u$$

Boussinesq equation

$$u_t = \nabla \cdot (u \nabla u) = \frac{1}{2} \Delta u^2$$

3° Porous medium equation

$$\frac{\partial}{\partial t} u = \nabla \cdot \left(u \nabla u^{m-1} \right)$$

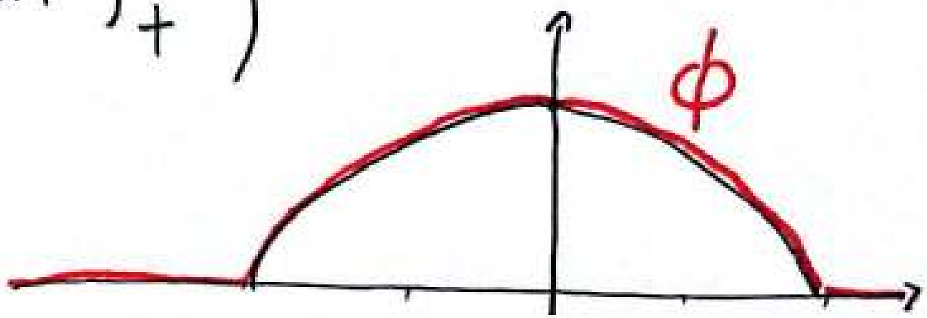
$$= \frac{1}{m} \Delta u^{m}$$

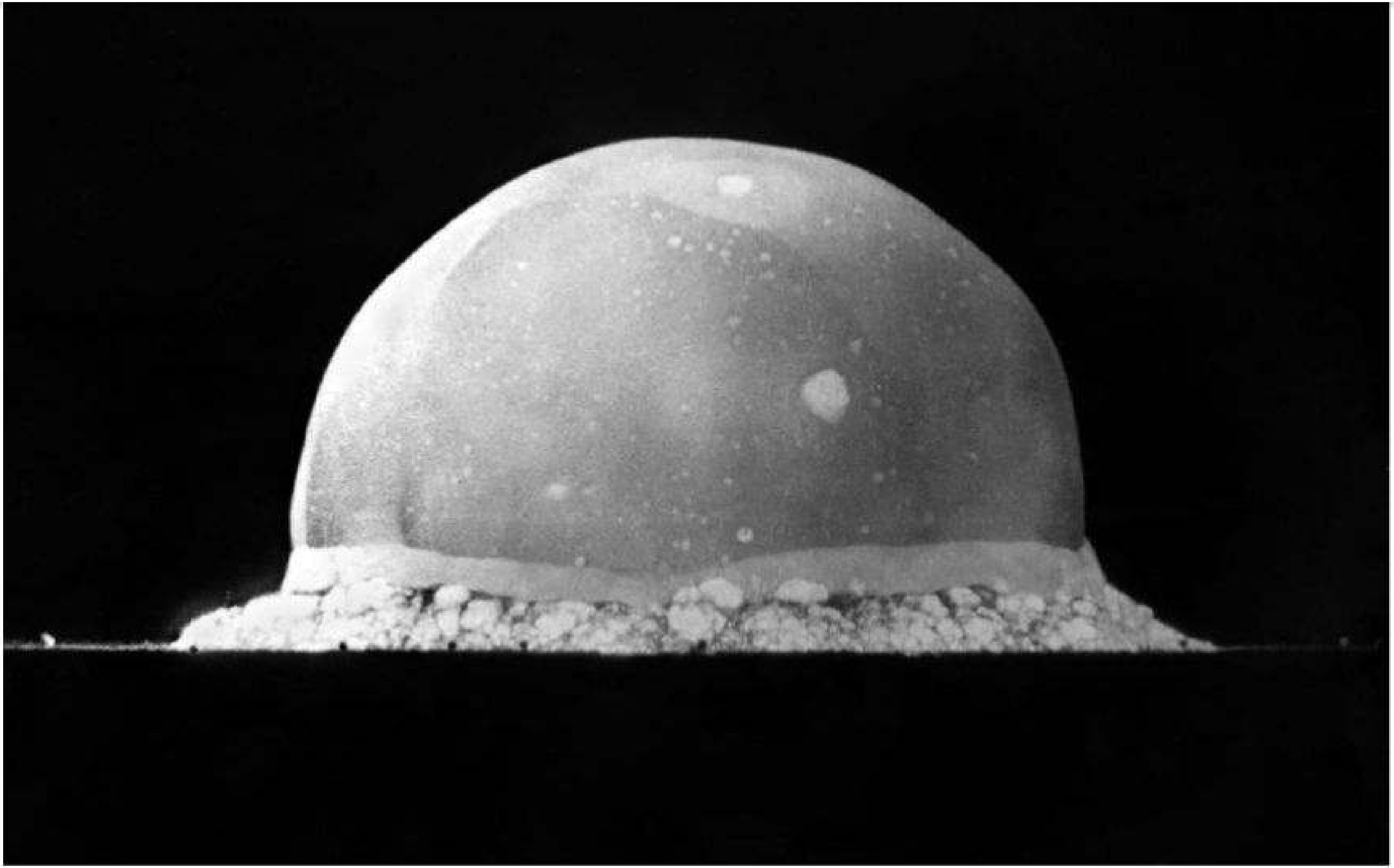
BARENBLATT Solution

$$u(x,t) = t^{\frac{-d}{d(m-1)+2}} \phi_R \left(\frac{x}{t^{1/d(m-1)+2}} \right)$$

where

$$\phi_R(x) = \left(C(d) \left(R - |x|^2 \right)_+ \right)^{\frac{1}{m-1}}$$





The Trinity explosion, 16 milliseconds after detonation.

The fireball is about 600 feet (200 m) wide.

Nonlocal porous medium equation

In the conservation law

$$\frac{\partial u}{\partial t} = -\nabla \cdot J$$

we assume that

$$J = -u \nabla p \quad \text{where}$$

the preassure satisfies

$$p = (-\Delta)^{\frac{\alpha}{2}-1} u^{m-1} \qquad \alpha \in (0,2] \quad m > 1$$

Motivation: DISLOCATION DYNAMICS

P. Biler, G. K., R. Monneau, Comm. Math. Ph. (2010)

FRACTIONAL GRADIENT

Nonlocal porous medium equation

$$\frac{\partial}{\partial t} u = \nabla \cdot (u \nabla p) = \nabla \cdot \left(u \nabla (-\Delta)^{\frac{\alpha}{2}-1} u^{m-1}\right)$$

$$= \nabla \cdot \left(u \nabla^{\alpha-1} u^{m-1} \right)$$

where

