Faculty of Mathematics,
Computer Science and Econometrics
University of Zielona Góra

APPLICATION OF SEMIMARTINGALE MEASURE TO THE INVESTIGATION OF STOCHASTIC INCLUSIONS

Joachim Syga

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CONTENT:

1. Doléans-Dade measure
and semimartingale measure.

2. Itô inclusion.

3. Stratonovich inclusion.

$$\dot{x}(t) = f(x(t))$$

$\swarrow$ $\searrow$

$$\dot{x}(t) \in F(x(t)) \qquad\qquad dx_t = f(x_t)dt + g(x_t)dW_t$$

$$dx_t = h(x_t)dZ_t$$

K. Kuratowski,

C. Ryll-Nardzewski (1965)

C. Castaing (1967)

J.P.Aubin,

A. Cellina (1984)

K. Itô (1946)

P.A.Meyer (1967)

I.I. Gihman,

A.V. Skorohod (1972)

P. Protter (1977)

$\searrow$ $\swarrow$

$$dx_t \in F(x_t)dZ_t$$

M. Kisielewicz (1993)

N.U. Ahmed (1994)

M. Michta (1995)

M. Motyl (1995)

J.P. Aubin, G. Da Prato (1998)

1. Doléans-Dade measure and semimartingale measure.

$(\Omega, \mathcal{F}, \mathbb{F}, P)$ – a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual hypothesis.

DOLÉANS-DADE MEASURE

M – a square-integrable martingale,

$$s, t \in [0,T], \qquad s \leq t;$$

$B \subset \Omega$ – an $\mathcal{F}_s$-measurable set

$$[0,T] \times \Omega \longrightarrow (s,t] \times B$$

$$\mu_M((s,t] \times B) = E(\mathbb{1}_B (M_t - M_s)^2)$$

$$(s,t] \times B;$$

$$\downarrow$$

$$\mathcal{P}$$

$L^2_M = \{f \in \mathcal{P} : E(\int_0^T |f_\tau|^2 d[M,M]_\tau) < \infty\}$

with $\|f\|_{L^2_M} = (\int_{[0,T]\times\Omega} |f|^2 d\mu_M)^{1/2}$

$G : [0,T] \times \Omega \to 2^{\mathbb{R}^n}$, $G = (G_t)_{t\in[0,T]}$ – a predictable set-valued process

$$\mathcal{S}_M(G) := \{f \in L^2_M : \ f(t,\omega) \in G(t,\omega), \\ \mu_M\text{– a.e.}\}.$$

Let G be M-integrably bounded:
that exists process $m \in L^2_M$ such that
$H_{\mathbb{R}^n}(G,0) \le m \ \mu_M$ – a.e.

A set-valued stochastic integral of G with respect to M is defined as a set

$$\int_s^t G_\tau dM_\tau = \{\int_s^t g_\tau dM_\tau \ : g \in \mathcal{S}_M(G)\},$$

for $0 \le s < t \le T$ and

$$\int G_\tau dM_\tau = (\int_0^t G_\tau dM_\tau)_{t\in[0,T]}.$$

Theorem 1 (J.Motyl, J.S., [5] 2006)
Let M be a square-integrable martingale, $M_0 = 0$, and let G be an M-integrably bounded and predictable set-valued process. Then

$$dist_{L^2(\Omega)}\Big(\int_0^t f_\tau dM_\tau, \int_0^t G_\tau dM_\tau\Big) = \Big(E\int_0^t dist^2_{\mathbb{R}^n}(f_\tau, G_\tau)d[M,M]_\tau\Big)^{1/2}$$

for $f \in L^2_M$ and $t \geq 0$.

Theorem 2 (J.Motyl, J.S., [5] 2006)
Let M be a square-integrable martingale, $M_0 = 0$, and let F, G be M-integrably bounded and predictable set-valued processes. Then

$$H_{L^2(\Omega)}\Big(\int_0^t G_\tau dM_\tau, \int_0^t F_\tau dM_\tau\Big) \leq \Big(E\int_0^t H^2_{\mathbb{R}^n}(G_\tau, F_\tau)d[M,M]_\tau\Big)^{1/2}$$

each $t \geq 0$.

