Extinction of Fleming-Viot-type particle systems with strong drift

Mariusz Bieniek

Institute of Mathematics
University of Maria Curie-Skłodowska
Lublin, Poland
mariusz.bieniek@umcs.lublin.pl

Będlewo, September 2012

Coauthors

- Krzysztof Burdzy (University of Washington)
- Soumik Pal (University of Washignton)

Outline of the talk

1. **Fleming-Viot type process**
   - Description of the F-V particle system
   - Some earlier results
2. **Two-particle F-V driven by Bessel process on $(0, \infty)$**
   - Main result
   - Sketch of the proof
3. **N-particle F-V driven by a diffusion with strong drift**
   - Main result
   - Sketch of the proof

Outline of the talk

1. **Fleming-Viot type process**
   - Description of the F-V particle system
   - Some earlier results
2. **Two-particle F-V driven by Bessel process on $(0, \infty)$**
   - Main result
   - Sketch of the proof
3. **N-particle F-V driven by a diffusion with strong drift**
   - Main result
   - Sketch of the proof

Outline of the talk

1. **Fleming-Viot type process**
   - Description of the F-V particle system
   - Some earlier results
2. **Two-particle F-V driven by Bessel process on $(0, \infty)$**
   - Main result
   - Sketch of the proof
3. ***N*-particle F-V driven by a diffusion with strong drift**
   - Main result
   - Sketch of the proof

Outline of the talk

1. **Fleming-Viot type process**
2. Two-particle F-V driven by Bessel process on $(0, \infty)$
3. N-particle F-V driven by a diffusion with strong drift

Definition of Fleming-Viot process

- Consider a diffusion X in $\mathbb{R}^d$ and an open set $D \subset \mathbb{R}^d$.
- Fix $N \in \mathbb{N}$ and define $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$, $t \geq 0$, **driven by** X as follows:
- $\mathbf{X}_0 = (x^1, \ldots, x^N) \in D^N$
- $X_t^1, \ldots, X_t^N$ move as N independent copies of X until time τ_1, where

$$\tau_1 = \inf \left\{ t > 0 : \exists_{1 \leq j \leq N} X_t^j \in \partial D \right\}.$$

- At τ_1 the particle X^j which hit the boundary, jumps onto one of the remaining particles, uniformly chosen at random.

Definition of Fleming-Viot process

- Consider a diffusion X in $\mathbb{R}^d$ and an open set $D \subset \mathbb{R}^d$.
- Fix $N \in \mathbb{N}$ and define $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$, $t \geq 0$, **driven by** X as follows:
- $\mathbf{X}_0 = (x^1, \ldots, x^N) \in D^N$
- $X_t^1, \ldots, X_t^N$ move as N independent copies of X until time τ_1, where

$$\tau_1 = \inf \left\{ t > 0 : \exists_{1 \leq j \leq N} X_t^j \in \partial D \right\}.$$

- At τ_1 the particle X^j which hit the boundary, jumps onto one of the remaining particles, uniformly chosen at random.

Definition of Fleming-Viot process

- Consider a diffusion X in $\mathbb{R}^d$ and an open set $D \subset \mathbb{R}^d$.
- Fix $N \in \mathbb{N}$ and define $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$, $t \geq 0$, **driven by** X as follows:
- $\mathbf{X}_0 = (x^1, \ldots, x^N) \in D^N$
- $X_t^1, \ldots, X_t^N$ move as N independent copies of X until time τ_1, where

$$\tau_1 = \inf \left\{ t > 0 : \exists_{1 \leq j \leq N} X_t^j \in \partial D \right\}.$$

- At τ_1 the particle X^j which hit the boundary, jumps onto one of the remaining particles, uniformly chosen at random.

Definition of Fleming-Viot process

- Consider a diffusion X in $\mathbb{R}^d$ and an open set $D \subset \mathbb{R}^d$.
- Fix $N \in \mathbb{N}$ and define $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$, $t \geq 0$, **driven by** X as follows:
- $\mathbf{X}_0 = (x^1, \dots, x^N) \in D^N$
- $X_t^1, \dots, X_t^N$ move as N independent copies of X until time τ_1, where

$$\tau_1 = \inf \left\{ t > 0 : \exists_{1 \leq j \leq N} X_t^j \in \partial D \right\}.$$

- At τ_1 the particle X^j which hit the boundary, jumps onto one of the remaining particles, uniformly chosen at random.

