Poincaré inequality on fractals

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University of Warsaw

joint work with Andrzej Stós (Clermont-Ferrand)

Będlewo, September 11th, 2012

Outline of the talk

1. Introduction.
2. Poincaré inequalities on simple fractals.
3. Definition of Sobolev spaces on fractals.
4. Inclusions between various types of Sobolev spaces on fractals.

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Katarzyna Pietruska-Pałuba, Andrzej Stós, *Poincaré Inequality and Hajłasz-Sobolev spaces on nested fractals*, arXiv:1201.3493v1

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Examples:

- Gaussian measure on $\mathbb{R}^n$,
- distributions of exponential type on $\mathbb{R}$: $\mu(\mathrm{d}x) = \mathrm{e}^{-|x|^\alpha}\mathrm{d}x$, $\alpha \geqslant 1$.

Poincaré inequality in analysis

- local Poincaré inequality for balls in $\mathbb{R}^n$, analytic version:

lat $p \geqslant 1$. for $B = B(x_0, r)$ one has

$$\fint_B |u(x) - u_B| \mathrm{d}x \leqslant Cr \left(\fint_B |\nabla u(x)|^p \mathrm{d}x \right)^{1/p}, \qquad (1)$$

where $u \in W^{1,2}(\mathbb{R}^n)$, $u_B = \frac{1}{|B|} \int_B u(x) \mathrm{d}x$ is the mean of u over the ball B.

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- $g-$the gradient of f.

Nested fractals (embedded in $\mathbb{R}^n$)

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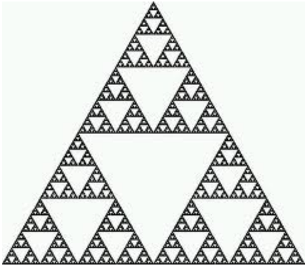

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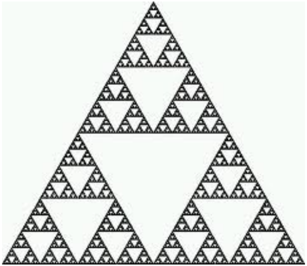

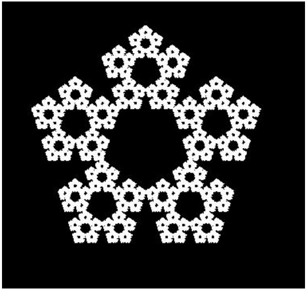

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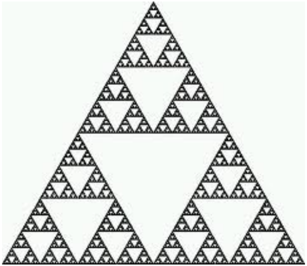
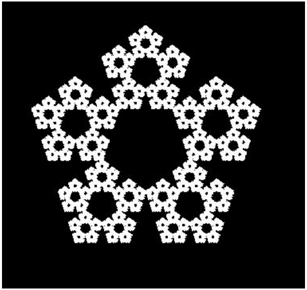


the Sierpiński gasket



the snowflake

Nested fractals

- Embedded in $\mathbb{R}^n$,
- Satisfying the Open Set Condition,
- With lots of symmetries,
- Finitely ramified.

The Brownian Dirichlet form

The Brownian Dirichlet form

Let $p(t, x, y)$ be the transition density of the Brownian motion on the nested fractal $\mathcal{K}$.

1. Markovian definition: the limit as $t \to 0$

$$\lim_{t\to 0} \frac{1}{2t} \int_{\mathcal{K}} \int_{\mathcal{K}} (f(x) - f(y))^2 p(t, x, y) \, \mathrm{d}\mu(x)\mathrm{d}\mu(y)$$

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2. Another definition (equivalent): as a limit of discrete forms.

