Revisiting Clark's robustness problem

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I. CLARK'S ROBUSTNESS PROBLEM
II. ROBUSTNESS VIA ZAKAI SPDE (joint with Friz)
III. ROBUSTNESS VIA KALLIANPUR-STRIEBEL FUNCTIONAL (joint with Crisan,Diehl,Friz)

I. CLARK'S ROBUSTNESS PROBLEM

Nonlinear filtering

Fix $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Consider the pair (X, Z) where

$$\begin{aligned} dX_t &= \mu(X_t)\, dt + V(X_t) \circ dB_t \text{ (signal in } \mathbb{R}^{dx}) \\ dZ_t &= h(X_t)\, dt + d\tilde{B}_t \text{ (observation in } \mathbb{R}^{dz}) \end{aligned}$$

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Goal: Given real-valued function f, compute

$$\pi_t(f) = \mathbb{E}\left[f(X_t) \,|\, \sigma(Z_s, s \in [0, t])\right].$$

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There exists a measurable map $\theta_t^f : C\left([0,t], \mathbb{R}^{d_Z}\right) \to \mathbb{R}$

$$\pi_t(f) = \theta_t^f\left(Z|_{[0,t]}\right) \ \mathbb{P}\text{-a.s.}$$

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Clark's robustness problem

- only discrete observations $0 \leq t_1 \leq \cdots \leq t_n \leq t$ of Z availabe
  - BV path Z^n which approximates Z
  - BUT $\theta_t^f : C\left([0,t], \mathbb{R}^{d_Y}\right) \to \mathbb{R}$ not unique, every $\tilde{\theta}_t^f$ s.t. $\theta_t^f(.) = \tilde{\theta}_t^f(.)$ $\mathbb{P} \circ \left(Z|_{[0,t]}\right)^{-1} - a.s$ fulfills

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Solution. If B and $\tilde{B}$ independent (Clark78, ClarkCrisan05, Davies 80/81,...) then

$$\exists!\ \ \theta_t^f : \left(C\left([0,t], \mathbb{R}^{d_Z}\right), |.|_\infty\right) \to \mathbb{R}$$

continuous in supremums norm.

Correlated noise

$$
\begin{aligned}
dX_t &= \mu(X_t)\,dt + V(X_t)\circ dB_t + \sigma(X_t)\circ d\tilde{B}_t \ \text{(signal)} \\
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Bad news!

$$\nexists \theta_t^f : C\left([0,t], \mathbb{R}^{d_Z}\right) \to \mathbb{R}$$

s.t.

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Our main result. $\exists! \theta_t^f : C\left([0,t], G^2\left(\mathbb{R}^{d_Z}\right)\right) \to \mathbb{R}$

- continuous in a rough path metric
- $\theta_t^f\left(1 + Z + \int Z \otimes dZ\right) = \pi_t(f)$ $\mathbb{P}$-a.s

II. ROBUSTNESS OF THE ZAKAI SPDE
(joint with P. Friz)

- Work in unnormalized form

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- $\rho_t(.)$ solution of (measure-valued) SDE (*Zakai equation*)
- Assume density $\rho_t(f) = \int_{\mathbb{R}^{d_X}} f(x)\, u_t(x)\, dx$
- Described via a linear, parabolic SPDE (*dual Zakai equation*)

$$\begin{aligned} du &= G^\star u dt + \sum_i N_i^\star u dZ^i \\ &= \left(G^\star - \frac{1}{2} \sum_i N_i N_i^\star \right) u dt + \sum_i N_i^\star u \circ dZ_t^i \end{aligned}$$

with

$$\begin{aligned} G &\quad \ldots \quad \text{generator of } X \\ N_j^\star u &= \sigma_j \cdot Du + h_j u. \end{aligned}$$

General SPDE

Find $u : [0, T] \times \mathbb{R}^e \to \mathbb{R}$ which solves

$$
\begin{aligned}
du + L\left(t, x, u, Du, D^2 u\right) dt &= \sum_{i=1}^{d} \Lambda_i \left(t, x, u, Du\right) \circ dZ_t^i, \quad (0, T) \times \mathbb{R}^e \\
u(0, .) &= u_0(.) \text{ on } \mathbb{R}^e
\end{aligned}
$$

where

$$
\begin{aligned}
L : [0, T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e \times \mathbb{S}^e &\to \mathbb{R} \\
\Lambda_i : [0, T] \times \mathbb{R}^e \times \mathbb{R} \times \mathbb{R}^e &\to \mathbb{R}
\end{aligned}
$$

are (affine) linear.
Question: Regularity of

$$Z \mapsto u$$

Heuristic explanation

Take $L \equiv 0$, only gradient noise (corresponds to correlation in the filtering set-up!!!), i.e.

