Asymptotic independence and limit laws for Wiener chaos

Jan Rosiński, University of Tennessee

Joint work with Ivan Nourdin

6th International Conference on Stochastic Analysis and its Applications

Bedlewo, Poland, September 2012

Outline

1. Introduction and motivation

$B = (B_t)_{t \geq 0}$: a standard one-dimensional Brownian motion;
f: a symmetric function in $L^2(\mathbb{R}_+^q)$.

$I_q(f)$: the q-tuple Wiener-Itô integral of f with respect to B.

$$
\begin{aligned}
I_q(f) &= \int_{\mathbb{R}_+^d} f(t_1, \ldots, t_q)\, dB_{t_1} \ldots dB_{t_q} \\
&= q! \int_0^\infty \Big(\int_0^{t_q} \cdots \int_0^{t_2} f(t_1, \ldots, t_q)\, dB_{t_1} \ldots dB_{t_{q-1}} \Big) dB_{t_q}
\end{aligned}
$$

where the latter is the usual Itô integral. Thus

$$
E[I_q(f)] = 0, \quad E[I_q(f)^2] = q!\, \|f\|^2_{L^2(\mathbb{R}_+^q)},
$$

$$
E\big[I_q(f) I_r(g)\big] = \begin{cases} 0 & q = r, \\ q! \langle f, g \rangle_{L^2(\mathbb{R}_+^q)} & q = r. \end{cases}
$$

Wiener chaos expansion:

$\forall\ X \in L^2(\Omega, \mathcal{F}^B, P)\ \exists\ f_q \in L^2(\mathbb{R}_+^q)$ symmetric:

$$X = EX + \sum_{q=1}^{\infty} I_q(f_q) \quad \text{in } L^2(\Omega).$$

$I_q(f_q) \in \mathcal{H}_q$, the space of qth Wiener chaos.

1.a. CLT for Wiener chaos

Nualart and Peccati discovered the following surprising CLT:

Theorem (Nualart-Peccati (2005))

Let $F_n = I_q(f_n)$, where $q \geq 2$ is fixed and $f_n \in L^2(\mathbb{R}_+^q)$ are symmetric. Assume also that $E[F_n^2] = 1$ for all n. Then, as $n \to \infty$,

$$F_n \overset{\text{law}}{\to} N(0,1) \quad \Longleftrightarrow \quad E[F_n^4] \to 3.$$

F_n belong to $I_q(f_q)H_q$, the qth Wiener chaos.
It is well-known that $N(0,1)^3$ is not determined by its moments.
Moreover, there are elements of $\mathcal{H}_q$ are not determined by their moments when $q \geq 3$, q odd.

What is the special property of the chaos space $\mathcal{H}_q$ which makes this theorem hold?

1.b. Independence in the Wiener space

Üstünel and Zakai gave the following characterization of the independence of multiple Wiener-Itô integrals.

Theorem (Üstünel-Zakai (1989))

Let $p, q \geq 1$ be integers and let $f \in L^2(\mathbb{R}_+^p)$ and $g \in L^2(\mathbb{R}_+^q)$ be symmetric. Then, random variables $I_p(f)$ and $I_q(g)$ are independent if and only if

$$\int_{\mathbb{R}_+} f(x_1, \ldots, x_{p-1}, u) g(x_p, \ldots, x_{p+q-2}, u)\, du = 0$$

for almost all $(x_1, \ldots, x_{p+q-2}) \in \mathbb{R}_+^{p+q-2}$.

Rosiński and Samorodnitsky (1999) observed the following

Proposition (J.R. and Samorodnitsky (1999))

$$I_p(f) \perp\!\!\!\perp I_q(g) \iff \mathrm{Cov}(I_p(f)^2, I_q(g)^2) = 0. \tag{1}$$

Remark

1. Condition (1) *can be viewed as a generalization of the usual covariance criterion for the independence of jointly Gaussian random variables (the case of $p = q = 1$).*

2. Interestingly, squares of multiple Wiener-Itô integrals are always non negatively correlated.

1.c. The link

The link between the Nualart and Peccati CLT and independence structure in $\mathcal{H}_q$, the qth Wiener chaos, comes via a simple formula:

F and G i.i.d. with zero mean and unit variance

$$\frac{1}{2}\mathrm{Cov}((F+G)^2,(F-G)^2) = E[F^4]-3,$$

AND

Bernstein's theorem: $F+G \perp\!\!\!\perp F-G \iff F$ Gaussian.