$$\nabla^{\alpha-1} u(x) = \nabla (-\Delta)^{\frac{\alpha}{2}-1} u(x) =$$

$$= C_{d,\alpha} \int_{\mathbb{R}^d} \left(u(x) - u(x+z)\right) \frac{z}{|z|^{d+\alpha}} \, dz$$

Self-similar solution

THEOREM (P. BILER, G.K. C. IMBERT)

The equation

$$u_t = \nabla \cdot \left(u \nabla^{\alpha-1} u^{m-1} \right) \qquad x \in \mathbb{R}^d$$

has a one parameter family of solutions

$$u(x,t) = t^{-\frac{d}{d(m-1)+\alpha}} \phi\left(\frac{x}{t^{\frac{1}{d(m-1)+\alpha}}} \right)$$

where

$$\phi(y) = \left(K(\alpha,d) \left(R^2 - |y|^2 \right)_+^{\frac{\alpha}{2}} \right)^{\frac{1}{m-1}}$$

for every $R > 0$

IDEA OF THE PROOF

$$u_t = \nabla \cdot (u \nabla^{d-1} u^{m-1})$$

Scaling:

If $u(x,t)$ is a solution, then

$$L^{d\lambda} u(L^{\lambda} x, Lt) \quad \text{with } \lambda = \frac{1}{d(m-1)+d}$$

is a solution for every $L>0$.

We look for a self-similar solution

$$u(x,t) = \frac{1}{t^{d\lambda}} \phi\left(\frac{x}{t^{\lambda}}\right)$$

After substituting, we obtain

$$-\lambda \nabla \cdot (y\phi) = \nabla \cdot (\phi \nabla^{d-1} \phi^{m-1})$$

NONLOCAL DIRICHLET PROBLEM

We look for solutions to the equation

$$-\lambda \nabla \cdot (y \phi) = \nabla \cdot \left(\phi \, \nabla^{d-1} \phi^{m-1} \right)$$

