Analytic characterizations of gaugeability for generalized Feynman-Kac functionals

D. Kim and K. Kuwae

(Kumamoto University)

6ICSAA

September 10, 2012, Będlewo Poland

1 Setting

$\mathbf{X} = (\Omega, X_t, \mathbf{P}_x, \zeta)$: m-symmetric transient Hunt process on E satisfying the following conditions :

(I) : X is irreducible.

(AC) : $P_t(x, dy) \ll m(dy)$ for all $t > 0$ and $x \in E$.

(AC) follows the (resolvent) strong Feller (RSF) property:

(RSF) : $R_\alpha f \in C_b(E)$ for any $f \in \mathcal{B}(E)_b$ and $\forall \alpha > 0$.

But we do not assume (RSF) in our main results!

$Q_t f(x) := \mathrm{E}_x[e_A(t) f(X_t)]$: **Feynman-Kac semigroup**

$e_A(t) := \exp(A_t)$,

$A_t := A_t^{u,\mu,F} := N_t^u + A_t^\mu + A_t^F$.

$\mu = \mu_1 - \mu_2 \in S_1(X) - S_1(X)$

: **signed smooth measure in the strict sense.**

$S_1(X) \ni \nu \leftrightarrow A_t^\nu \in \mathrm{A}_{c,1}^+$: **Revuz correspondence,**

i.e.

$$\mathrm{E}_{g\nu}\left[\int_0^\infty e^{-\alpha t} f(X_s) ds\right] = \mathrm{E}_{fm}\left[\int_0^\infty e^{-\alpha t} g(X_s) dA_s^\nu\right].$$

$F_1, F_2 \geq 0$: sym bdd fts. on E^2 with $F_i(x, \partial) = 0$,

$F_i(x, x) = 0$, $(i = 1, 2)$. We set $F := F_1 - F_2$.

$F_i \in J_1(X)$ $(\overset{\text{def}}{\Leftrightarrow} N(|F|)\mu_H \in S_1(X))$, $(i = 1, 2)$.

(N, H): Lévy system, $N(|F|)(x) := \int_E |F(x,y)| N(x,dy)$

$A_t^{F_i} := \sum_{0<s\leq t} F_i(X_{s-}, X_s)$, $\quad A_t^F := A_t^{F_1} - A_t^{F_2}$.

N_t^u: CAF of 0-energy part for $u \in \mathcal{F}_{\text{loc}} \cap C(E_\partial)$

with $\mu_{\langle u\rangle} \in S_D^1(X)$:

$$u(X_t) - u(X_0) = M_t^u + N_t^u \quad t \in [0, \infty[\ \mathrm{P}_x\text{-a.s. for } {}^\forall x \in E$$

Schrödinger operator $\mathcal{L}^{u,\mu,F} :=$ "$\mathcal{L}+\mathcal{L}u+\mu+\mu_H \mathrm{F}$",

where $\mathrm{F}f(x) := \int_{E_\partial}(e^{F(x,y)}-1)f(y)N(x,dy)$.

$\mathcal{Q}(f,g) = (-\mathcal{L}^{u,\mu,F} f, g)_m \Leftrightarrow Q_t f(x) = \mathbf{E}_x[e_A(t)f(X_t)]$.

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u,fg) - \mathcal{H}(f,g)$$

is well-defined for $f,g \in \mathcal{F}$ provided $\mu_{\langle u\rangle} + |\mu| + N(|F|)\mu_H \in S^1_D(\mathrm{X})$, where

$$\mathcal{E}(u,fg) := \frac{1}{2}\int_E f d\mu_{\langle u,g\rangle} + \frac{1}{2}\int_E g d\mu_{\langle u,f\rangle},$$

$$\mathcal{H}(f,g) := \int_E f(x)g(x)\mu(dx) + \iint_{E\times E} f(x)N(g(e^F-1))(x)\mu_H(dx).$$