2. Notation

$(A, \mathcal{A}, \mu)$: an atomless σ-finite measure space.

$\mathfrak{H} = L^2(A, \mathcal{A}, \mu)$.

W: Gaussian white noise on $(A, \mathcal{A}, \mu)$.

$\mathfrak{H}^{\odot q} = L^2_s(A^q, \mathcal{A}^{\otimes q}, \mu^{\otimes q})$: the space of symmetric square integrable functions on A^q.

Multiple Wiener-Itô integral:

$$I_q(f) := \int_{A^q} f(t_1, \ldots, t_q)\, W(dt_1) \cdots W(dt_q), \quad f \in \mathfrak{H}^{\odot q}.$$

The r-contaction:

$$(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) := \int_{A^r} f(t_1, \dots, t_{p-r}, s_1, \dots, s_r) \\ \times\ g(t_{p-r+1}, \dots, t_{p+q-2r}, s_1, \dots, s_r)\, d\mu(s_1) \dots d\mu(s_r).$$

$f \widetilde{\otimes}_r g$, the symmetrization of $f \otimes_r g$:

$$(f \widetilde{\otimes}_r g)(t_1, \dots, t_{p+q-2r}) \\ = \frac{1}{(p+q-2r)!} \sum_{\pi} (f \otimes_r g)(t_{\pi(1)}, \dots, t_{\pi(p+q-2r)})$$

where π runs over all permutations of $\{1, \dots, p+q-2r\}$.

The multiplication formula: if $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, then

$$I_p(f)I_q(g) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \widetilde{\otimes}_r g).$$

Theorem (I. Nourdin and J.R.)

Let $d \geq 2$, and let $q_1, \ldots, q_d$ be positive integers. Consider vectors

$$(F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geq 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Assume that for some random vector $(U_1, \ldots, U_d)$,

$$(F_{1,n}, \ldots, F_{d,n}) \overset{\text{law}}{\to} (U_1, \ldots, U_d) \quad \text{as } n \to \infty.$$

Then U_i's admit moments of all orders and the following three conditions are equivalent:

Theorem (I. Nourdin and J.R., cont.)

(a) $E[U_1^{k_1} \dots U_d^{k_d}] = E[U_1^{k_1}] \dots E[U_d^{k_d}]$ *for all* $k_1, \dots, k_d \in \mathbb{N}$ *(i.e.,* $U_1, \dots, U_d$ *are* *<u>moment</u>* *independent);*

(b) $\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *for all* $i \neq j$*;*

(c) $\lim_{n\to\infty} \|f_{i,n} \otimes_r f_{j,n}\| = 0$ *for all* $i \neq j$ *and all* $r = 1, \dots, q_i \wedge q_j$*;*

Moreover, if the distribution of each U_i *is determined by its moments, then (a) is equivalent to that*

(d) $U_1, \dots, U_d$ *are independent.*

Remark

Condition (**a**) can also be stated in terms of cumulants. Recall that the joint cumulant of random variables $X_1, \ldots, X_n$ is defined by

$$\kappa(X_1, \ldots, X_n) = (-i)^n \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \log E[e^{i(t_1 X_1 + \cdots + t_n X_n)}]\Big|_{t_1=0,\ldots,t_n=0},$$

provided $E|X_1 \cdots X_n| < \infty$. When all X_i are equal to X, $\kappa(X, \ldots, X) = \kappa_n(X)$, the usual nth cumulant of X. Then (**a**) is equivalent to

