A multilayer extension of the KPZ equation.

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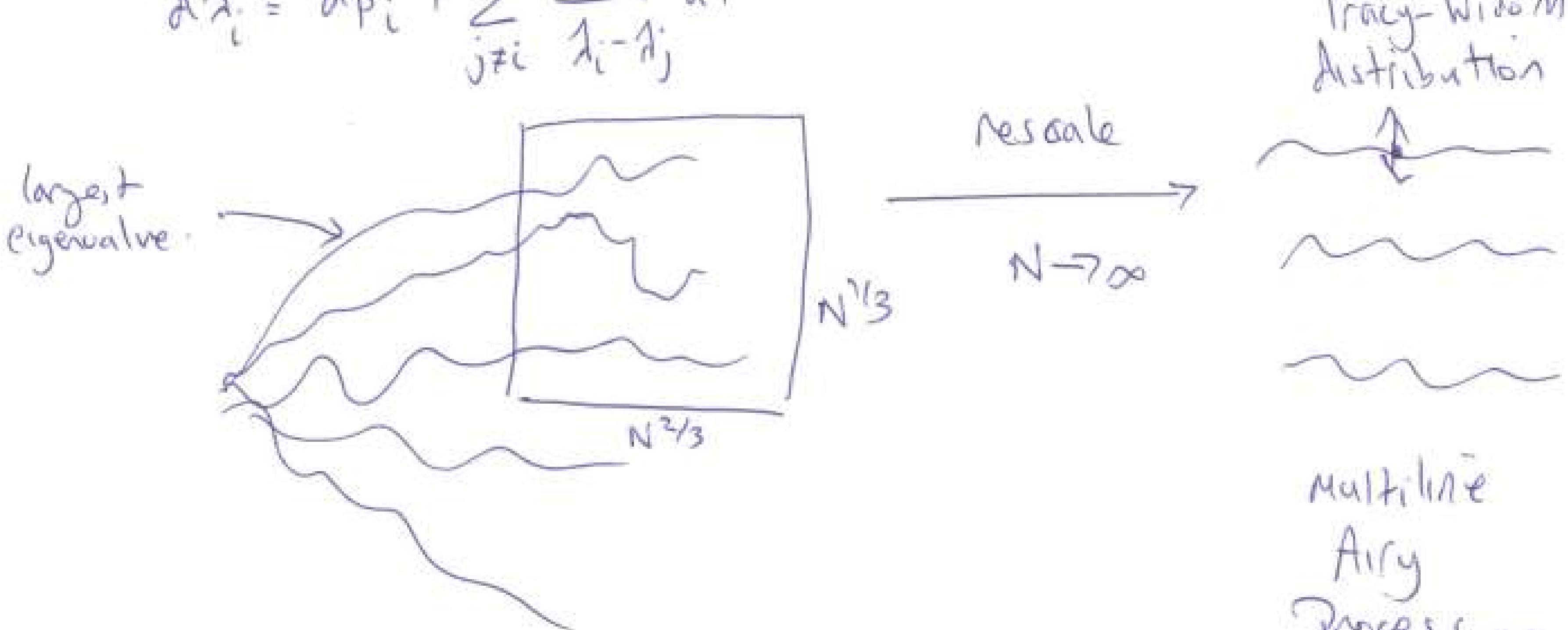
(joint work with Neil O'Connell)

BM in N×N Hermitian Matrices

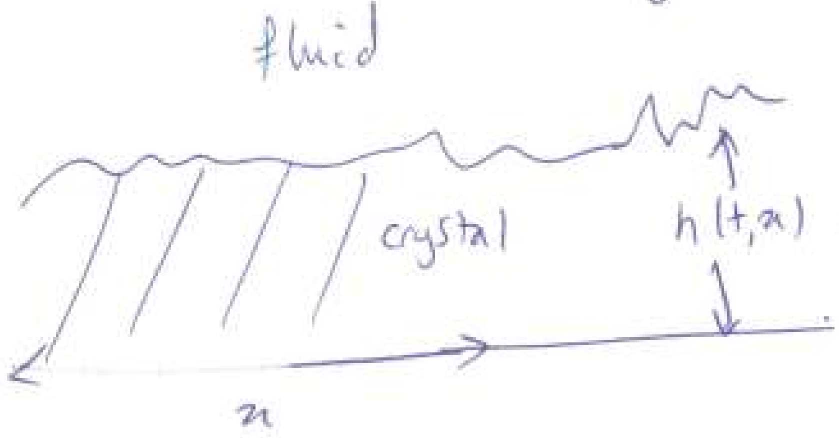
$\Big\downarrow$ Eigenvalues : Dyson's BM

$$d\lambda_i = d\beta_i + \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}\, dt$$



Multiline Airy Process — infinite system of repelling BMs

KPZ equation (Kardar, Parisi, Zhang
Dynamic scaling of growing interfaces, 1986)



$h(t,x)$ = height of interface at time t above point $x \in \mathbb{R}$

one space dimension

$$\partial_t h = \tfrac{1}{2}\,\partial_{xx} h + \tfrac{1}{2}(\partial_x h)^2 + \dot{W}$$

space time white noise.