2 Kato class, Green-tight Kato class

$$\nu \in S^1_K(X) \overset{\text{def}}{\iff} \nu \in S_1(X), \lim_{\alpha\to\infty} \sup_{x\in E} \int_E R_\alpha(x,y)\nu(dy)=0.$$

$$\nu \in S^1_D(X) \overset{\text{def}}{\iff} \nu \in S_1(X), \sup_{x\in E} \int_E R_\alpha(x,y)\nu(dy) < \infty$$

$$\exists/\forall \alpha > 0.$$

$\nu \in S^1_{D_0}(X)$ (Green-bounded)

$$\overset{\text{def}}{\iff} \nu \in S_1(X), \sup_{x\in E} \int_E R(x,y)\nu(dy) < \infty.$$

$\nu \in S^1_{K_\infty}(X)$ (Green-tight Kato class)

$\overset{\text{def}}{\iff} \nu \in S^1_K(X)$ and $\forall \varepsilon > 0, \exists K(\subset E)$: compact set s.t. $\displaystyle\sup_{x\in E} \int_{K^c} R(x,y)\nu(dy) < \varepsilon.$

$\nu \in S^1_{CK_\infty}(X)$ (Chen's Green-tight Kato measure)

$\overset{\text{def}}{\iff} \nu \in S_1(X)$, $\forall \varepsilon > 0$, $\exists K(\subset E)$: Borel set s.t. $\nu(K) < \infty$ & $\exists \delta > 0$ s.t. $\forall$ mea'ble set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{x \in E} \int_{K^c \cup B} R(x,y)\nu(dy) < \varepsilon.$$

We can define $S^1_{K^+_\infty}(X), S^1_{CK^+_\infty}(X)$ by 1-subprocess.

Lem 2.1 *(RSF) is stable under time change by $\nu \in S^1_K(X)$. By using this, under (RSF), we have $S^1_{K_\infty}(X) = S^1_{CK_\infty}(X)$, $S^1_{K^+_\infty}(X) = S^1_{CK^+_\infty}(X)$.*

We prepare conditions (GT) and (GT)*:

$$(\mathrm{GT}):\ m\in S^1_{K^+_\infty}(X),\qquad (\mathrm{GT})^*:\ m\in S^1_{CK^+_\infty}(X).$$

λ_p: the spectral radius of the semigroup $(P_t)_{t\geq 0}$ on $L^p(E;m)$: $\lambda_p=-\lim_{t\to\infty}\frac{1}{t}\log\|P_t\|_{p,p}$, $p\in[0,\infty]$.

Thm 2.1 (Chen (02) TAMS, Takeda (02) JFA)

Assume (I), (AC) and (GT)* (or (RSF) and (GT)). Then $\lambda_\infty=\lambda_2$. In particular,

$$\sup_{x\in E}\mathrm{E}_x[e^{\lambda\zeta}]<\infty \overset{\text{Sato(1985)}}{\iff} \lambda_\infty=\lambda_2>\lambda.$$

2.1 Gaugeability: Result

Thm 2.2 *Suppose* $\nu := \mu_{\langle u\rangle} + \mu_1 + N(F_1)\mu_H \in S^1_{CK_\infty}(X)$ $\mu_2 + N(F_2)\mu_H \in S^1_{D_0}(X)$. *Then TFAE:*

(1) (X, A) *is gaugeable*, *i.e.*, $\sup_{x\in E} \mathbf{E}_x\left[e_A(\zeta)\right] < \infty$.

(2) $\lambda(\mu_V) := \inf\left\{\mathcal{Q}(f,f) \mid f \in \mathcal{F} \cap C_0(E), \int_E f^2 d\mu_V^1 = 1\right\} > 0$

$\mu_V^1 := N(V)\mu_H + \mu_1 + \frac{1}{2}\mu^c_{\langle u\rangle}$, $\mu_V^2 := N(F_2)\mu_H + \mu_2$,

$V := (G^u - F^u) + F_1 (\Rightarrow V \geq 0)$,

$G^u := e^{F^u} - 1, F^u(x,y) := F(x,y) + u(x) - u(y)$,

$0 \leq G^u - F^u \leq \frac{1}{2}e^{\|F^u\|_\infty}(F^u)^2 \Rightarrow \mu_V^1 \leq {}^\exists C\nu$.