(a') *for all integers* $1 \leq j_1 < \cdots < j_k \leq d$, $k \geq 2$, *and* $m_1, \ldots, m_k \geq 1$

$$\kappa(\underbrace{U_{j_1}, \ldots, U_{j_1}}_{m_1}, \ldots, \underbrace{U_{j_k}, \ldots, U_{j_k}}_{m_k}) = 0.$$

Short proof of the 4th moment theorem of Nualart-Peccati

Suppose that $E[F_n^4] \to 3$, $E[F_n^2] = 1$, $F_n = I_q(f_n)$. Thus (F_n) is tight. WLOG assume that $F_n \overset{\text{law}}{\to} Y$. Must show that $Y \sim N(0,1)$.

Let G_n be an independent copy of F_n of the form $G_n = I_q(g_n)$ with $f_n \otimes_1 g_n = 0$. Then have

$$(I_q(f_n + g_n), I_q(f_n - g_n)) = (F_n + G_n, F_n - G_n) \overset{\text{law}}{\to} (Y + Z, Y - Z)$$

where Z stands for an independent copy of Y. Since

$$\frac{1}{2}\mathrm{Cov}[(F_n + G_n)^2, (F_n - G_n)^2] = E[F_n^4] - 3 \to 0,$$

$Y + Z$ and $Y - Z$ are moment-independent. If they are actually independent, then by the classical Bernstein Theorem are done.

In general, for all $m_1, m_2 \geq 1$ we have

$$\kappa(\underbrace{Y+Z,\ldots,Y+Z}_{m_1},\underbrace{Y-Z,\ldots,Y-Z}_{m_2}) = 0.$$

Taking $m_1 = n-2$, $m_2 = 2$, where $n \geq 3$, we get

$$\begin{aligned} 0 &= \kappa(\underbrace{Y+Z,\ldots,Y+Z}_{n-2}, Y-Z, Y-Z) \\ &= \kappa(\underbrace{Y,\ldots,Y}_{n}) + \kappa(\underbrace{Z,\ldots,Z}_{n}) = 2\kappa_n(Y), \end{aligned}$$

where we used the multilinearity of κ and that $Y \perp\!\!\!\perp Z$. Therefore,

$$\kappa_1(Y) = EY = 0, \quad \kappa_2(Y) = \mathrm{Var}(Y) = 1, \quad \kappa_n(Y) = 0 \ \forall n \geq 3$$

implying $Y \sim N(0,1)$. $\square$

We will give an extension of the above theorem to the vector version (needed for stochastic processes).

Definition

For each $n \geq 1$, let $F_n = (F_{i,n})_{i\in I}$ be a family of real-valued random variables indexed by a finite set I. Consider partition of I into disjoint blocks I_k, so that $I = \cup_{k=1}^{d} I_k$. We say that vectors $(F_{i,n})_{i\in I_k}$, $k = 1, \ldots, d$ are asymptotically moment-independent if each $F_{i,n}$ admits moments of all orders and for any sequence $(\ell_i)_{i\in I}$ of non-negative integers,

$$\lim_{n\to\infty} \Big\{ E\big[\prod_{i\in I} F_{i,n}^{\ell_i}\big] - \prod_{k=1}^{d} E\big[\prod_{i\in I_k} F_{i,n}^{\ell_i}\big] \Big\} = 0.$$

Theorem (I. Nourdin and J.R.)

Let I be a finite set and $(q_i)_{i\in I}$ be a sequence of non-negative integers. For each $n \geq 1$, let $F_n = (F_{i,n})_{i\in I}$ be a family of multiple Wiener-Itô integrals, where $F_{i,n} = I_{q_i}(f_{i,n})$ with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Assume that for every $i \in I$

$$\sup_n E\left[F_{i,n}^2\right] < \infty.$$

Given a partition of I into disjoint blocks I_k, the following conditions are equivalent:

- **(a)** *random vectors $(F_{i,n})_{i\in I_k}$, $k = 1, \ldots, d$ are asymptotically moment-independent;*
- **(b)** $\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *for every i, j from different blocks;*
- **(c)** $\lim_{n\to\infty} \|f_{i,n} \otimes_r f_{j,n}\| = 0$ *for every i, j from different blocks and $r = 1, \ldots, q_i \wedge q_j$.*