Hence

$$-\lambda \, y \, \phi = \phi \, \nabla^{d-1} \phi^{m-1}$$

Thus, we consider the Dirichlet problem

$$-\lambda y = \nabla^{d-1} \left(\phi^{m-1} \right) \qquad \text{in } B(0,1)$$

$$\phi = 0 \qquad \text{in } \mathbb{R}^d - B(0,1)$$

We check that

$$\phi(y) = \left(K(d,d) \left(1 - |y|^2 \right)_+^{\frac{d}{2}} \right)^{\frac{1}{m-1}}$$

is THE EXPLICTT SOLUTION

GETOOR FUNCTION

THEOREM (Getoor, 1961)

For all $\alpha \in (0,2)$

$$K \cdot (-\Delta)^{\alpha/2} \left(1 - |y|_+^2\right)^{\frac{\alpha}{2}} = 1 \quad \text{in } B(0,1)$$

where $K = K(\alpha, d)$ is a constant.

- For $d = 1$, integrating this equation we obtain the self-similar profile ϕ
- For $d \geq 1$, we use the following Lemma.

Lemma

For all $\beta \in (0,2)$ and $\gamma > 0$, we have

$$(-\Delta)^{-\beta/2}\left((1-|y|^2)_+^{\frac{\gamma}{2}}\right)$$
$$= \begin{cases} C_{\gamma,\beta,d} \times {}_2F_1\left(\frac{d-\beta}{2}, -\frac{\gamma+\beta}{2}; \frac{d}{2}; |y|^2\right) & \text{for } |y| \leq 1, \\ \tilde{C}_{\gamma,\beta,d}\, |y|^{\beta-d} \times {}_2F_1\left(\frac{d-\beta}{2}, \frac{2-\beta}{2}; \frac{d+\gamma}{2}; \frac{1}{|y|^2}\right) & \text{for } |y| > 1, \end{cases}$$

with

$$C_{\gamma,\beta,d} = 2^{-\beta}\frac{\Gamma\left(\frac{\gamma}{2}+1\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{\beta+\gamma}{2}+1\right)} \quad \text{and} \quad \tilde{C}_{\gamma,\beta,d} = 2^{-\beta}\frac{\Gamma\left(\frac{\gamma}{2}+1\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{d}{4}\right)\Gamma\left(\frac{d+\gamma}{2}+1\right)}.$$

The hypergeometric function:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \quad \text{for } |z| < 1,$$

where $(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}$.

SUMMARY : "(α, m) - law"

The nonlocal porous medium equation

$$u_t = \nabla \cdot \left(u \, \nabla^{\alpha - 1} u^{m-1} \right)$$

has • the "fundamental" solution

• the source solution

$$u(x,t) = \frac{1}{t^{d\lambda}} \, \phi\left(\frac{x}{t^{\lambda}} \right) \qquad \lambda = \frac{1}{d(m-1)+\alpha}$$

where

$$\phi(y) = \left(k \left(R - |y|^2 \right)_+^{\alpha/2} \right)^{\frac{1}{m-1}}$$

and $R > 0$ is chosen to have $\displaystyle\int_{\mathbb{R}^d} \phi = \underline{1}$

UNIQUENES is not known.

THE INITIAL VALUE PROBLEM

THEOREM (P. Biler, G.K., C. Imbert)

Let $\alpha \in (0,2)$ and $m > 1$.

For every $0 \le u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the problem

$$u_t = \nabla \cdot \left(u \, \nabla^{\alpha-1} u^{m-1} \right)$$

$$u(x,0) = u_0(x)$$

has a solution.

This solution satisfies

- $u(x,t) \ge 0$, $\int_{\mathbb{R}^d} u(x,t)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$
- $\| u(\cdot,t) \|_{L^p(\mathbb{R}^d)} \le C(u_0)\, t^{-\frac{d}{d(m-1)+\alpha}\left(1-\frac{1}{p}\right)}$

UNIQUENESS is not known.

IDEA of THE PROOF

Regularized problem

$$u_t - \delta \Delta u = \nabla \cdot \left(u \nabla^{d-1} G_\varepsilon(u) \right)$$

$u(x,0) = u_0(x)$

where $G_\varepsilon(u) = \operatorname{sgn} u \left(\left(|u|^2 + \varepsilon^2 \right)^{\frac{m-1}{2}} - \varepsilon^{m-1} \right)$

with $\delta > 0$, $\varepsilon > 0$.

- Fixed point argument to have a sequence $\{ u^{\delta,\varepsilon} \}_{\delta,\varepsilon > 0}$.

- Compactness argument to pass to the limit.


WORK IN
PROGRESS
CHECK BACK SOON!

Thank you
for your attention!