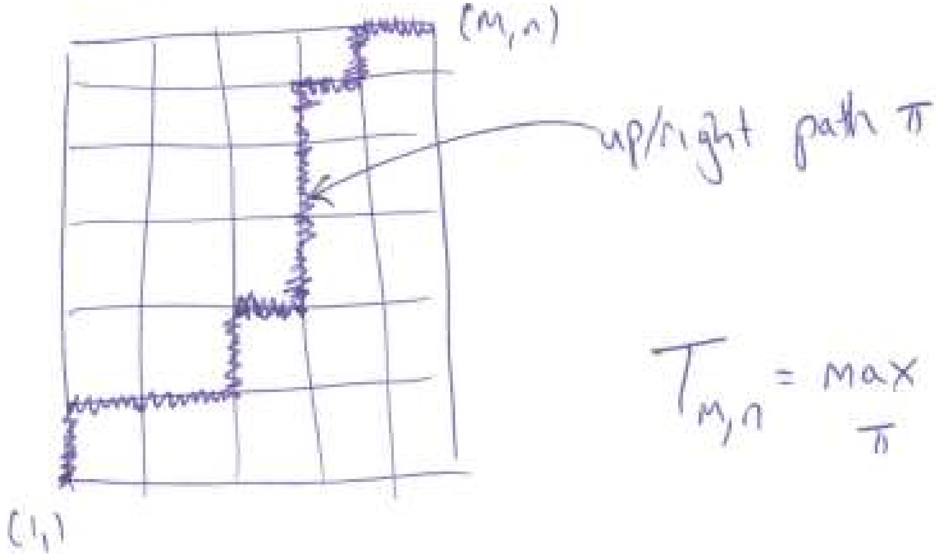
Amir, Corwin, Quastel, Sasamoto, Spohn.

narrow wedge initial condition

$$\big(h(t,x)\,;\, x \in \mathbb{R} \big) \xrightarrow[\text{rescaled.}]{t \to \infty} \text{top line of Airy process}$$

A discrete model (Johansson 2000)

w_{ij} $(i,j)\in\mathbb{Z}^2$ independent $P(w_{ij}>x)=e^{-x}$, $x\geq 0$.



$$T_{m,n}=\max_{\pi}\sum_{(i,j)\in\pi} w_{ij}$$

Then .....

$$T_{m,n}\overset{law}{=}\lambda_1$$

λ_1 largest eigenvalue of AA^*

A $m\times n$ matrix with iid complex Gaussian entries!

the other eigenvalues ?

R.S.K $\Rightarrow$ k disjoint paths $(\pi_1, \pi_2 \cdots, \pi_k)$

if $T^{(k)}_{m,n} = \sup_{(\pi_1, \pi_2 \cdots, \pi_k)} \sum_{w_{ij} \in \cup \pi_r} w_{ij}$

then

$$\left(T^{(1)}_{m,n}, T^{(2)}_{m,n}, T^{(3)}_{m,n} \cdots \right) \overset{law}{=} \left(\lambda_1, \lambda_1+\lambda_2, \lambda_1+\lambda_2+\lambda_3 \cdots \right)$$

λ_i : ith largest eigenvalue

Is the limit as $\beta \to \infty$ of following finite temp. analogue

$${}^{\beta}T_{m,n} = \frac{1}{\beta} \log \left(\sum_{\pi} \exp \left(\beta \sum_{(i,j) \in \pi} w_{ij} \right) \right)$$

Seppalainen

. Directed Polymer

KPZ is a continuum directed Polymer

$\dot{W}(t,x)$ space time white noise $\sim$ w_{ij} exp random variables

B a one dimensional Brownian path conditioned to end at (t,x) $\sim$ π up/right path to (m,n).

heat kernel

$$u(t,x) = p(t,x)\,\mathbb{E}_B\left[:\exp \int_0^t dW(s,B_s):\right]$$

"Feynman-Kac formula"

B independent of W

$$\sim \quad {}^{\beta}T_{mn} = \frac{1}{\beta}\log\Big(\sum_{\pi} \exp\Big(\beta \sum_{(i,j)} w_{ij}\Big)\Big)$$

$h(t,x) = \log u(t,x)$

Cole-Hopf soln to KPZ

The Feynman-Kac formula isn't rigorous ....

u solves Stochastic Heat equation

$$\partial_t u = \tfrac{1}{2}\partial_{xx} u + u\dot{W} \qquad u(0,\cdot) = \delta_0(\cdot)$$