$\mathcal{H}^p$, $1 \le p \le \infty$ – a space of one-dimensional semimartingales $Z : [0,T] \times \Omega \to \mathbb{R}$, $Z = (Z_t)_{t\in[0,T]}$, with a canonical decomposition $Z = N + A$, and a norm

$$\|Z\|_{\mathcal{H}^p} = \|[N,N]_T^{1/2} + \int_0^T |dA_\tau|\,\|_{L^p(\Omega)}.$$

$\mathcal{H}^p_n$, $1 \le p \le \infty$ – a space of n-dimensional semimartingales $Z = (Z^1,\ldots,Z^n)$, $Z^i \in \mathcal{H}^p$, $i = 1,\ldots,n$, with a norm

$$\|Z\|_{\mathcal{H}^p_n} = (\textstyle\sum_{i=1}^n \|Z^i\|^2_{\mathcal{H}^p})^{1/2}.$$

SEMIMARTINGALE MEASURE

Z – an $\mathcal{H}^2$-semimartingale, $Z = N + A$.

For a local martingale $N \in \mathcal{H}^2$ we define a Doléans-Dade measure μ_N,
(applying Cor. II.6.4 of P.Protter [6], 2005).

For an FV-process $A \in \mathcal{H}^2$ we define a measure ν_A on $\mathcal{P}$:

$D \subset [0, T] \times \Omega$ – a predictable set

$$\nu_A(D) = \int_\Omega \int_0^T \mathbb{1}_D(\omega, t)\alpha(\omega, dt)P(d\omega)$$

$$\alpha(\omega, dt) = |dA_t(\omega)| \cdot \int_0^T |dA_t(\omega)|$$

For a semimartingale $Z \in \mathcal{H}^2$ with a canonical decomposition $Z = N + A$ we define a measure μ_Z as:

$$\mu_Z = \mu_N + \nu_A$$

$Z \in \mathcal{H}^2$, $Z = N + A$

$$L_Z^2 = \{f \in \mathcal{P} : \int_{[0,T]\times\Omega} |f|^2 d\mu_Z < \infty\}.$$

with $\|f\|_{L_Z^2} = (\int_{[0,T]\times\Omega} |f|^2 d\mu_Z)^{1/2}$

Theorem 3 (J.S., [7], 2012)
For $Z \in \mathcal{H}^2$ and $f \in L_Z^2$ we have

$$\| \int f_\tau dZ_\tau \|_{\mathcal{H}_n^2}^2 \leq 2\|f\|_{L_Z^2}^2.$$

$Z \in \mathcal{H}^2$, $Z_0 = 0$; $G : [0,T] \times \Omega \to 2^{\mathbb{R}^n}$,
$G = (G_t)_{t\in[0,T]}$ – a predictable set-valued process

$$\mathcal{S}_Z(G) := \{f \in L^2_Z : \ f(t,\omega) \in G(t,\omega), \quad \mu_Z\text{– a.e.}\}.$$

Definition 1 Let $Z = (Z_t)_{t\in[0,T]}$ be an $\mathcal{H}^2$–semimartingale, $Z_0 = 0$.
Let $G = (G_t)_{t\in[0,T]}$ be a predictable Z-integrably bounded set-valued process.
We define set-valued integrals

$$\int_s^t G_\tau dZ_\tau = \{\int_s^t g_\tau dZ_\tau \ : g \in \mathcal{S}_Z(G)\},$$

for $0 \le s < t \le T$ and

$$\int G_\tau dZ_\tau = (\int_0^t G_\tau dZ_\tau)_{t\in[0,T]}.$$

Remark 1 *A set-valued process G is Z-integrably bounded, if there exists a process $m \in L^2_Z$ such that*

$$H_{\mathbb{R}^n}(G,0) \le m \ \mu_Z \text{ – a.e.}$$

Theorem 4 (J.Motyl, J.S., [5], 2006)
Let $Z = (Z_t)_{t\in[0,T]}$ *be an* $\mathcal{H}^2$*–semimartingale.*
Let $F = (F_t)_{t\in[0,T]}, G = (G_t)_{t\in[0,T]}$
be predictable Z*-integrably bounded set-valued processes.*
Then there exists a constant $K \geq 0$ *such that*

$$H^2_{\mathcal{H}^2_n}(\int G_\tau dZ_\tau, \int F_\tau dZ_\tau)$$

$$\leq K \cdot \| \int H^2_{\mathbb{R}^n}(G_\tau, F_\tau) dZ_\tau \|_{\mathcal{H}^2},$$

where
$K = 2 \cdot \max\{\| \int_0^T |dA_t(\omega)| \|_{L^2(\Omega)}, E[N,N]_T^{1/2}\}$

