The differential transform method for solving random differential models

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Będlewo, 2012

Main Problem

$$\begin{cases} \dot{X}(t) &=& P_n(t)X(t) + B(t); \quad P_n(t) := \sum_{i=0}^{n} a_i t^i, \quad t \in T, \\ X(0) &=& X_0 \end{cases} \tag{1}$$

where $B(t)$ is a stochastic process and a_i, X_0 are random variables.

1. Construction of a solution of the form:

$$X(t) = \sum_{k=0}^{\infty} \hat{X}(k)(t - t_0)^k, \quad X : T \times \Omega \to \mathbb{R}, \quad T \subset \mathbb{R},$$

   by means of Differential Transform Method.

2. Numerical example and main statistical properties (expectation and standard deviation)

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2. Numerical example and main statistical properties (expectation and standard deviation)

- A real random variable X defined on $(\Omega, \mathscr{F}, P)$ is called of order p (p-r.v.), if

$$\mathrm{E}[|X|^p] < \infty, \qquad p \geq 1, \quad (L_p, \|X\|_p := (\mathrm{E}[|X|^p])^{1/p}); \qquad p = 2$$

- Let $\{X_n : n \geq 0\}$ be a sequence of p-r.v.'s. We say that it is convergent in the p-th mean to the p-r.v. $X \in L_p$, if

$$\lim_{n \to +\infty} \|X_n - X\|_p = 0.$$

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- If $\mathrm{E}[|X(t)|^p] < +\infty$ for all $t \in T$, then it is called a stochastic process of order p (p-s.p.)
- If there exists a stochastic process $\frac{dX(t)}{dt}$ of order p, such that $\left\| \frac{X(t+h)-X(t)}{h} - \frac{dX(t)}{dt} \right\|_p \to 0 \quad \text{as} \quad h \to 0$, then we say that $\{X(t) : t \in T\}$ is p-th mean differentiable at $t \in T$.

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Random DTM

The random differential transform of the process $X(t)$ is defined as

$$\hat{X}(k) = \frac{1}{k!}\left[\frac{d^k(X(t))}{dt^k}\right]_{t=t_0}, \tag{2}$$

where $\hat{X}$ is the transformed s.p and $\frac{d}{dt}$ denotes de m.s. derivative. The inverse transform of $\hat{X}$ is defined as

$$X(t) = \sum_{k=0}^{\infty} \hat{X}(k)(t-t_0)^k. \tag{3}$$

Table: Mathematical operations

Original s.p.	Transformed process[1]
$X(t) = Y(t) \pm Z(t)$	$\hat{X}(k) = \hat{Y}(k) \pm \hat{Z}(k)$
$X(t) = \lambda Y(t)$	$\hat{X}(k) = \lambda \hat{Y}(k)$
$Y(t) = \frac{dX(t)}{dt}$	$\hat{Y}(k) = (k+1)\hat{X}(k+1)$
$X(t) = Y(t)Z(t)$	$\hat{X}(k) = \sum_{r=0}^{k} \hat{Y}(r)\hat{Z}(k-r)$

[1] L. Villafuerte, C.A. Braumann, J.C. Cortés, L. Jódar, Random differential operational calculus: Theory and applications, Comput. Math. Appl. 59 (2010) 115–125.

$\dot{X}(t) = P_n(t)X(t) + B(t)$

- A p-s.p. $\{X(t) : |t| < c\}$ is p-th mean analytic on $|t| < c$ if it can be expanded in the p-th mean convergent Taylor series

$$X(t) = \sum_{n=0}^{\infty} (t - t_0)X^{(n)}(t_0)/n!$$

- Let $\{H(t) : |t| < c\}$ be a p-th analytic s.p. given by

$$H(t) = \sum_{k=0}^{\infty} H_k t^k, \qquad H_k = \frac{H^{(k)}(0)}{k!}, \quad |t| < c,$$

where the derivatives are considered in the p-th sense. Then there exists $M > 0$ such that $\|H_k\|_p \leq \frac{M}{\rho^k}$, $0 < \rho < c$, $\forall k \geq 0$

Application to our problem-$X(t)=\sum_{k=0}^{\infty}\hat{X}(k)(t-t_0)^k$.