$$\begin{aligned} du &= \langle Du, \sigma_1(x) \rangle \circ dZ_t^1 + \langle Du, \sigma_2(x) \rangle \circ dZ_t^2 \\ u(0,.) &= u_0(.) \end{aligned}$$

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If ϕ^Z denotes the SDE flow of

$$dY_t = \sigma_1(Y_t) \circ dZ_t^1 + \sigma_2(Y_t) \circ dZ_t^2$$

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then (formally) $u(t,x) = u_0\left(\phi^Z(t,x)\right)$.
Question: Robustness of SDE solutions,

$$Z \mapsto \phi^Z.$$

Answer: Poor robustness in uniform norm (except for degenerate situations where vectorfields σ_1,σ_2 commute). Not even continuous!

INTERMEZZO: ROUGH PATH THEORY

Theorem

(Wong–Zakai). Let B^n be the piecewise linear approximation to B along the dyadics of $[0, T]$. Then the ODE solutions (Y^n) of

$$Y_t^n = V(Y_t^n)\, dB_t^n, Y_0^n = y$$

converge uniformly to Y, the solution of the Stratonovich SDE

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BUT there are approximations (B^n)

$$|B^n - B|_{\infty,[0,T]} \to_n 0 \text{ s.t. } \left|Y^n - \overline{Y}\right| \to_n 0$$

where

$$d\overline{Y} = V\left(\overline{Y}_t\right) \circ dB_t + c\left(\overline{Y}_t\right) dt$$

and c is any linear combination of

$$\left[V_{i_1}, \left[V_{i_2} \ldots \left[V_{i_{N-1}}, V_{i_N}\right] \ldots\right]\right].$$

- **Lyons '98.** Let $(z^n) \subset C^1([0,T], \mathbb{R}^d)$ be Cauchy in rough path metric with limit $\mathbf{z}$. Assume

$$\text{(ODE)} \quad y^n = V(y^n)\,\dot{z}^n, \quad y^n(0) = y_0 \in \mathbb{R}^e$$

then y^n converges uniformly to some $y = y^{\mathbf{z}} \in C([0,T], \mathbb{R}^e)$ which is **independent** of the approximating sequence.

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- **Interpretation:** y is the solution of a *rough differential equation* driven by the *rough path* $\mathbf{z}$. Write

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- What are *rough path metrics* and *rough paths?*
- First example (not applicable to Brownian motion): take

$$\rho_{\alpha-Hol}(z, \bar{z}) = \frac{|z_{s,t} - \bar{z}_{s,t}|}{(t-s)^{\alpha}} \text{ for } \alpha \in \left(\frac{1}{2}, 1\right]$$

and *rough paths* are just α-Hölder paths, RDEs are "Young" ODEs.

- Better example (applicable to Brownian paths): for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ take

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where we introduced the generalized increments of $z \in C^1$ as

$$z_{s;t} := \left(z^1_{s,t}, z^2_{s,t}\right) := \left(\int_s^t dz, \int_s^t \int_s^{r_2} dz_{r_1} \otimes dz_{r_2}\right) \in \mathbb{R}^d \oplus \left(\mathbb{R}^d\right)^{\otimes 2}$$

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- The abstract completion of C^1-paths wrt to $\rho_{\alpha-Hoel}$ leads to *a rough path space* which can be identified as a subset of

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- From $d\left(z^i z^j\right) = z^i dz^j + z^j dz^i$ it follows $Sym\left(z^2\right) = \frac{1}{2} z^1 \otimes z^1$ and

$$\left(z^1_{s,t}; z^2_{s,t}\right) \leftrightarrow \left(z^1_{s,t}, a_{s,t}\right) \text{ with } a_{s,t} := Anti\left(z^2_{s,t}\right)$$

Advantages to a Probabilist

- RDE solution of $dy = V(y)\, d\boldsymbol{B}(\omega)$ is solved for fixed ω; depends continuosly on $\boldsymbol{B} = \left(B, \int B \circ dB\right)$ and coincides with (Stratonovich) solution of $dY = V(Y) \circ dB$.