Lem 2.2 $\lambda(\mu_V) > 0$ *is equivalent to*

$$\lambda(\mu_{V_1}) := \inf\{\mathcal{Q}(f,f) \mid f \in \mathcal{F} \cap C_0(E), \int_E f^2 d\mu^1_{V_1} = 1\} > 0,$$

$$\mu^1_{V_1} := N(V_1)\mu_H + \mu_1 + \tfrac{1}{2}\mu^c_{\langle u \rangle}, \ \mu^2_{V_1} := \mu_2,$$

$$V_1 := (G^u_1 - F^u_1) + F_1 (\Rightarrow V_1 \geq 0),$$

$$G^u_1 := e^{F^u_1} - 1, F^u_1(x,y) := F_1(x,y) + u(x) - u(y).$$

Proof. $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$: sym. closed form on $L^2(E;m)$.

$$\lambda(\nu_2, \nu_1) := \inf\{\mathcal{A}(f,f) + \|f\|^2_{\nu_2} \mid f \in \mathcal{C}, \|f\|^2_{\nu_1} = 1\}.$$

By Lem 3.2 in Takeda-Uemura (04), if $\lambda(\nu_2, \nu) > 0$, then $\lambda(\nu_2 + \nu, \nu_1 + \nu) > 1 \Leftrightarrow \lambda(\nu_2, \nu_1) > 1$. $\square$

Thm 2.2 is new even if X is a Brownian motion!

Ex 2.1 $X = (\Omega, X_t, \mathrm{P}_x)$: BM on $\mathbb{R}^d$, $d \geq 3$.

$u \in H^1(\mathbb{R}^d) \cap C_\infty(\mathbb{R}^d)$ with $|\nabla u|^2 \in K_d (\Rightarrow |\nabla u|^2 \in K_d^\infty)$.

Then TFAE.

(1) (X, N^u) is gaugeable, that is,

$$\sup_{x \in \mathbb{R}^d} \mathbf{E}_x \left[e^{N^u_\infty} \right] < \infty.$$

(2) $\lambda(|\nabla u|^2 dx) := \inf \{ \frac{1}{2}(\|\nabla f\|_2^2 + (\nabla u, \nabla f^2)_2) \mid$

$$f \in C_0^\infty(\mathbb{R}^d), \frac{1}{2} \int_{\mathbb{R}^d} f^2 |\nabla u|^2 dx = 1 \} > 0.$$

Thm 2.2 also improves the previous known result for $u = \mu = 0$. ($S^1_{CK_\infty}(X) \subset S^1_{D_0}(X)$).

Thm 2.3 (Takeda (02) JFA)

Suppose $u = F = 0$, $|\mu| \in S^1_{K_\infty}(X)$. Under (RSF), the assertion of Thm 2.2 holds.

Thm 2.4 (Chen (02) TAMS)

Suppose $u = F = 0$, $\mu_1 = \mu_+ \in S^1_{CK_\infty}(X)$ and $\mu_2 = \mu_- \in S^1_{D_0}(X)$. Then the assertion of Thm 2.2 holds.

Thm **2.5** (**Chen** (03) JFA)

Suppose $u = 0$, $|\mu| \in S^1_{CK_\infty}(X)$ *and* $|F| \in A_2(X) := A^1_{CS_\infty}(X) \cap J^1_{CS_\infty}(X)$. *Then the assertion of* **Thm** *2.2 holds.*

Rem: $F \in A_2(X) \Longrightarrow N(|F|)\mu_H \in S^1_{CK_\infty}(X) \subset S^1_{D_0}(X)$.

X^z: Doob's h-transformed process by $h := R(\cdot, z)$.

$\nu \in$ semi-$S^1_{DS_0}(X) := \bigcap_{z \in E} S^1_{D_0}(X^z)$ (Semi-conditionally Green-bounded measure in the sense of Chen).

$\nu \in$ semi-$S^1_{CS_\infty}(X) := \bigcap_{z \in E} S^1_{CK_\infty}(X^z)$ (Semi-conditionally Green-tight measure of Kato class in the sense of Chen).