- $\dot{X}(t)=P_n(t)X(t)+B(t);\; P_n(t):=\sum_{i=0}^{n}a_i t^i$

$$(k+1)\hat{X}(k+1)=\sum_{r=0}^{k}\hat{P}_n(r)\hat{X}(k-r)+\hat{B}(k) \qquad (4)$$

$$\hat{P}_n(r)=\begin{cases} a_r, & 0\le r\le n,\\ 0, & r>n.\end{cases} \qquad (5)$$

Thus equation (**??**) becomes

$$(k+1)\hat{X}(k+1)=\sum_{r=0}^{n}a_r\hat{X}(k-r)+\hat{B}(k), \qquad k=n,n+1,\dots \qquad (6)$$

for which the r.v's $\hat{X}(0),\dots,\hat{X}(n)$ have to be computed from (**??**).

$\vec{Z}(k) := (Z_1(k), Z_2(k), ..., Z_n(k), Z_{n+1}(k))^T =$
$(\hat{X}(k-n), \hat{X}(k-n+1), ..., \hat{X}(k-1), \hat{X}(k))^T,$

$$\vec{Z}(k+1) = A(k)\vec{Z}(k) + \vec{H}(k) \tag{7}$$

where

$$A(k) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ \frac{a_n}{k+1} & \frac{a_{n-1}}{k+1} & \frac{a_{n-2}}{k+1} & \cdots & \cdots & \frac{a_0}{k+1} \end{pmatrix}, \qquad \vec{H}(k) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{\hat{B}(k)}{k+1} \end{pmatrix}$$

Conditions for the m.s. convergence of $X(t) = \sum_{k=0}^{\infty} \hat{X}(k)(t-t_0)^k$

$$\vec{Z}(k) = \left(\prod_{i=n}^{k-1} A(i)\right) \vec{Z}(n) + \sum_{r=n}^{k-1} \left(\prod_{i=r+1}^{k-1} A(i)\right) \vec{H}(r) \qquad (8)$$

for $k = n, n+1, \ldots$ Here

$$\prod_{i=n_0}^{m} A(i) = \begin{cases} A(m)A(m-1)\cdots A(n_0) & \text{if } m \geq n_0 \ , \\ 1 & \text{otherwise.} \end{cases} \qquad (9)$$

Because of $\|\vec{Z}(k)\|_{4,v} \geq \|\hat{X}(k)\|_4$; $(L_p^m, \|\vec{X}\|_{p,v} := \max \|X_j\|_p)$

$$\sum_{k=0}^{\infty} \|\vec{Z}(k)\|_{4,v}(t-t_0)^k < \infty \Longrightarrow \sum_{k=0}^{\infty} \|\hat{X}(k)\|_4 (t-t_0)^k < \infty$$

- Computing the p norm:

$$\left\|\vec{Z}(k)\right\|_{p,v} \leq \left\|\left(\prod_{i=n}^{k-1} A(i)\right) \vec{Z}(n)\right\|_{p,v} + \sum_{r=n}^{k-1} \left\|\left(\prod_{i=r+1}^{k-1} A(i)\right) \vec{H}(r)\right\|_{p,v}$$

- Using that $\|A\|_{p,m} = \sum_{i,j} \|A_{ij}\|_p$

$$\mathrm{E}[|XY|] \leq \|X\|_2 \|Y\|_2 \quad \Longrightarrow \quad \|XY\|_p \leq \|X\|_{2p} \|Y\|_{2p}$$

and

$$\left\|\prod_{i=1}^{m} X_i\right\|_p \leq \prod_{i=1}^{m} \left(\|(X_i)^m\|_p\right)^{\frac{1}{m}},\ m \geq 1,\ \mathrm{E}[(X_i)^{mp}] < \infty,\ \forall i.$$ [2]

[2] J.-C. Cotés et al, Solving the random Legendre differential equation: Mean square power series solution and its statistical functions, Computers and Mathematics with Applications 61 (2011): 2782 - 2792

$$\left\|\vec{Z}(k)\right\|_{p,v} \leq \left\|\left(\prod_{i=n}^{k-1} A(i)\right)\vec{Z}(n)\right\|_{p,v} + \sum_{r=n}^{k-1}\left\|\left(\prod_{i=r+1}^{k-1} A(i)\right)\vec{H}(r)\right\|_{p,v}$$

- It follows

$$\left\|\prod_{i=n}^{k-1} A(i)\right\|_{2p,m} \leq \sum_{r,s,s_1,s_2,\ldots,s_{k-n}=1}^{n+1} \left(\left\|\left(A_{rs_1}(k-1)\right)^{k-n}\right\|_{2p}\right)^{\frac{1}{k-n}} \cdots \left(\left\|\left(A_{s_{k-n-1}\,s}(n)\right)^{k-n}\right\|_{2p}\right)^{\frac{1}{k-n}}$$