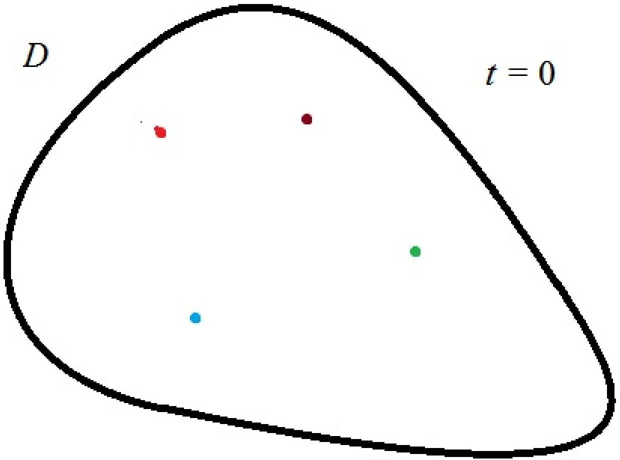
Definition of Fleming-Viot process

- Consider a diffusion X in $\mathbb{R}^d$ and an open set $D \subset \mathbb{R}^d$.
- Fix $N \in \mathbb{N}$ and define $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$, $t \geq 0$, **driven by** X as follows:
- $\mathbf{X}_0 = (x^1, \ldots, x^N) \in D^N$
- $X_t^1, \ldots, X_t^N$ move as N independent copies of X until time τ_1, where

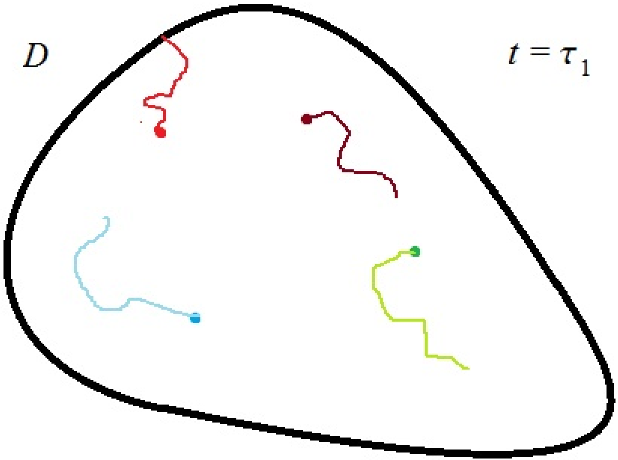
$$\tau_1 = \inf \left\{ t > 0 : \exists_{1 \leq j \leq N} X_t^j \in \partial D \right\}.$$

- At τ_1 the particle X^j which hit the boundary, jumps onto one of the remaining particles, uniformly chosen at random.

An example: $D \subset \mathbb{R}^2$, $N = 4$



An example: $D \subset \mathbb{R}^2$, $N = 4$



Definition (cntd.)

- Then after time τ_1 all particles move as N independent copies of X until the time

$$\tau_2 = \inf\left\{t > \tau_1 : \exists_{1\le j\le N} X_t^j \in \partial D\right\}.$$

- The subsequent evolution of $\mathbf{X}$ proceeds in the same way.

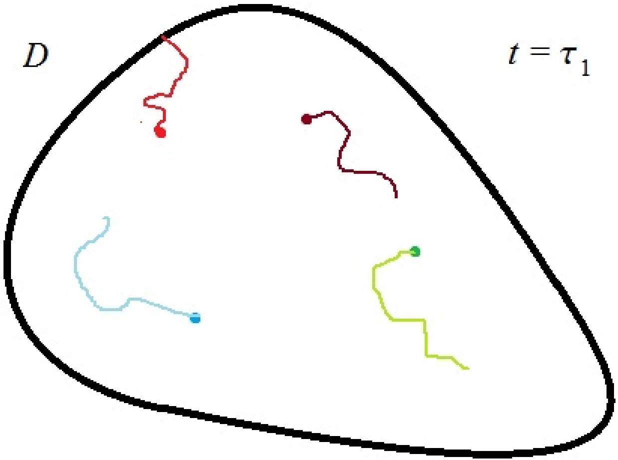
Definition (cntd.)