$\nu \in S^1_{DS_0}(X)$ (Conditionally Green-bounded measure in the sense of Chen) $\overset{\text{def}}{\iff} \mathsf{d} := \{(x,y) \mid R(x,y) = \infty\}$

$$\sup_{(x,z)\in E\times E\setminus \mathsf{d}} \int_E R^z(x,y)\nu(dy) < \infty.$$

$\nu \in S^1_{CS_\infty}(X)$ (Conditionally Green-tight measure in the sense of Chen) $\overset{\text{def}}{\iff} \forall \varepsilon > 0$, $\exists K(\subset E)$: Borel set s.t. $\nu(K) < \infty$ and $\exists \delta > 0$ s.t. $\forall$ mea'ble set $B \subset K$ with $\nu(B) < \delta$,

$$\sup_{(x,z)\in E\times E\setminus \mathsf{d}} \int_{K^c\cup B} R^z(x,y)\nu(dy) < \varepsilon,$$

where $R^z(x,y) := \frac{R(x,y)R(y,z)}{R(x,z)}$, $x, y \in E^z := \{R(\cdot, z) < \infty\}$.

$$F \in J^1_{DS_0}(X) \overset{\text{def}}{\iff} N(|F|)\mu_H \in S^1_{DS_0}(X).$$

$$F \in J^1_{CS_\infty}(X) \overset{\text{def}}{\iff} N(|F|)\mu_H \in S^1_{CS_\infty}(X).$$

$$F \in A^1_{DS_0}(X) \overset{\text{def}}{\iff}$$

$$\sup_{(x,z)\in E^2\setminus \mathsf{d}} \int_{E^2} \frac{R(x,y)|F(y,w)|R(w,z)}{R(x,z)} N(y,dw)\mu_H(dy) < \infty.$$

$$F \in A^1_{CS_\infty}(X) \overset{\text{def}}{\iff} \forall \varepsilon > 0,\ \exists K(\subset E) : \text{ Borel set s.t.}$$

$$N(|F|)\mu_H(K) < \infty \text{ and } \exists \delta > 0 \text{ s.t. } \forall \text{ mea'ble set}$$

$$B \subset K \text{ with } N(|F|)\mu_H(B) < \delta,$$

$$\sup_{(x,z)\in E^2\setminus \mathsf{d}} \int_{E^2\setminus (K\setminus B)^2} \frac{R(x,y)|F(y,w)|R(w,z)}{R(x,z)} N(y,dw)\mu_H(dy) < \varepsilon.$$

2.2 Semi-conditional Gaugeability: Result

Thm 2.6 $\mu_{\langle u\rangle} + \mu_1 + N(F_1)\mu_H \in \textit{semi-}S^1_{CS_\infty}(X)$, $\mu_2 + N(F_2)\mu_H \in \textit{semi-}S^1_{DS_0}(X)$. *Then TFAE.*

(1) $\sup_{x\in E} \mathbf{E}_x\left[e_A(\zeta)\right] < \infty$ (2) $\sup_{x\in E^y} \mathbf{E}^y_x\left[e_A(\zeta^y)\right] < \infty$, $\forall y \in E$

(3) $\lambda(\mu_V) := \inf\{\mathcal{Q}(f,f) \mid f \in \mathcal{F} \cap C_0(E), \int_E f^2 d\mu^1_V = 1\} > 0$

(4) *Subcriticality*; $R^A(x,y) < \infty$ $^\forall(x,y) \in E \times E \setminus \mathsf{d}$.

(5) *For* $\forall y \in E$, $\exists C = C^y > 0$ *s.t.*

$$C^{-1}R(x,y) \le R^A(x,y) \le CR(x,y) \quad x \in E^y.$$

R^Z: *Green kernel of the transformed process* Z *by* $Z_t := Y_t^1 e^{-A_t^{F_2}}$ *with* $Y_t^1 := \mathsf{Exp}(M^{G_1^u} + M^{-u,c})_t$.