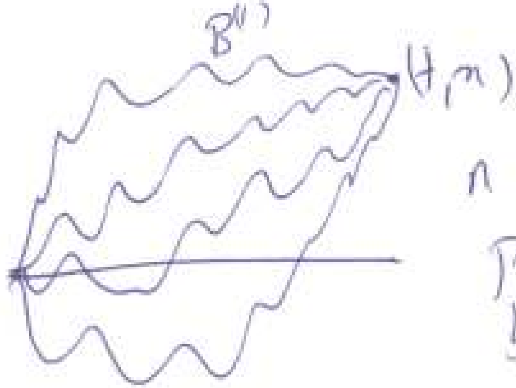
FK corresponds to Chaos expansion

$$u(t,x) = p(t,x) \sum_{m=0}^{\infty} \int_{0 \le t_1 < t_2 \cdots < t_m < t} \int_{\mathbb{R}^m} R_m\big((t_1,x_1)\cdots(t_m,x_m)\big)\, W(dt_1, dx_1) \cdots W(dt_m, dx_m)$$

m^{th} space time correlation function for path of B.

Where are the other "eigenvalues" again?

$(B^{(1)}, \dots, B^{(n)})$ a watermelon



n Brownian paths independent but conditioned not to collide and to end at (t,x)

In F-K replace B by $(B^{(1)}, \dots, B^{(k)})$

At chaos expansion level define

$$u^{(n)}(t,x) = p(t,x)^n \sum_{m=0}^{\infty} \int_{0 \le t_1 < \dots < t_m \le t} \int_{\mathbb{R}^m} R_m^{(n)}(\ ) \, W(dt_1, dx_1) \dots W(dt_m, dx_m)$$

mth correlation function of watermelon.

n^{th} layer of KPZ is then defined to be

$$h^{(n)}(t,x) = \log\left(\frac{u^{(n)}(t,x)}{u^{(n-1)}(t,x)}\right)$$

recall $T_{M,n}^{(n)}$ sum of eigenvalues

Can prove L^2-convergence of expansion, but no proof of cts version, no proof of positivity.

Evolution equations

Replace $\dot{W}$ by smooth time dependent potential ϕ then

$u = u^{(1)}$ solves $\partial_t u = \frac{1}{2}\partial_{xx} u + \phi u$

Introduce dependence on initial condition $u(0,\cdot) = \delta_z(\cdot)$
$z \in \mathbb{R}$

then $u^{(n)} = C_{n,t} \det \{ \partial_z^i \partial_x^j u \}_{0 \le i,j \le n-1}$

Follows from theory of Darboux transforms that

$v^{(n)} = \dfrac{u^{(n)}}{u^{(n-1)}}$ solves $\partial_t v^{(n)} = \frac{1}{2}\partial_{xx} v^{(n)} + \left\{ \phi + \partial_{xx} \log \dfrac{u^{(n-1)}}{p^{n-1}} \right\} v^{(n)}$

perturbed potential

In fact $v^{(n)}$ tau function for 2-d toda.

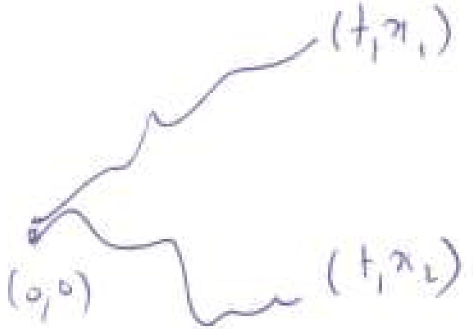
But $\phi \to \dot{W}$ and the spde for $v^{(n)}$ is bad....

Nevertheless line ensemble

$$\{ u^{(1)}(t,x),\ u^{(2)}(t,x) \cdots u^{(n)}(t,x) : x \in \mathbb{R} \}$$

should have Markovian evolution as t varies

<u>$n=2$ case</u>



use F-K to define

$$\{ u(t,x_1,x_2) : (x_1,x_2) \in \mathbb{R}^2 \}$$

This is Markovian.

Surprisingly $\{ u(t,x_1,x_2) ; (x_1,x_2) \in \mathbb{R}^2 \} \equiv \{ u^{(1)}(t,x), u^{(2)}(t,x), x \in \mathbb{R} \}$

uses 2-d toda structure.

Implies Markov Property.

Can we go further?

- For a fixed t does $\{ u^{(k)}(t,x) \,;\, x\in\mathbb{R},\ k\geq 1\}$ determine $\dot{W}$ on $[0,t]\times\mathbb{R}$?

- For a fixed t does $\{ u^{(k)}(t,x) \,;\, k\geq 1\}$ have a Markovian ~~...~~ evolution in x? BM with "soft" repulsion.

- As $t\to\infty$ should recover multiline Airy process.