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For e.g. $V \in Lip^{3+\epsilon}$ can see that $\phi, D\phi, D^2\phi$ exist and depend continuously on $\boldsymbol{z}$, also $\phi^{-1}, D\phi^{-1}, D^2\phi^{-1}$. Limit theorems on the level of stochastic flows!

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- No restriction to semimartingales as noise (as long as we can construct the higher levels)
- Continuity of solution map $\boldsymbol{z} \mapsto y$ makes it easier to prove large deviations, Freidlin-Wentzell estimates, support theorems etc.

BACK TO ZAKAI SPDE

Viscosity solutions

Theorem (Friz-Caruana-O,Friz-O)

Let the coefficients in L and Λ fulfill (TC). Let $(z^n) \subset C^1([0,T], \mathbb{R}^d)$ and consider the viscosity solutions (u^n) of

$$\begin{aligned} du^n + L\left(t, x, u^n, Du^n, D^2 u^n\right) dt &= \Lambda\left(t, x, u^n, Du^n\right) dz^n \\ u^n(0,.) &= u_0(.) \in BUC(\mathbb{R}^e) \end{aligned}$$

If (z^n) converges to a geometric rough path z then $\exists!$ $u^z \in BUC([0,T] \times \mathbb{R}^e, \mathbb{R})$ such that $u^n \to u^z$ (loc. uniformly on compacts). Further,

1. *u^z **independent** of the choice of (z^n),*
2. *$(z, u_0) \mapsto u^z$ is continuous,*
3. $|u^z - v^z|_{\infty;[0,T]\times\mathbb{R}^n} \le e^{cT} |u_0 - v_0|_{\infty,\mathbb{R}^n}$

Remark

Motivated by Lions-Souganidis theory of viscosity SPDEs

Conditions **(TC)**:

$$
\begin{aligned}
L(t,x,r,p,M) &= -Tr\left[a(t,x)\cdot a^T(t,x)M\right] + b(t,x)\cdot p + c(t,x,r)\\
\Lambda &= (\Lambda_1,\ldots,\Lambda_d)\\
\Lambda_k(t,x,r,p) &= \langle p,\sigma_k(t,x)\rangle + r\cdot\nu_k(t,x) + g_k(t,x)
\end{aligned}
$$

- a, b bounded, continuous in t, Lipschitz in x (uniformly in t)
- c continuous and bounded for bounded r with a lower Lipschitz constant
- All coefficients in Λ are Lip^{γ} for $\gamma > \frac{1}{\alpha} + 2$

- L can be semilinear and degenerate elliptic (i.e. first order case no problem)
- If $z = \left(B, \int B \circ dB\right)$ get approximations theorems, support results, large deviations for SPDEs
- SPDEs with non-Brownian or non-semimartingale (e.g. fBM) noise
- etc.

L^2 solutions

Apply with $z = \boldsymbol{B}(\omega) = \left(B(\omega), \int B \circ dB(\omega)\right)$.

Proposition (Friz-O)

If L is uniformly elliptic, (TC) and $\widetilde{L}^\star$ exists, then $u^{\boldsymbol{B}}$ is "the" unique $L^2(\mathbb{R}^n)$-solution: $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$

$$\langle u_t, \varphi\rangle_{L^2} - \langle u_0, \varphi\rangle_{L^2} = \int_0^t \left\langle u_r, \widetilde{L}^\star \varphi \right\rangle_{L^2} dr + \int_0^t \langle u_r, \Lambda_k^\star \varphi\rangle_{L^2} dB_r^k$$

with $\widetilde{L}\varphi = \left(L + \frac{1}{2}\sum_{k=1}^d \Lambda_k \Lambda_k^\star\right)\varphi$.

L^2 solutions

Apply with $z = \boldsymbol{B}(\omega) = \left(B(\omega), \int B \circ dB(\omega)\right)$.