Remark

Under notation of the above theorem, $(F_{i,n})_{i \in I_k}$, $k = 1, \dots, d$ are asymptotically moment-independent if and only if

(b') *for every* $1 \leq k \neq l \leq d$

$$\lim_{n \to \infty} \mathrm{Cov}(\|(F_{i,n})_{i \in I_k}\|^2, \|(F_{i,n})_{i \in I_l}\|^2) = 0,$$

where $\|\cdot\|$ denotes Euclidean norms.

Corollary (Joint convergence)

Under notation of the above theorem, let $(U_i)_{i \in I}$ *be a random vector such that*

- $\forall\, k \ \ (F_{i,n})_{i \in I_k} \stackrel{\text{law}}{\to} (U_i)_{i \in I_k}$;
- $(U_i)_{i \in I_k}$, $k = 1, \dots, d$ *are independent;*
- $\lim_{n \to \infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0 \ \ \forall\, i, j$ *from different blocks;*
- $\mathcal{L}(U_i)$ *is determined by its moments* $\forall\, i \in I$.

Then

$$(F_{i,n})_{i \in I} \stackrel{\text{law}}{\to} (U_i)_{i \in I}, \quad n \to \infty.$$

4.1. Generalizing a result by Peccati and Tudor.

Theorem (Peccati-Tudor)

Let $d \geq 2$, and let $q_1, \ldots, q_d$ be positive integers. Consider vectors

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geq 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$. Assume moreover that, for all $i, j = 1, \ldots, d$,

$$E[F_{i,n} F_{j,n}] \to \sigma_{ij}.$$

Then, as $n \to \infty$, the following two conditions are equivalent:

(i) $F_n \stackrel{\text{law}}{\to} N \sim N_d(0, \Sigma)$, *where* $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq d}$;

(ii) $E[\|F_n\|^4] \to E[\|N\|^4]$.

Proof.

Let $G_n = (G_{1,n}, \ldots, G_{d,n}) = (I_{q_1}(g_{1,n}), \ldots, I_{q_d}(g_{d,n}))$ be and independent copy of F_n.

$$\begin{aligned}
&\frac{1}{2}\mathrm{Cov}[\|F_n + G_n\|^2, \|F_n - G_n\|^2] \\
&\qquad = E[\|F_n\|^4] - \left(E[\|F_n\|^2]\right)^2 - 2\sum_{i,j=1}^{d} \mathrm{Cov}(F_{i,n}, G_{j,n}) \\
&\qquad = E[\|F_n\|^4] - E[\|N\|^4] + o(1).
\end{aligned}$$

Hence $F_n + G_n, F_n - G_n$ are moment-independent $\Leftrightarrow$ $E[\|F_n\|^4] \to E[\|N\|^4]$. Taking one-dimensional projections we proceed by cumulants as above. $\square$

The following result associates neat estimates to the above theorem. Consider a vector

$$F = (F_1, \dots, F_d) = (I_{q_1}(f_1), \dots, I_{q_d}(f_d))$$

with $f_i \in \mathfrak{H}^{\odot q_i}$. Let $\Sigma = (\sigma_{ij})_{1 \le i,j \le d}$ be the covariance matrix of F, that is, $\sigma_{ij} = E[F_i F_j]$.

Theorem (I. Nourdin and J.R.)

Let $N \sim N_d(0, \Sigma)$ and assume that Σ is invertible.
(i) Then, for any Lipschitz function $h : \mathbb{R}^d \to \mathbb{R}$ we have

$$\begin{aligned} &\big| E[h(F)] - E[h(N)] \big| \\ &\qquad \le \sqrt{d}\, \|\Sigma\|_{op}^{1/2} \|\Sigma^{-1}\|_{op} \|h\|_{Lip} \sqrt{E\|F\|^4 - E\|N\|^4}. \end{aligned}$$

Theorem (I. Nourdin and J.R., cont.)