- Then after time τ_1 all particles move as N independent copies of X until the time

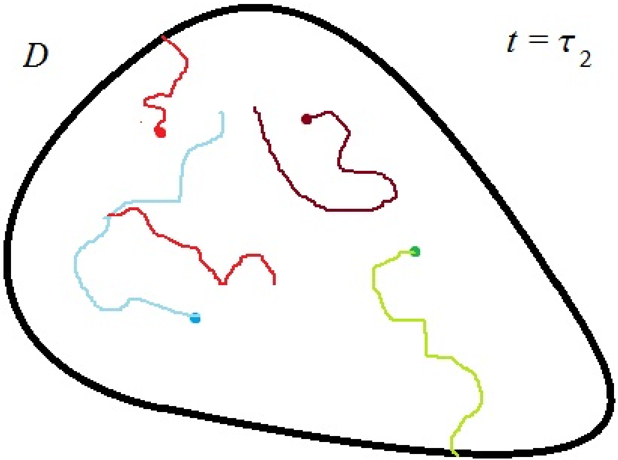
$$\tau_2 = \inf \left\{ t > \tau_1 : \exists_{1 \le j \le N} X_t^j \in \partial D \right\}.$$

- The subsequent evolution of $\mathbf{X}$ proceeds in the same way.

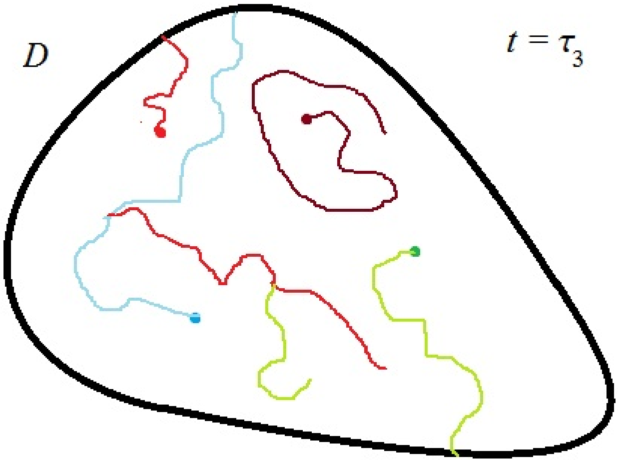
An example cntd: $D \subset \mathbb{R}^2$, $N = 4$



An example cntd: $D \subset \mathbb{R}^2$, $N = 4$



An example cntd: $D \subset \mathbb{R}^2$, $N = 4$



Extinction or non-extinction of F-V process

- Let τ_k denote the time of the k-th jump of the process $\mathbf{X}_t$

$$\tau_{k+1} = \inf\left\{t > \tau_k : \exists_{1 \le j \le N} X_t^j \in \partial D\right\}.$$

- Define $\tau_\infty = \lim_{k \to \infty} \tau_k$ — 'the time of extinction of $\mathbf{X}_t$'

Problem

Is the process $\mathbf{X}_t$ well defined for all $t > 0$?
Is it true that almost surely $\tau_\infty = \infty$?

In other words, is it possible that all N particles hit ∂D at the same finite time?

Extinction or non-extinction of F-V process

- Let τ_k denote the time of the k-th jump of the process $\mathbf{X}_t$

$$\tau_{k+1} = \inf \left\{ t > \tau_k : \exists_{1 \le j \le N} X_t^j \in \partial D \right\}.$$

- Define $\tau_\infty = \lim_{k \to \infty} \tau_k$ — 'the time of extinction of $\mathbf{X}_t$'

Problem

Is the process $\mathbf{X}_t$ well defined for all $t > 0$?
Is it true that almost surely $\tau_\infty = \infty$?

In other words, is it possible that all N particles hit ∂D at the same finite time?

Extinction or non-extinction of F-V process

- Let τ_k denote the time of the k-th jump of the process $\mathbf{X}_t$

$$\tau_{k+1} = \inf\left\{t > \tau_k : \exists_{1\le j\le N} X_t^j \in \partial D\right\}.$$

- Define $\tau_\infty = \lim_{k\to\infty} \tau_k$ — 'the time of extinction of $\mathbf{X}_t$'

Problem

Is the process $\mathbf{X}_t$ well defined for all $t > 0$?
Is it true that almost surely $\tau_\infty = \infty$?

In other words, is it possible that all N particles hit ∂D at the same finite time?

Extinction or non-extinction of F-V process

- Let τ_k denote the time of the k-th jump of the process $\mathbf{X}_t$

$$\tau_{k+1} = \inf \left\{ t > \tau_k : \exists_{1 \le j \le N} X_t^j \in \partial D \right\}.$$

- Define $\tau_\infty = \lim_{k \to \infty} \tau_k$ — 'the time of extinction of $\mathbf{X}_t$'

Problem

Is the process $\mathbf{X}_t$ well defined for all $t > 0$?
Is it true that almost surely $\tau_\infty = \infty$?