Proposition (Friz-O)

If L is uniformly elliptic, (TC) and $\widetilde{L}^\star$ exists, then $u^{\boldsymbol{B}}$ is "the" unique $L^2(\mathbb{R}^n)$-solution: $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$

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Remark

- Connects RPDEs to classic L^2-theory
- $u^{\boldsymbol{B}}$ is a robust version (in the equivalence class) of unique L^2-solution
- no Sobolev embedding needed

III. ROBUSTNESS VIA THE KALLIANPUR-STRIEBEL FUNCTIONAL
(joint with Crisan,Diehl,Friz)

Pathwise filtering

$$
\begin{aligned}
dX_t &= \mu(X_t)\,dt + V(X_t) \circ dB_t \ \text{(signal)} \\
dZ_t &= h(X_t)\,dt + d\tilde{B}_t \ \text{(observation)}
\end{aligned}
$$

Pathwise filtering

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- $B = (B^i)_{i=1}^d$ and $\tilde{B} = \left(\tilde{B}^j\right)_{j=1}^e$ standard BM
- $\left\langle B^i, \tilde{B}^j \right\rangle = \rho_{ij}\, t$

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- $B = \left(B^i\right)_{i=1}^d$ and $\tilde{B} = \left(\tilde{B}^j\right)_{j=1}^e$ standard BM
- $\left\langle B^i, \tilde{B}^j \right\rangle = \rho_{ij}\,t$

Goal: Find a robust version of

$$\pi_t(f) = \mathbb{E}\left[f(X_t)\,|\sigma(Z_s, s\in[0,t])\right].$$

Transformation

(For simplicity of presentation $X, Z, B, \tilde{B}$ 1-dimensional)

- Define $\mathbb{P}_0$ via
$$\frac{d\mathbb{P}_0}{d\mathbb{P}} = \exp\left(- \int_0^T h(X_r)\, d\tilde{B} - \frac{1}{2} \int_0^T |h(X_r)|^2\, dr \right)$$
- Set $v_t := \rho h(X_t) + B_t$ and $w_t := v - \rho Z_t$

Transformation

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- Set $v_t := \rho h(X_t) + B_t$ and $w_t := v - \rho Z_t$

Then under $\mathbb{P}_0$,

1. Z and v are standard BM and $\langle v_t, Z_t\rangle = \rho t$
2. $W_t := \frac{1}{\sqrt{1-\rho^2}} w_t$ is standard BM, independent of Z

Transformation

Using this, the signal X becomes

$$dX_t = L_0(X_t)\,dt + L(X_t)\circ dZ_t + M(X_t)\circ dW_t$$

- W, Z independent BM under $\mathbb{P}_0$
- $M = \sqrt{1-\rho^2}V,\ L_0 = \mu - \rho h V,\ L = \rho V$

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- W, Z independent BM under $\mathbb{P}_0$
- $M = \sqrt{1-\rho^2}V,\ L_0 = \mu - \rho h V,\ L = \rho V$
- **This is the key formula to robust filtering**

Kallianpur-Striebel

KS-formula

$$\pi_t(f) = \frac{\rho_t(f)}{\rho_t(1)}$$

with

$$\rho_t(f) = \mathbb{E}_{\mathbb{P}_0}\left[\underbrace{f(X_t)}_{(i)}\underbrace{\exp\left(\int_0^t h(X_r)\,dZ_r - \frac{1}{2}\int_0^t |h(X_r)|^2\,dr\right)}_{(ii)}\middle| Z\right].$$

Kallianpur-Striebel

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Express (i) and (ii) as functionals of Z and show regularity of

$$Z \quad \mapsto \quad (i) \text{ and } Z \mapsto (ii)$$

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A rough path result

Theorem (Crisan, Diehl, Friz, O)

Let $(z^n) \subset C^1\left([0,T],\mathbb{R}^d\right)$ and $z^n \to z$ in rough path metric. If W is a standard BM, then for a.s. the solutions of the SDE

$$dX_t^n = L_0\left(X_t^n\right)dt + L\left(X_t^n\right)dz_t^n + M\left(X_t^n\right)\circ dW_t.$$

converge uniformly to a continuous path $X(\omega)$. We write formally

$$dX_t = L_0\left(X_t\right)dt + L\left(X_t\right)dz_t + M\left(X_t\right)\circ dW_t.$$

Further, $z \mapsto X$ is continuous wrt $|X|_{S^q} := \mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^q\right]^{1/q}$ any $q \geq 1$.