(ii) For any C^2-function $h : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\|h''\|_\infty = \max_{1\le i,j\le d} \sup_{x\in\mathbb{R}^d} \left| \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \right| < \infty,$$

we have

$$\left|E[h(F)] - E[h(N)]\right| \le \frac{1}{2}\|h''\|_\infty \sqrt{E\|F\|^4 - E\|N\|^4}.$$

4.2. A multivariate version of the convergence towards χ^2

$G(\nu)$ will denote a random variable distributed according to the centered χ^2 distribution with $\nu > 0$ degrees of freedom.

If $\nu > 0$ is an integer, then $G(\nu) \stackrel{\text{law}}{=} \sum_{i=1}^{\nu}(Z_i^2 - 1)$, where $Z_1, \ldots, Z_\nu$ are i.i.d. $N(0,1)$ random variables.

In general, $G(\nu)$ is a centered gamma random variable with a shape parameter $\nu/2$ and scale parameter 2.

$G(\nu)$ is determined by its moments, for any $\nu > 0$ (it's easy to check the Carleman's condition).

The following is a multivariate extension of a result of Nourdin and Peccati [1]. This was an open problem.

Theorem (I. Nourdin and J.R.)

Let $d \geq 2$, let $\nu_1, \ldots, \nu_d$ be positive reals, and let $q_1, \ldots, q_d \geq 2$ be even integers. Consider vectors

$$F_n = (F_{1,n}, \ldots, F_{d,n}) = \big(I_{q_1}(f_{1,n}), \ldots, I_{q_d}(f_{d,n})\big), \quad n \geq 1,$$

with $f_{i,n} \in \mathfrak{H}^{\odot q_i}$, and assume that $\lim_{n\to\infty} E[F_{i,n}^2] = 2\nu_i$ for every i. Assume also that:

(i) $E[F_{i,n}^4] - 12E[F_{i,n}^3] \to 12\nu_i^2 - 48\nu_i \;\; \forall i$;

(ii) $\lim_{n\to\infty} \mathrm{Cov}(F_{i,n}^2, F_{j,n}^2) = 0$ *whenever* $q_i = q_j$ *and* $i \neq j$;

(ii) $\lim_{n\to\infty} E[F_{i,n}^2 F_{j,n}] = 0$ *whenever* $q_j = 2q_i$.

Theorem (I. Nourdin and J.R., cont.)

Then

$$(F_{1,n}, \dots, F_{d,n}) \stackrel{\text{law}}{\to} (G(\nu_1), \dots, G(\nu_d))$$

where $G(\nu_1), \dots, G(\nu_d)$ *are independent random variables having centered* χ^2 *distributions with* $\nu_1, \dots, \nu_d$ *degrees of freedom, respectively.*

Example

Consider $F_n = (F_{1,n}, F_{2,n}) = (I_{q_1}(f_{1,n}), I_{q_2}(f_{2,n}))$, where $2 \leq q_1 \leq q_2$. Suppose that

$$E[F_{1,n}^4] - 6E[F_{1,n}^3] \to -3 \quad \text{and}$$
$$E[F_{2,n}^4] - 6E[F_{2,n}^3] \to 0, \quad n \to \infty.$$

When $q_1 = q_2$ or $q_2 = 2q_1$ we require additionally:

$$\mathrm{Cov}(F_{1,n}^2, F_{2,n}^2) \to 0 \ (q_1 = q_2), \quad E[F_{1,n}^2 F_{2,n}] \to 0 \ (q_2 = 2q_1).$$

Then

$$F_n \stackrel{\text{law}}{\to} (V_1 - 1, V_2 + V_3 - 2)$$

where V_1, V_2, V_3 are i.i.d. standard exponential random variables.

Theorem (I. Nourdin and J.R.)