Some notation

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- For a nonvertex x, $\Delta_m(x)$: the 'small copy' of $\mathcal{K}$ that contains x.
- Points $x, y \in V^{(m)}$ are called $m-$neighbours (denoted: $x \overset{m}{\sim} y$) if they are vertices of a common 'small copy' of $\mathcal{K}$, scaled down m times (scale L^{-m})

The first estimate

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Theorem (Barlow 1996)

Suppose $x, y \in V^{(m)}$ *are* $m-$*neighbours i.e.* $x \overset{m}{\sim} y$. *Take* $f \in \mathcal{D}(\mathcal{E})$. *Then*

$$|f(x) - f(y)|^2 \leqslant C\rho^{-m}\mathcal{E}(f,f),$$

where the constant ρ *equals to* $L^{d_w - d}$ *(*$L-$*the length scaling factor of* $\mathcal{K}$, $d-$*the Hausdorff dimension of* $\mathcal{K}$, d_w-*the walk dimension od* $\mathcal{K}$*).*

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Want: to get an expression with a *local* version of $\mathcal{E}$ on the right-hand side.

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Theorem (Barlow-Bass-Kumagai 2006)

On several 'regular' fractals, one has

$$\int_B |f - f_B|^2 d\mu \leqslant c\Psi(R) \int_B d\Gamma(f, f),$$

for $f \in \mathcal{D}(\mathcal{E})$, *where* $B = B(x_0, R)$ *is a ball, and* $\Psi(R) = R^\sigma$, $\sigma = d_w/2$.

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$$\int_X \phi \, \mathrm{d}\Gamma(u,v) = \frac{1}{2}[\mathcal{E}(u,\phi v) + \mathcal{E}(v,\phi u) - \mathcal{E}(uv,\phi)].$$

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(2) obtain a pointwise estimate, not an integral one.

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where: ν is the so-called Kusuoka measure on $\mathcal{K}$

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Recall: one can write, for x, y–neighbouring points in $V^{(m)}$ and $f \in \mathcal{D}(\mathcal{E})$:

$$|f(x) - f(y)|^2 \leqslant C\rho^{-m} \int_{\mathcal{K}} \langle \nabla f, Z \nabla f \rangle \, d\nu, \quad [\rho = L^{d_w - d}]. \qquad (2)$$

How to get a local version of (2)?

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Easy to get a statement with expressions that approximate $\mathcal{E}$, ∇, Z, ν (these notions are designed so that everything works fine), a statement with limiting object – delicate.

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Lemma

Let $f \in \mathcal{D}(\mathcal{E})$, and let Δ be an m-simplex. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a similitude that maps $\mathcal{K}$ onto Δ. Then

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$$\int_{\mathcal{K}} \langle \nabla (f \circ \phi), Z\,\nabla (f \circ \phi) \rangle d\nu = L^{-m(d_w - d)} \int_{\Delta} \langle \nabla f, Z\,\nabla f \rangle d\nu$$

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where Δ^* denotes the union of Δ and all m-simplices adjacent to Δ.

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Sierpiński gaskets, snowflakes, Vicsek set...

Euclidean version cont'd

(index between two points)

For $x, y \notin V^{(\infty)}$, let

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where $S(x,y)$ was introduced above.

Euclidean version cont'd

Theorem (integral)

Suppose that $\mathcal{K}$ satisfies property **(P)**. *Let $f \in \mathcal{D}(\mathcal{E})$. Let $x_0 \in \mathcal{K} \setminus V^{(\infty)}$ be a nonvertex point and let $r > 0$ be given.*

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$$\begin{aligned} \fint_B |f - f_B| \, \mathrm{d}\mu &\leqslant \left(\fint_B |f - f_B|^2 \, \mathrm{d}\mu \right)^{1/2} \\ &\leqslant C r^{\frac{d_w}{2}} \left(\frac{1}{r^d} \int_{B(x_0, Ar)} \langle \nabla f, Z \nabla f \rangle \, \mathrm{d}\nu \right)^{1/2} . \end{aligned}$$

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- the Hajłasz-Sobolev space $M^{1,p}_\sigma(\mathcal{K}, \mu)$, when there exists a nonnegative function $g \in L^p(\mathcal{K}, \mu)$ such that for μ-a.e. $x, y \in \mathcal{K}$,

$$|f(x) - f(y)| \leqslant \rho(x, y)^\sigma (g(x) + g(y));$$

Such a function g is called an upper gradient of f.