Suppose $\mu_{\langle u \rangle} + \mu_1 + N(F_1)\mu_H \in S^1_{CS_\infty}(X)$, $\mu_2 + N(F_2)\mu_H \in S^1_{DS_0}(X)$ *and* $\exists C > 0$ *s.t.*

$$\text{(RC)} : C^{-1}R(x,y) \leq R^Z(x,y) \leq CR(x,y), \ (x,y) \in E \times E \setminus \mathsf{d}.$$

Then (1)–(5) are equivalent to

(6) $\sup_{(x,z)\in E\times E\setminus \mathsf{d}} \mathbf{E}_x^z[e_A(\zeta^z)] < \infty$.

(7) $\exists C > 0$ *independent of* x, y *s.t.*

$$C^{-1}R(x,y) \leq R^A(x,y) \leq CR(x,y), \ (x,y) \in E \times E \setminus \mathsf{d}.$$

Rem **2.1 Examples satisfying (RC):**

(1) $u = 0$, $F_1 \in A^1_{CS_\infty}(X) \cap J^1_{DS_0}(X)$, $F_2 \in A^1_{DS_0}(X)$

(**Chen-Song** (03) for $F \in A_2(X)$).

(2) **Brownian motion on** $\mathbb{R}^d$: (**Song** (06)).

(3) **(Relativistic) stable-like process on** $\mathbb{R}^d$:

(**Chen-Kim-Kumagai** (11)).

3 Rough sketch of the proof of Thm 2.2

- **Gauge Theorem**:

$$^{\exists}x \in E \text{ s.t. } \mathrm{E}_x[e_A(\zeta)] < \infty \Leftrightarrow \sup_{x\in E} \mathrm{E}_x[e_A(\zeta)] < \infty$$

- **Time Change Method**:

  Set $\eta := \mu^1_{V_1} \in S^1_{CK_\infty}(X) \subset S^1_{D_0}(X)$. Stollmann-Voigt inequality, $\mathcal{F}_e \subset L^2(E;\eta)$ and

$$\int_E f^2 d\eta \leq \|R\eta\|_\infty \mathcal{E}(f,f).$$

  $(\check{\mathrm{X}}, \eta)$: the time changed process of X w.r.t. η.

For simplicity, we assume $\mathrm{supp}[\eta] = E$. Then $(\check{\mathcal{E}}^\eta, \check{\mathcal{F}}^\eta) = (\mathcal{E}, \mathcal{F}_e)$, $(\check{\mathcal{Q}}^\eta, \check{\mathcal{F}}^\eta) = (\mathcal{Q}, \mathcal{F}_e)$ on $L^2(E; \eta)$ $\check{Q}^\eta_t f(x) = \check{\mathrm{E}}^\eta_x[\check{e}_A(t) f(\check{X}_t)] := \mathrm{E}_x[e_A(\tau_t) f(X_{\tau_t})]$. $\check{e}_A(t) := e_A(\tau_t)$ and $\tau_t := \inf\{s > 0 \mid A^\eta_s > t\}$.

- We may assume $m \in S^1_{CK_\infty}(X)$ in proving Thm 2.2 in view of the time changed process $(\check{\mathrm{X}}, \eta)$ of X because $\eta \in S^1_{CK_\infty}(\check{\mathrm{X}}, \eta)$, $\check{\zeta}^\eta := A^\eta_\zeta$ and

$$\check{\mathrm{E}}^\eta_x[\check{e}_A(\check{\zeta}^\eta)] = \mathrm{E}_x[e_A(\tau_{A^\eta_\zeta})] = \mathrm{E}_x[e_A(\zeta)].$$

- $Y^1_t := \mathsf{Exp}(M^{G^u_1} + M^{-u,c})_t$, $Z_t := Y^1_t e^{-A^{F_2}_t}$

Z: e^{-2u}-sym process; $\mathbf{E}^Z_x[f(X_t)] := \mathbf{E}_x[Z_t f(X_t)]$.

$$e^{u(x)}\mathbf{E}_x[e^{-u(X_{\zeta-})}e_A(\zeta)] = \mathbf{E}^Z_x[\exp(A^{\mu_{V_1}}_\zeta)]$$

$$\Longrightarrow \sup_{x\in E}\mathbf{E}_x[e_A(\zeta)] < \infty \Leftrightarrow \sup_{x\in E}\mathbf{E}^Z_x[\exp(A^{\mu_{V_1}}_\zeta)] < \infty.$$

- If we get $e^{-2u}\mu^1_{V_1} \in S^1_{CK_\infty}(Z)$, $e^{-2u}\mu^2_{V_1} \in S^1_{D_0}(Z)$, then the proof can be reduced to Chen (02)! To get this, we need more arguments.