In other words, is it possible that all N particles hit ∂D at the same finite time?

Solution - first attempt

Theorem (Burdzy, March, Hołyst (2000))

If X is a Brownian motion then for any $D \subset \mathbb{R}^d$ open

$$\lim_{k\to\infty} \tau_k = \infty, \quad a.s.$$

Proof

Incorrect. If it was correct, it would apply to a very wide class of Markov processes.

Solution - first attempt

Theorem (Burdzy, March, Hołyst (2000))

If X is a Brownian motion then for any $D \subset \mathbb{R}^d$ open

$$\lim_{k \to \infty} \tau_k = \infty, \quad a.s.$$

Proof

Incorrect. If it was correct, it would apply to a very wide class of Markov processes.

Partial solution — Lipschitz domains

Let $\mathbf{X} = (X^1, \ldots, X^N)$ be Fleming-Viot particle system driven by a Brownian motion in $\mathbb{R}^d$.

Theorem (B, Burdzy, Finch (2012))

There exists the constant $c(N, d)$ such that if $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain with the Lipschitz constant $L(D) < c(N, d)$, then the N-particle process $\mathbf{X} = (X^1, \ldots, X^N)$ in D is well defined.

Example

The square in the plane has too sharp angles!!!

Partial solution — Lipschitz domains

Let $\mathbf{X} = (X^1, \ldots, X^N)$ be Fleming-Viot particle system driven by a Brownian motion in $\mathbb{R}^d$.

Theorem (B, Burdzy, Finch (2012))

There exists the constant $c(N, d)$ such that if $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain with the Lipschitz constant $L(D) < c(N, d)$, then the N-particle process $\mathbf{X} = (X^1, \ldots, X^N)$ in D is well defined.

Example

The square in the plane has too sharp angles!!!

Partial solution — Lipschitz domains

Let $\mathbf{X} = (X^1, \dots, X^N)$ be Fleming-Viot particle system driven by a Brownian motion in $\mathbb{R}^d$.

Theorem (B, Burdzy, Finch (2012))

There exists the constant $c(N, d)$ such that if $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain with the Lipschitz constant $L(D) < c(N, d)$, then the N-particle process $\mathbf{X} = (X^1, \dots, X^N)$ in D is well defined.

Example

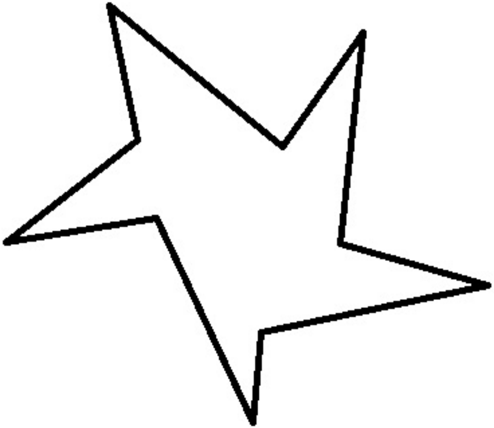
The square in the plane has too sharp angles!!!

Two examples of B, Burdzy and Finch

Example 1 - Theorem

If D is arbitrary polyhedral domain in $\mathbb{R}^d$, then the two-particle F-V process $\mathbf{X}_t = (X_t^1, X_t^2)$ driven by Brownian motion, is well defined for all t (i.e. $\tau_\infty = \infty$).

Example of polyhedral domain in $\mathbb{R}^2$



Two examples of B, Burdzy and Finch - cntd

Example 1 - Theorem

If D is arbitrary polyhedral domain in $\mathbb{R}^d$, then the two-particle F-V process $\mathbf{X}_t = (X_t^1, X_t^2)$ driven by Brownian motion, is well defined for all t (i.e. $\tau_\infty = \infty$).

Example 2

If $D = (0, \infty)$ then for two particle F-V process driven by a diffusion

$$dX_t = dW_t - \frac{5}{2X_t}\, dt, \quad X_0 = 1,$$

we have $\tau_\infty < \infty$.

Outline of the talk

1. Fleming-Viot type process
2. **Two-particle F-V driven by Bessel process on $(0, \infty)$**
3. N-particle F-V driven by a diffusion with strong drift

F-V driven by Bessel processes

Theorem

Let $\mathbf{X}$ be a Fleming-Viot process with N particles on $(0, \infty)$ driven by Bessel process of dimension $\nu \in \mathbb{R}$.