Sobolev spaces on fractals cont.

$f \in L^p(\mathcal{K}, \mu)$ belongs to:

- the Korevaar-Schoen Sobolev space $KS^{1,p}_{\sigma}(\mathcal{K})$, when

$$\limsup_{\epsilon \to 0} \int_{\mathcal{K}} \fint_{B(x,\epsilon)} \frac{|f(x) - f(y)|^p}{\epsilon^{p\sigma}} d\mu(x) d\mu(y) < \infty,$$

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They coincide with the Besov-Lipschitz spaces $Lip(\sigma, p, \infty)(\mathcal{K})$, and in particular $\mathcal{D}(\mathcal{E}) = KS^{1,2}_{d_w/2}(\mathcal{K})$.

Sobolev spaces on fractals cont.

$f \in L^p(\mathcal{K}, \mu)$ belongs to:

- the Poincaré-Sobolev space $\mathcal{P}^{1,p}_{\sigma}(\mathcal{K})$, when there exists a nonnegative function $g \in L^p(\mathcal{K}, \nu)$ such that for any $x \in \mathcal{K}$ and $0 < r < \operatorname{diam} \mathcal{K}$,

$$⨍_{B(x,r)} |f - f_{B(x,r)}| d\mu \leqslant r^{\sigma} \left(\frac{1}{\mu(B(x, Ar))} \int_{B(x,Ar)} g^p d\nu \right)^{1/p}.$$

On some metric spaces other than fractals, with $\sigma = 1$, e.g. on Riemannian manifolds, one typically has inclusions

$$M^{1,p}(X) \subset \mathcal{P}^{1,p}(X) \subset KS^{1,p}(X)$$

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The inequalities can be reversed for example on $\mathbb{R}^n$.

Inclusions on fractals: Korevaar-Schoen vs. Hajłasz

Theorem (Hu 2003)

On nested fractals one has:

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- $KS^{1,p}_\sigma(\mathcal{K}) \subset M^{1,p}_{\sigma'}(\mathcal{K})$, *for all* $0 < \sigma' < \sigma$.

Inclusions on fractals: Korevaar-Schoen vs. Poincaré

Theorem (A.Stos, KPP 2011)

Suppose that the fractal $\mathcal{K}$ *satisfies property* **(P)**. *Let* $p \geqslant 1$ *and* $\sigma > 0$ *be given.*

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(1) *If $\sigma > d/p$, then $\mathcal{P}^{1,p}_{\sigma}(\mathcal{K}) \subset KS^{1,p}_{\sigma}(\mathcal{K})$.*

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Idea of the proof of (1) – fractal version of Koskela/McManus: for a function that satisfies the Poincaré inequality with function g, introduce a fractal version of Riesz potentials:

$$J_p(g,n,x) = \sum_{m=0}^{\infty} L^{-(m+n)\sigma} \left(\frac{1}{\mu(\Delta^*_{n+m}(x))} \int_{\Delta^*_{n+m}(x)} g^p(z)\, d\nu(z) \right)^{1/p},$$

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To conclude, sum they up and estimate the Korevaar-Schoen norm.

Inclusions on fractals: Poincaré vs. Hajłasz

Proposition (KPP, A. Stos 2011)

Suppose that the nested fractal $\mathcal{K}$ *satisfies Property* **(P)**. *Assume* $p \geqslant 1$, $\sigma > 0$. *Then one has:*

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(1) $\mathcal{P}^{1,p}_{\sigma}(\mathcal{K}) \subset (M^{1,p}_{\sigma})_w(\mathcal{K}) \subset M^{1,p'}_{\sigma}(\mathcal{K})$, *with any $1 \leqslant p' < p$ (the last inclusion requires $p > 1$).*

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Here $(M^{1,p})_w$ is the 'weak' Hajlasz-Sobolev space, i.e. the function g from the definition belongs to the weak-L^2 space (the Marcinkiewicz space).

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then:

- use the Vitali covering lemma to obtain $g \in L_w^p(\mathcal{K})$,
- and the estimates for the Riesz kernel to obtain the inequality from the definition of the Hajłasz-Sobolev space.

Thank You!

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