Let $p_1, \dots, p_r, q_1, \dots, q_s$ be positive integers such that $\min p_i \geq \max q_j$. *Consider*

$$F_n = (F_{1,n}, \dots, F_{r,n}) = (I_{p_1}(f_{1,n}), \dots, I_{p_r}(f_{r,n})),$$

$$G_n = (G_{1,n}, \dots, G_{s,n}) = (I_{q_1}(g_{1,n}), \dots, I_{q_s}(g_{s,n})),$$

with $f_{i,n} \in \mathfrak{H}^{\odot p_i}$ and $g_{j,n} \in \mathfrak{H}^{\odot q_j}$. Suppose that as $n \to \infty$

$$F_n \overset{\text{law}}{\to} N \sim N(0, \Sigma) \quad \textit{and} \quad G_n \overset{\text{law}}{\to} V,$$

where $N \perp\!\!\!\perp V$ and $\mathcal{L}(V)$ is determined by its joint moments. If $E[F_{i,n}G_{j,n}] \to 0$ (trivially holds when $p_i \neq q_j$) for all i, j, then

$$(F_n, G_n) \overset{\text{law}}{\to} (N, V) \quad \textit{jointly, as } n \to \infty.$$

4.4. Partial sums associated with Hermite polynomials

Consider a centered stationary Gaussian sequence $\{G_k\}_{k\in\mathbb{Z}}$ with unit variance and covariance

$$r(k) = E[G_k G_0] = k^{-D} L(k), \quad k \geq 1,$$

where $D > 0$ and $L : (0, \infty) \to (0, \infty)$ a function which is slowly varying at infinity and bounded away from 0 and infinity on every compact subset of $[0, \infty)$. For any integer $q \geq 1$, we write

$$S_{q,n}(t) = \sum_{k=1}^{\lfloor nt \rfloor} H_q(G_k), \quad t \geq 0,$$

where H_q denotes the qth Hermite polynomial. As an application of the previous theorem we give the following result.

Proposition (I. Nourdin and J.R.)

Let $q \geq 3$ *and* $D \in (\frac{1}{q}, \frac{1}{2})$. *Then*

$$\left(\frac{S_{q,n}}{\sqrt{n}}, \frac{S_{2,n}}{n^{1-D} L(n)} \right) \stackrel{\text{f.d.d.}}{\to} (a\, B, b\, R_{1-D}),$$

with B *a Brownian motion and* R_{1-D} *a Rosenblatt process of parameter* $1 - D$ *independent of* B. *Here*

$$a = \Big[q! \sum_{k \in \mathbb{Z}} r(k)^q \Big]^{1/2} \quad \text{and} \quad b = \Big[(1-D)(1-2D) \Big]^{-1/2}.$$

The convergence of marginals was proved by Breuer and Major (1983) and Taqqu (1975). Since the Rosenblatt process lives in the homogeneous chaos of order 2, which is determined by its moments, our theorem applies.

Thank you!

References

I. Nourdin and G. Peccati (2009). Non-central convergence of multiple integrals. *Ann. Probab.* **37**(4), 1412-1426.

I. Nourdin and J. Rosiński (2012). Asymptotic independence of multiple Wiener-Ito integrals and the resulting limit laws. Preprint. `http://arxiv.org/pdf/1112.5070v2.pdf`

D. Nualart and G. Peccati (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33**, no. 1, 177-193.

G. Peccati and C.A. Tudor (2005). Gaussian limits for vector-valued multiple stochastic integrals. In: *Séminaire de Probabilités XXXVIII* , 247-262. Lecture Notes in Math. **1857**, Springer-Verlag, Berlin.

J. Rosiński (2010). Conference talk.
www.ambitprocesses.au.dk/fileadmin/pdfs/ambit/Rosinski.pdf

J. Rosiński and G. Samorodnitsky (1999). Product formula, tails and independence of multiple stable integrals. *Advances in stochastic inequalities* (Atlanta, GA, 1997), 169-194, *Contemp. Math.* **234**, Amer. Math. Soc., Providence, RI.

A.S. Üstünel and M. Zakai (1989). On independence and conditioning on Wiener space. *Ann. Probab.* **17** no. 4, 1441-1453.