- $\lambda(\mu_{V_1}) := \inf\{\mathcal{Q}(f,f) \mid f \in \mathcal{C}, \int_E f^2 d\mu^1_{V_1} = 1\} > 0$

$$\overset{[\text{TU (04)}]\&m\in S^1_{CK_\infty}(X)}{\Longrightarrow} \check{\lambda}_2 := \inf\Big\{\mathcal{Q}(f,f) + \|f\|^2_{m+\mu^1_{V_1}} \mid f \in \mathcal{C}, \int_E f^2 d(m+\mu^1_{V_1}) = 1\Big\} > 1$$

$$\Longrightarrow \check{\lambda}_2 = \inf\Big\{\mathcal{E}^Z(e^u f, e^u f) + \|f\|^2_{m+\mu^2_{V_1}} \mid f \in \mathcal{C}, \int_E f^2 d(m+\mu^1_{V_1}) = 1\Big\} > 1$$

$$\overset{[\text{Chen,Takeda (02)}]}{\Longrightarrow} \sup_{x\in E} \mathrm{E}_x^{Z,m+\mu^2_{V_1}}[e^{\zeta + A_\zeta^{\mu^1_{V_1}}}] < \infty.$$

$\because)$ $(\check{\mathrm{X}}, m+\mu^1_{V_1})$: time change of the killed process $\mathrm{Z}^{m+\mu^2_{V_1}}$ of Z by $e^{-t-A_t^{\mu^2_{V_1}}}$ satisfies (I), (AC), (GT)*.

$$\Rightarrow \sup_{x\in E} \mathrm{E}_x^Z[e^{A_\zeta^{\mu_{V_1}}}] \le \sup_{x\in E} \mathrm{E}_x^{Z,m+\mu^2_{V_1}}[e^{\zeta + A_\zeta^{\mu^1_{V_1}}}] < \infty.$$

- $\boldsymbol{\lambda}(\mu_V) > 0 \Longrightarrow \boldsymbol{\lambda}^{(p)}(\mu_V^{(p)}) > 0$ for p close to 1.

  $\lambda^{(p)}(\mu_V^{(p)}) := \inf\{\mathcal{Q}^{(p)}(f,f) \mid f \in \mathcal{C}, \int_E f^2 d\mu^1_{V^{(p)}} = 1\}.$

  $\mathcal{Q}^{(p)}$ and $V^{(p)}$ are defined for $pu, p\mu, pF$ as well

  as $\mathcal{Q}$ and V are defined for u, μ, F.

  $\Longrightarrow$ **Supergaugeability** holds, i.e.

  $$\sup_{x \in E} \mathrm{E}_x[e_A(\zeta)^p] < \infty \text{ for } p \text{ close to } 1.$$

- $\sup_{x \in E} \mathrm{E}_x[(Y^1_\zeta)^p] < \infty$ for p close to 1,

  because $Y^1_t = e_{A - A^{\mu_V}}(t) e^{u(X_t) - u(X_0)}$ for $F_2 = 0$.

  Then $\overline{\mu}_V = 0$ and $\lambda(\overline{\mu}_V) > 0$ always hold.