(i) If $N = 2$ then $\tau_\infty < \infty$, a.s., if and only if $\nu < 0$.

(ii) If $N\nu \geq 2$ then $\tau_\infty = \infty$, a.s.

F-V driven by Bessel processes

Theorem

Let $\mathbf{X}$ be a Fleming-Viot process with N particles on $(0, \infty)$ driven by Bessel process of dimension $\nu \in \mathbb{R}$.

(i) If $N = 2$ then $\tau_\infty < \infty$, a.s., if and only if $\nu < 0$.

(ii) If $N\nu \geq 2$ then $\tau_\infty = \infty$, a.s.

F-V driven by Bessel processes

Theorem

Let $\mathbf{X}$ be a Fleming-Viot process with N particles on $(0, \infty)$ driven by Bessel process of dimension $\nu \in \mathbb{R}$.

(i) If $N = 2$ then $\tau_\infty < \infty$, a.s., if and only if $\nu < 0$.

(ii) If $N\nu \geq 2$ then $\tau_\infty = \infty$, a.s.

Sketch of the proof of (i)

- For $\nu \in \mathbb{R}$ and $x > 0$ let $X \sim \text{Bes}^{\nu}(x)$ denote ν-dimensional Bessel process on $(0, \infty)$ killed at 0, i.e. the solution to SDE

$$dX_t = dW_t + \frac{\nu - 1}{2X_t} dt,$$

where W is the standard Brownian motion.

- Let T_0 denote the hitting time of 0.
- Scaling of Bessel processes: If $X \sim \text{Bes}^{\nu}(x)$ is a Bessel process on $[0, T_0)$, then for all $c > 0$,

$$cX_{c^{-2}t} \sim \text{Bes}^{\nu}(cx) \quad \text{on} \quad [0, c^2 T_0]$$

Sketch of the proof of (i)

- For $\nu \in \mathbb{R}$ and $x > 0$ let $X \sim \mathrm{Bes}^{\nu}(x)$ denote ν-dimensional Bessel process on $(0, \infty)$ killed at 0, i.e. the solution to SDE

$$dX_t = dW_t + \frac{\nu - 1}{2X_t} dt,$$

where W is the standard Brownian motion.

- Let T_0 denote the hitting time of 0.
- Scaling of Bessel processes: If $X \sim \mathrm{Bes}^{\nu}(x)$ is a Bessel process on $[0, T_0)$, then for all $c > 0$,

$$cX_{c^{-2}t} \sim \mathrm{Bes}^{\nu}(cx) \quad \text{on} \quad [0, c^2 T_0]$$

Sketch of the proof of (i)

- For $\nu \in \mathbb{R}$ and $x > 0$ let $X \sim \text{Bes}^{\nu}(x)$ denote ν-dimensional Bessel process on $(0, \infty)$ killed at 0, i.e. the solution to SDE

$$dX_t = dW_t + \frac{\nu - 1}{2X_t} dt,$$

where W is the standard Brownian motion.

- Let T_0 denote the hitting time of 0.
- Scaling of Bessel processes: If $X \sim \text{Bes}^{\nu}(x)$ is a Bessel process on $[0, T_0)$, then for all $c > 0$,

$$cX_{c^{-2}t} \sim \text{Bes}^{\nu}(cx) \quad \text{on} \quad [0, c^2 T_0]$$

Alternative construction of FV

- Let $\mathbf{Y}_t = (Y_t^1, Y_t^2)$, where Y^1 and Y^2 are independent copies of $X \sim \text{Bes}^\nu(1)$
- Let $\mathbf{Y}_t^i = \left(Y_t^{i,1}, Y_t^{i,2}\right)$, $i = 1, 2, \ldots$, be a sequence of independent copies of $\mathbf{Y}$.
- For $i = 1, 2, \ldots$ we set

$$\sigma_i = \inf\left\{t > 0 \,:\, Y_t^{i,1} \wedge Y_t^{i,2} = 0\right\}, \qquad \alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

- By scaling of Bessel processes

$$\tau_n = \sum_{j=1}^{n} \xi_{j-1}^2 \sigma_j, \quad \text{with} \quad \xi_j = \prod_{i=1}^{j} \alpha_i$$

