An extension of Wasserstein contraction associated with the curvature-dimension condition

Kazumasa Kuwada

(Ochanomizu University)

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1. Introduction

Framework

M: cpl., stoch. cpl. Riem. mfd., $\dim \geq 2$, $\partial M = \emptyset$
P_t: heat semigr. on M

Goal

Characterize

$$\mathrm{Ric} \geq K \ \& \ \dim M \leq N$$

in terms of heat distributions $P_t^* \mu$ ($\mu \in \mathcal{P}(M)$)

lower Ricci curv. bound

Known facts [von Renesse & Sturm '05] etc.

For $K \in \mathbb{R}$, TFAE:

(i) $\mathrm{Ric} \geq K$

(ii) $W_2(P_t^* \mu_0, P_t^* \mu_1) \leq e^{-Kt} W_2(\mu_0, \mu_1)$

(iii) $|\nabla P_t f|^2 \leq e^{-Kt} P_t(|\nabla f|^2)$

(iv) $\frac{1}{2}(\Delta |\nabla f|^2 - 2\langle \nabla f, \nabla \Delta f \rangle) \geq K |\nabla f|^2$

(v) Ent: K-convex w.r.t. W_2

How important?

- **(iii)(iv)** has rich applications
  in functional ineq. & differential geometry
  (Bakry & Émery etc.)

  $\Rightarrow$ More applications if "$\mathbf{dim}\, \boldsymbol{M}$" is involved

- **(v)** makes sense well even on singular spaces
  & stable under measured Gromov-Hausdorff conv.
  [Sturm '06, Lott & Villani '09]

  $\Rightarrow$ extension of **(ii)(iii)(iv)** to singular spaces
  [Ambrosio, Gigli & Savaré]

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On non-smooth sp.:

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- identification of $P_t^* \mu$ with the gradient flow of Ent in $(\mathcal{P}(M), W_2)$
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Summary of implications

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What we did for $\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$:

- Formulate a missing condition corresponding to **(ii)**
- Extension of the implication **(ii)** $\Leftrightarrow$ **(iii)**

2. Curvature-dimension condition

Known conditions

(i) $\mathrm{Ric} \geq K$

$\Updownarrow$

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(iii)' $|\nabla P_t f|^2 \leq e^{-2Kt} P_t(|\nabla f|^2) + \frac{1 - e^{-2Kt}}{NK}(\Delta P_t f)^2$

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(i)' $\Leftrightarrow$ **(v)' CD** (K, N) cond. of Sturm/Lott-Villani

Theorem 1 ([K.])

For $K \in \mathbb{R}$ *and* $N \in [2, \infty]$,
(iii)' *is equivalent to the following* **(ii)'***:*

(ii)' $W_2(P_s^* \mu_0, P_t^* \mu_1)^2$

$$\leq \left(\fint_s^t e^{2Kr} \xi(dr) \right)^{-1} W_2(\mu_0, \mu_1)^2 + \frac{N}{2} \xi([s,t])^2$$

where $\xi(dr) = \left(\frac{2K}{1 - e^{-2Kr}} \right)^{-1/2} dr$

The case $K = 0$

Corollary 2 ([K.])

For $N \in [2, \infty]$, TFAE:

(i)' $\mathrm{Ric} \geq 0 \;\&\; \dim M \leq N$

(ii)' $$W_2(P_s^* \mu_0, P_t^* \mu_1)^2 \leq W_2(\mu_0, \mu_1)^2 + \frac{N}{2}(\sqrt{t} - \sqrt{s})^2$$

(iii)' $$|\nabla P_t f|^2 \leq P_t(|\nabla f|^2) + \frac{2}{N}(\Delta P_t f)^2$$

$\star$ **(ii)'** $\Rightarrow$ the sharp Laplacian comparison thm

Rem. on the proof

- **(ii)' $\Rightarrow$ (iii)'** & **(iii)' $\Rightarrow$ (ii)'**

  $\Uparrow$ extension

  [K.'10], which covers the case $N = \infty$
  (Kantorovich duality, Hoph-Lax semigroup,...)

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- **(i)' $\Rightarrow$ (ii)'**: Coupling of Brownian motions with different time-scale

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  ⇒ (possibly) sharper estimate

  (work in progress)

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  $\Rightarrow$ Extension to more general situation

- **(i)' $\Rightarrow$ (ii)'**: Coupling of Brownian motions with different time-scale

  $\Rightarrow$ (possibly) sharper estimate

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Extended duality

Theorem 3 ([K.])

M: *Polish geod. met. sp.*, $P_t = e^{t\mathcal{L}}$: *Feller semigr.*
Then for $p, q \in (1, \infty)$ *with* $p^{-1} + q^{-1} = 1$
& $a, b : [0, \infty) \to (0, \infty)$, *TFAE:*

(A) $W_p(P_s^* \mu_0, P_t^* \mu_1)^2$
$$\leq \left(\fint_s^t \frac{\xi(dr)}{a(r)} \right)^{-1} W_p(\mu_0, \mu_1)^2 + \xi([s,t])^2$$

(B) $|\nabla P_t f|^2 \leq a(t) \Big[P_t(|\nabla f|^q)^{2/q} + b(t)(\mathcal{L} P_t f)^2 \Big]$

where $\xi(dr) := b(r)^{-1/2} dr$

Questions

- Sturm/Lott & Villani's $\mathsf{CD}(K, N) \Rightarrow$ **(ii)'**?
- Sharper formulation of **(ii)'** which (directly) implies the Laplacian comparison even when $K \neq 0$?
- Connection with the monotonicity of normalized $\mathcal{L}$-transp. cost under a backward Ricci flow?
[cf. Topping '09, K.-Philipowski '11]