- Now:

$$\left(\left\|\left(A_{rs}(l)\right)^{k-n}\right\|_{2p}\right)^{\frac{1}{k-n}} = \begin{cases} 0 & \text{or,} \\ 1 & \text{or,} \\ \left(\mathrm{E}\left[\left(\frac{a_j}{l+1}\right)^{2p(k-n)}\right]\right)^{\frac{1}{2p(k-n)}} \end{cases} \quad (10)$$

- Recalling:

$$\left\|\vec{Z}(k)\right\|_{p,v} \leq \left\|\left(\prod_{i=n}^{k-1} A(i)\right)\vec{Z}(n)\right\|_{p,v} + \sum_{r=n}^{k-1}\left\|\left(\prod_{i=r+1}^{k-1} A(i)\right)\vec{H}(r)\right\|_{p,v}$$

- and $\vec{H}(r) = (0, \ldots, 0, \frac{\hat{B}(k)}{k+1})^T$

$$\left\|\vec{H}(r)\right\|_{2p,v} = \left\|\frac{\hat{B}(r)}{r+1}\right\|_{2p} \leq \frac{M_2}{(r+1)\rho^r}, \tag{11}$$

for $0 < \rho < c$.

Main result

Theorem

Consider the problem

$$\left\{ \begin{array}{rcl} \dot{X}(t) & = & P_n(t)X(t) + B(t); \quad P_n(t) := \sum_{i=0}^{n} a_i t^i, \quad t \in T, \\ X(0) & = & X_0 \end{array} \right. \tag{12}$$

Assume that the random variables a_i satisfy the condition

$$E\left[|a_j^m|\right] \leq KM^m < \infty, \qquad \forall m \geq 0, \quad j = 0, 1, \ldots, n$$

X_0 is 4-r.v and $B(t)$ is an 2-th analytic stochastic process. Then there exists a solution of the form

$$X(t) = \sum_{k=0}^{\infty} \hat{X}(k) t^k; \quad \hat{X}(k+1) = \frac{1}{k+1} \left(\sum_{r=0}^{k} \hat{P}_n(r) \hat{X}(k-r) + \hat{B}(k) \right)$$

$|t| < c = min(\rho, \frac{1}{n+1})$

Approximations of the mean and standard deviation functions

1. The truncated process and its main moments

$$X_N(t) = \sum_{k=0}^{N} \hat{X}(k) t^k; \qquad t_0 = 0.$$

$$\mathrm{E}[X_N(t)] = \sum_{k=0}^{N} \mathrm{E}\left[\hat{X}(k)\right] t^k$$

$$\mathrm{E}\left[(X_N(t))^2\right] = \sum_{k=0}^{N} \mathrm{E}\left[\left(\hat{X}(k)\right)^2\right] t^{2k} + 2 \sum_{k=1}^{N} \sum_{l=0}^{k-1} \mathrm{E}\left[\left(\hat{X}(k)\hat{X}(l)\right)\right] t^{k+l}$$

2. Advantages of the mean square convergence
   $X_N(t) \xrightarrow{m.s} X(t) \Rightarrow E[X_N(t)] \rightarrow E[X(t)]$
   $X_N(t) \xrightarrow{m.s} X(t) \Rightarrow Var[X_n] \rightarrow Var[X(t)]$

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Example

Consider:

$$\begin{cases} \dot{X}(t) &= atX(t)+e^{-bt}; \qquad t\in[0,1/2], \\ X(0) &= X_0 \end{cases} \tag{13}$$

where a is a standard normal r.v. $(a \sim N(0,1))$, b is a exponential r.v. with parameter $\lambda = 1$ $(b \sim Exp[\lambda = 1])$, and X_0 is a Beta r.v. with parameters $\alpha = 2$ and $\beta = 3$, $(X_0 \sim Be(\alpha = 2, \beta = 3))$

Numerical results

t	$E[X(t)]$	$E[X_N(t)]$	$\sigma[X(t)]$	$\sigma[X_N(t)]$
0.00	0.250000	0.25000 0	0.193649	0.193389
0.10	0.345314	0.345314	0.193711	0.193448
0.20	0.432392	0.432392	0.194511	0.194243
0.30	0.512766	0.512766	0.197620	0.197344
0.40	0.587884	0.587884	0.205386	0.205102
0.50	0.659248	0.659248	0.220871	0.220580

Table: Comparison of the expectation and standard function

THANK YOU!