Proof (sketch)

Two possible approaches:

1. Use a "Kunita flow decomposition"
2. Construct a joint rough path of

$$z = (z^1, z^2) \text{ and } \boldsymbol{W} = \left(W, \int dW \otimes dW \right)$$

use rough path continuity

1. Flow decomposition

Lemma

Take $z \in C^1([0,T], \mathbb{R}^e)$, W a standard d-dimensional BM. Let X be the unique SDE solution of

$$dX_t = L_0(X_t)\,dt + L(X_t)\,dz + M(X_t) \circ dW.$$

Consider the transformation

$$\phi(t,x) = x + \int_0^t L(\phi(t,x))\,dz.$$

Then $\overline{X}_t := \phi^{-1}(t, X_t)$ solves the SDE

$$d\overline{X}_t = \overline{L}_0(\overline{X}_t)\,dt + \overline{M}(\overline{X}_t) \circ dW_t$$

with $\overline{M}_{ij} := \sum_k \partial_k \phi_i^{-1}(t, \phi_t(t,x)) M_{k,j}(t, \phi(t,x))$, $\overline{L}_0 := \dots$.

Proof.

Ito! ∎

1. Flow decomposition

- Construct the "rough path flow"

$$\phi^z(t,x) = x + \int_0^t L(\phi^z(t,x))\,dz$$

Note: stable under smooth approximations to z in rough metric

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- Solve the ordinary SDE

$$d\overline{X}^{z}_t = \overline{L}^{z}_0\left(\overline{X}_t\right)dt + \overline{M}^{z}\left(\overline{X}_t\right)\circ dW_t$$

  and set

$$X^{z}_t := \phi^{z}\left(t, \overline{X}^{z}_t\right)$$

1. Flow decomposition

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Note: stable under smooth approximations to z in rough metric

- Solve the ordinary SDE

$$d\overline{X}_t^z = \overline{L}_0^z\left(\overline{X}_t\right)dt + \overline{M}^z\left(\overline{X}_t\right)\circ dW_t$$

and set

$$X_t^z := \phi^z\left(t, \overline{X}_t^z\right)$$

- By construction

$$z \mapsto X^z \text{ is continuous wrt } |X|_{S^q} := \mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^q\right]^{1/q}$$

A Rough&Stochastic DE

Theorem

Let $(z^n) \subset C^1([0,T], \mathbb{R}^d)$ and $z^n \to z$ in rough path metric. If W is a standard BM, then for a.e. ω the solutions of the SDE

$$dX_t^n = L_0(X_t^n)\, dt + L(X_t^n)\, dz_t^n + M(X_t^n) \circ dW_t$$

converge uniformly to a continuous path $X(\omega)$. We write formally

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Further, $z \mapsto X$ is continuous wrt $|X|_{S^q} := \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^q\right]^{1/q}$ any $q \geq 1$.

Executive summary

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\begin{aligned}
dX_t &= \mu(X_t)\,dt + V(X_t) \circ dB_t \text{ (signal)} \\
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Fact: If B and $\tilde{B}$ are correlated then

$$\nexists \theta_t^f : C([0,t], \mathbb{R}^e) \to \mathbb{R}$$

s.t.

- $\pi_t(f) = \theta_t^f \left(Z|_{[0,t]} \right)$ $\mathbb{P}$-a.s.
- θ_t^f is continuous in uniform norm

Executive summary

$$\begin{aligned} dX_t &= \mu(X_t)\,dt + V(X_t)\circ dB_t \text{ (signal)} \\ dZ_t &= h(X_t)\,dt + d\tilde{B}_t \text{ (observation)} \end{aligned}$$

Fact: If B and $\tilde{B}$ are correlated then

$$\exists!\ \theta_t^f : C\left([0,t], G^2(\mathbb{R}^e)\right) \to \mathbb{R}$$

s.t.

- $\pi_t(f) = \theta_t^f\left(\mathbf{Z}|_{[0,t]}\right)$ $\mathbb{P}$-a.s. with

$$\mathbf{Z} = 1 + \int dZ + \int dZ \otimes dZ = \exp\left(Z, \text{Area}(Z)\right)$$

- θ_t^f is continuous in rough path norm (even locally Lipschitz!)

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THANK YOU!