2. Itô inclusion.

S^2 – a space of adapted single-valued càdlàg processes
$x : [0, T] \times \Omega \to \mathbb{R}^n$, $x = (x_t)_{t\in[0,T]}$ with

$$\|x\|_{S^2} = \| \sup_{t\in[0,T]} |x_t| \|_{L^2(\Omega)}.$$

Let $Z = (Z_t)_{t\in[0,T]}$ be one-dimensional $\mathcal{H}^2$–semimartingale, $Z_0 = 0$,
$F : [0, T] \times \mathbb{R}^n \to Cl\,Conv(\mathbb{R}^n)$.

For $0 \le s < t \le T$ we consider a stochastic inclusion:

$$x_t - x_s \in cl_{L^2(\Omega)}(\int_s^t F(\tau, x_\tau)dZ_\tau)$$

(SI)

$$x_0 = \xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^n)$$

Definition 2 A process $x \in S^2$ is a solution of the stochastic inclusion (SI), if $x_0 = \xi$ and for any $0 \leq s < t \leq T$ a random variable $x_t - x_s$ belongs to the set

$$cl_{L^2(\Omega)}(\int_s^t F(\tau, x_\tau) dZ_\tau).$$

Assumption 1 Let $F : [0,T] \times \mathbb{R}^n \to ClConv(\mathbb{R}^n)$ be a multifunction satisfying:

(1) $F : [0,T] \times \mathbb{R}^n \to Cl\,Conv(\mathbb{R}^n)$ is a $(\beta, \mathcal{F})$-measurable multifunction;

(2) $F : [0,T] \times \mathbb{R}^n \to Cl\,Conv(\mathbb{R}^n)$ is a Lipschitz multifunction:
i.e. there exists a constant D such that for all $t \in [0,T]$ and $u, v \in \mathbb{R}^n$

$$H(F(t,u), F(t,v)) \leq D|u - v|;$$

(3) For any $x \in S^2$ a process $(F(t, x_{t-}))_{t \in [0,T]}$ is Z-integrably bounded.

Theorem 5 (J.S., [7], 2012)
Let $Z = (Z_t)_{t \in [0,T]}$ *be an* $\mathcal{H}^\infty$*–semimartingale,* $Z_0 = 0$,
$F : [0,T] \times \mathbb{R}^n \to Cl\, Conv(\mathbb{R}^n)$ *satisfies the Assumption 1.*

Then for any $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^n)$ *there exists a solution of the inclusion (SI).*

Theorem 6 (J.S., [7], 2012)
Let $Z = (Z_t)_{t \in [0,T]}$ *be an* $\mathcal{H}^\infty$*–semimartingale,* $Z_0 = 0$ *decomposed into a sum* $Z = N + A$*, where* N *is a local martingale and* A *is a deterministic FV-process.*
Let $F : [0,T] \times \mathbb{R}^n \to Cl\, Conv(\mathbb{R}^n)$ *satisfies the Assumption 1.*

Then for any $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^n)$ *the set of solutions of the inclusion (SI) is closed in* S^2.

Assumption 2 Let $F : [0,T]\times\mathbb{R}^n \to ClConv(\mathbb{R}^n)$ be a multifunction satisfying:

(1) $F : [0,T] \times \mathbb{R}^n \to Cl\, Conv(\mathbb{R}^n)$
is a Carathéodory-type multifunction;

(2) For any $x \in S^2$ a set-valued process $(F(t, x_{t-}))_{t\in[0,T]}$ is Z-integrably bounded.

Assumption 3 Let $Z = (Z_t)_{t\in[0,T]}$ be an $\mathcal{H}^2-$semimartingale such that the measure μ_Z is absolutely continuous with respect to $\lambda\otimes P$ on $\mathcal{P}$, where λ – a Lebesgue measure on $[0,T]$.

Theorem 7 (J.S., [7], 2012)
Let $Z = (Z_t)_{t\in[0,T]}$ be an $\mathcal{H}^2-$semimartingale, $Z_0 = 0$ satisfying Assumption 3.
Let $F : [0,T] \times \mathbb{R}^n \to Cl\, Conv(\mathbb{R}^n)$ satisfies Assumption 2.