- $\nu \in S^1_{CK_\infty}(X) \Longrightarrow e^{-2u}\nu \in S^1_{CK_\infty}(Z)$,

  $\nu \in S^1_{D_0}(X) \Longrightarrow e^{-2u}\nu \in S^1_{D_0}(Z)$.

  $$\because) \sup_{x\in E} \mathbf{E}^Z_x[\int_0^\zeta f(X_s)A^\nu_s] \le \sup_{x\in E} \mathbf{E}_x[Y^1_\zeta (f * A^\nu)_\zeta]$$
  $$\le \sup_{x\in E} \mathbf{E}_x[(Y^1_\zeta)^p]^{1/p} \sup_{x\in E} \mathbf{E}_x[((f * A^\nu)_\zeta)^q]^{1/q}$$
  $$\le (([q]+1)!)^{1/([q]+1)} \sup_{x\in E} \mathbf{E}_x[(Y^1_\zeta)^p]^{1/p}$$
  $$\cdot \sup_{x\in E} \mathbf{E}_x[(f * A^\nu)_\zeta]^{1/([q]+1)}.$$

  Then we get

  $e^{-2u}\mu^1_{V_1} \in S^1_{CK_\infty}(Z)$, $e^{-2u}\mu^2_{V_1} \in S^1_{D_0}(Z)$, because

  $\mu^1_{V_1} \in S^1_{CK_\infty}(X)$ and $\mu^2_{V_1} \in S^1_{D_0}(X)$.

4 Rough sketch of the proof of Thm 2.6

- $\lambda(\mu_V) > 0 \Leftrightarrow \sup_{x \in E^y} \mathrm{E}_x^y[e_A(\zeta^y)] < \infty \ {}^\forall y \in E$.

- $R^A(x,y) = \mathrm{E}_x^y[e_A(\zeta^y)] R(x,y)$ m-a.e. $x \in E^y$.

- $R^A(x,y) < \infty \Rightarrow \mathrm{E}_x^y[e_A(\zeta^y)] < \infty$ m-a.e. $x \in E^y$.

  $\Rightarrow \sup_{x \in E^y} \mathrm{E}_x^y[e_A(\zeta^y)] < \infty$

- Fine continuity of $E \ni y \mapsto R^A(x,y)$.

- $\sup_{x \in E^y} \mathrm{E}_x^y[e_A(\zeta^y)] < \infty \Rightarrow {}^\exists C > 0$ s.t.

  $C^{-1} R \le R^A \le C R$ m-a.e. on $E^y \Rightarrow$ on E^y!

- Under (RC), we have $e^{-2u}\mu^1_{V_1} \in S^1_{CS_\infty}(Z)$ and $e^{-2u}\mu^1_{V_2} \in S^1_{DS_0}(Z)$.
- From $\sup_{x\in E} \mathbf{E}^Z_x[\exp(A^{\mu_{V_1}}_\zeta)] < \infty$

$$\sup_{(x,y)\in E\times E\setminus \mathsf{d}} (\mathbf{E}^Z_x)^y\left[\exp\left(A^{\mu_{V_1}}_{\zeta^y}\right)\right] < \infty,$$

- We have the relation:

$$R^{Z,\mu_{V_1}}(x,y) = R^Z(x,y)(\mathbf{E}^Z_x)^y\left[e^{A^{\mu_{V_1}}_{\zeta^y}}\right] \; m\text{-a.e. } x \in E^y.$$

- ${}^\exists C > 0$ s.t.

$$C^{-1}R^Z(x,y) \le R^{Z,\mu_{V_1}}(x,y) \le CR^Z(x,y) \; m\text{-a.e. } x \in E^y.$$

- $C^{-1}R \le R^A \le CR$ m-a.e. on E^y for all $y \in E$.

$$\Rightarrow C^{-1}R \le R^A \le CR \text{ on } E \times E \setminus \mathsf{d}.$$

- $C^{-1}R \le R^A \le CR$ on $E \times E \setminus \mathsf{d}$

$$\Rightarrow C^{-1} \le \mathrm{E}_x^y[e_A(\zeta^y)] \le C \; m\text{-a.e. on } E^y.$$

- Fine lower semi continuity of

$$E^y \ni x \mapsto \mathrm{E}_x^y[e_A(\zeta^y)].$$

- We have

$C^{-1} \le \mathrm{E}_x^y[e_A(\zeta^y)] \le C$ on E^y ${}^\forall y \in E$.

Many thanks for your attention!

Dziękuję za uwagę!