Then for any $\xi \in L^2(\Omega, \mathcal{F}_0, P; \mathbb{R}^n)$ the set of solutions of the inclusion (SI) is closed in S^2.

3. Stratonovich inclusion

$(\Omega, \mathcal{F}, \mathbb{F}, P)$ – a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ satisfying the usual hypothesis.

Definition 3 (M.Errami, F.Russo, P.Vallois, [9], 2002) For a stochastic càdlàg process g we set

$$\tilde{g}_t = (g_t)^{\sim} = g_{(1-t)-},$$

which is called a time-reversed process.

Definition 4 Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider on Ω two filtrations $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le 1}$ and $\mathbb{H} = (\mathcal{H}_t)_{0\le t\le 1}$ satisfying usual hypothesis.

A càdlàg process x is $(\mathbb{F}, \mathbb{H})$–reversible if x is an $\mathbb{F}$–adapted process on $[0,1]$ and $\tilde{x}$ is an $\mathbb{H}$–adapted process on $[0,1]$.

A càdlàg process Z is an $(\mathbb{F}, \mathbb{H})$–reversible semimartingale, if Z is an $\mathbb{F}$–semimartingale on $[0,1]$ and $\tilde{Z}$ is an $\mathbb{H}$–semimartingale on $[0,1)$ (P.Protter [6], 2005).

Definition 5 (M.Errami, F.Russo, P.Vallois, [9], 2002) *Let* $\{\tau_n\}$ *denote a subdivision of* $[0,1]$, $\tau_n = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$.
We set $|\tau_n| = \sup_i (t_{i+1} - t_i)$.
Let g *and* Z *be càdlàg processes continuous for* $t = 0$ *and* $t = 1$. *We define*

$$I^-_{\tau_n}(g, dZ)(a)$$
$$= \sum_i g(t_i \wedge a)(Z(t_{i+1} \wedge a) - Z(t_i \wedge a)),$$
$$I^+_{\tau_n}(g, dZ)(a)$$
$$= \sum_i g(t_{i+1} \wedge a)(Z(t_{i+1} \wedge a) - Z(t_i \wedge a)),$$
$$I^o_{\tau_n}(g, dZ)(a)$$
$$= 1/2\,(I^+_{\tau_n}(g, dZ)(a) + I^-_{\tau_n}(g, dZ)(a)).$$

The corresponding limits of above sums are called forward, backward and Stratonovich integrals, respectively, and they are denoted by

$$\int_{(0,a]} g d^- Z, \qquad \int_{(0,a]} g d^+ Z, \qquad \int_{(0,a]} g \circ dZ.$$

(J.Motyl, J.S., [10] 2010)

Definition 6 A stochastic set-valued process G is càdlàg if it has right continuous sample paths with left limits with respect to the Hausdorff metric.
A stochastic set-valued process G is RV–càdlàg if it is càdlàg and continuous for $t = 1$.

Definition 7 For a stochastic set-valued càdlàg process G we set

$$\tilde{G}_t = (G_t)^{\sim} = G_{(1-t)-},$$

which is called a time-reversed process.
The limit of the set-valued map is taken with respect to the Hausdorff metric.

Definition 8 A set-valued càdlàg process G is $(\mathbb{F}, \mathbb{H})$–reversible if G is an $\mathbb{F}$–adapted process on $[0, 1]$ and $\tilde{G}$ is an $\mathbb{H}$–adapted process on $[0, 1]$.

Lemma 8 *Let G be a set-valued $(\mathbb{F}, \mathbb{H})$–reversible process. Then there exists a selection g of G being an $(\mathbb{F}, \mathbb{H})$–reversible process.*

Definition 9 Let G be a set-valued $(\mathbb{F}, \mathbb{H})$–reversible RV–càdlàg process and let Z be an $(\mathbb{F}, \mathbb{H})$–reversible semimartingale, $Z_0 = 0$.
Let $S(G)$ denote a family of all $(\mathbb{F}, \mathbb{H})$–reversible RV–càdlàg selections of G.
For every $0 \le a < b \le 1$ we define

$$\int_{(a,b]} G \circ dZ$$

$$= \{1/2\, (\int_{(a,b]} g d^{-} Z + \int_{(a,b]} g d^{+} Z) \ : \ g \in S(G)\}$$

$$= \{1/2\, (\int_{(a,b]} g_{\tau-} dZ_{\tau} - \int_{[1-b,1-a)} \tilde{g}_{\tau-} d\tilde{Z}_{\tau}) \quad g \in S(G)\}.$$

Definition 10 Let Z be an RV–càdlàg process. Let x be a stochastic process such that for every $0 \le a < b \le 1$ there exist RV–càdlàg processes $g^{a,b}$ and $h^{a,b}$ satisfying $x_b - x_a = \int_{(a,b]} g^{a,b} d^- Z + \int_{(a,b]} h^{a,b} d^+ Z.$

A process x is called decomposable if there exist RV–càdlàg processes u, v, $u_0 \in \mathcal{F}_0$ and $v_1 \in \mathcal{H}_0$ such that

(i) $u_b - u_a = \int_{(a,b]} g^{a,b} d^- Z$,
$v_b - v_a = \int_{(a,b]} h^{a,b} d^+ Z$,
for every $0 \le a < b \le 1$,

(ii) $x = u + v$.

(J.S., [8] 2012)

Theorem 9 *Let Z be an $(\mathbb{F},\mathbb{H})$–reversible semimartingale from H^2, $Z_0 = 0$. Let G be a set-valued $(\mathbb{F},\mathbb{H})$–reversible process left continuous for $t = 1$ and integrably bounded by a process m. If a decomposable RV–càdlàg process $x = u+v$ satisfies $x_b - x_a \in \int_{(a,b]} G \circ dZ$, for every $0 \leq a < b \leq 1$,*
then there exists a pair $(g, \tilde{h})$ of stochastic processes such that $g \in cl_{L^2_Z} S_Z(G_-)$, $\tilde{h} \in cl_{L^2_{\tilde{Z}}} S_{\tilde{Z}}(\tilde{G}_-)$ and for all $0 < t \leq 1$

$$x_t = x_0 + 1/2 \int_{(0,t]} g_\tau dZ_\tau - 1/2 \int_{[1-t,1)} \tilde{h}_\tau d\tilde{Z}$$

a.s.

Definition 11 *Let* Z *be an* $(\mathbb{F}, \mathbb{H})$*–reversible semimartingale from* $\mathcal{H}^\infty$*,* $Z_0 = 0$*.*
Let $F : [0,1] \times \mathbb{R}^n \to Comp\,Conv(\mathbb{R}^n)$*.*
For $s, t \in [0, T]$*,* $s < t$ *we consider the Stratonovich-type stochastic inclusion*

$$x_t - x_s \in cl_{L^2(\Omega)}(\int_{(s,t]} F(\tau, x_\tau) \circ dZ) \quad \text{(SSI)}$$

with $x_0 = \xi \in L^2(\Omega, [\mathcal{F}_0, \mathcal{H}_1], P; \mathbb{R}^n)$*.*

A process $x \in S^2([0,1])$ *is a solution of the stochastic inclusion (SSI), if* $x_0 = \xi$ *and for any* $s, t \in [0,1]$*,* $s < t$ *a random variable* $x_t - x_s$ *belongs to the set*

$$cl_{L^2(\Omega)}(\int_{(s,t]} F(\tau, x_\tau) \circ dZ).$$

Assumption 4
Let $F : [0, T] \times \mathbb{R}^n \to Comp\, Conv(\mathbb{R}^n)$ be a multifunction satisfying:

(1) $F : [0, T] \times \mathbb{R}^n \to Comp\, Conv(\mathbb{R}^n)$ is a $(\beta, \mathcal{F})$-measurable multifunction;

(2) $F : [0, T] \times \mathbb{R}^n \to Comp\, Conv(\mathbb{R}^n)$ is a Lipschitz multifunction:

(3) For any $x \in S^2$ a process $(F(t, x_{t-}))_{t \in [0,T]}$ is integrably bounded.

Theorem 10 *Let Z be an $(\mathbb{F}, \mathbb{H})$–reversible semimartingale from $\mathcal{H}^\infty$, $Z_0 = 0$.*
Let $F : [0, 1] \times \mathbb{R}^n \to CompConv(\mathbb{R}^n)$ satisfies the Assumption 4.
Then for any $\xi \in L^2(\Omega, [\mathcal{F}_0, \mathcal{H}_1], P; \mathbb{R}^n)$ the set of solutions of the inclusion (SSI) is nonempty.

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