- To check when $\tau_n \to \infty$ we use the following result

Alternative construction of FV

- Let $\mathbf{Y}_t = (Y_t^1, Y_t^2)$, where Y^1 and Y^2 are independent copies of $X \sim \text{Bes}^\nu(1)$
- Let $\mathbf{Y}_t^i = \left(Y_t^{i,1}, Y_t^{i,2}\right)$, $i = 1, 2, \ldots,$ be a sequence of independent copies of $\mathbf{Y}$.
- For $i = 1, 2, \ldots$ we set

$$\sigma_i = \inf\left\{t > 0 \,:\, Y_t^{i,1} \wedge Y_t^{i,2} = 0\right\}, \qquad \alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

- By scaling of Bessel processes

$$\tau_n = \sum_{j=1}^{n} \xi_{j-1}^2 \sigma_j, \quad \text{with} \quad \xi_j = \prod_{i=1}^{j} \alpha_i$$

- To check when $\tau_n \to \infty$ we use the following result

Alternative construction of FV

- Let $\mathbf{Y}_t = (Y_t^1, Y_t^2)$, where Y^1 and Y^2 are independent copies of $X \sim \mathrm{Bes}^{\nu}(1)$
- Let $\mathbf{Y}_t^i = \left(Y_t^{i,1}, Y_t^{i,2}\right)$, $i = 1, 2, \ldots,$ be a sequence of independent copies of $\mathbf{Y}$.
- For $i = 1, 2, \ldots$ we set

$$\sigma_i = \inf\left\{t > 0 \,:\, Y_t^{i,1} \wedge Y_t^{i,2} = 0\right\}, \qquad \alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

- By scaling of Bessel processes

$$\tau_n = \sum_{j=1}^{n} \xi_{j-1}^2 \sigma_j, \quad \text{with} \quad \xi_j = \prod_{i=1}^{j} \alpha_i$$

- To check when $\tau_n \to \infty$ we use the following result

Alternative construction of FV

- Let $\mathbf{Y}_t = (Y_t^1, Y_t^2)$, where Y^1 and Y^2 are independent copies of $X \sim \text{Bes}^\nu(1)$
- Let $\mathbf{Y}_t^i = \left(Y_t^{i,1}, Y_t^{i,2}\right)$, $i = 1, 2, \ldots,$ be a sequence of independent copies of $\mathbf{Y}$.
- For $i = 1, 2, \ldots$ we set

$$\sigma_i = \inf\left\{t > 0 \,:\, Y_t^{i,1} \wedge Y_t^{i,2} = 0\right\}, \qquad \alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

- By scaling of Bessel processes

$$\tau_n = \sum_{j=1}^{n} \xi_{j-1}^2 \sigma_j, \quad \text{with} \quad \xi_j = \prod_{i=1}^{j} \alpha_i$$

- To check when $\tau_n \to \infty$ we use the following result

Alternative construction of FV

- Let $\mathbf{Y}_t = (Y_t^1, Y_t^2)$, where Y^1 and Y^2 are independent copies of $X \sim \text{Bes}^\nu(1)$
- Let $\mathbf{Y}_t^i = \left(Y_t^{i,1}, Y_t^{i,2}\right)$, $i = 1, 2, \ldots$, be a sequence of independent copies of $\mathbf{Y}$.
- For $i = 1, 2, \ldots$ we set

$$\sigma_i = \inf\left\{t > 0 \,:\, Y_t^{i,1} \wedge Y_t^{i,2} = 0\right\}, \qquad \alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

- By scaling of Bessel processes

$$\tau_n = \sum_{j=1}^{n} \xi_{j-1}^2 \sigma_j, \quad \text{with} \quad \xi_j = \prod_{i=1}^{j} \alpha_i$$

- To check when $\tau_n \to \infty$ we use the following result

Theorem (e.g. Diaconis, Freedman)

Let $\{(A_n, B_n),\ n \geq 1\}$ be a sequence of independent and identically distributed random variables such that $A_n, B_n \in \mathbb{R}$ and

$$\mathbb{E}\left(\log^+ |A_1|\right) < \infty, \quad \mathbb{E}\left(\log^+ |B_1|\right) < \infty.$$

Then the infinite random series

$$\sum_{n=1}^{\infty}\Big(\prod_{j=1}^{n-1} A_j\Big) B_n$$

converges a.s. to a finite limit if and only if $\mathbb{E}\log|A_1| < 0$.

Theorem (e.g. Diaconis, Freedman)

Let $\{(A_n, B_n),\ n \geq 1\}$ be a sequence of independent and identically distributed random variables such that $A_n, B_n \in \mathbb{R}$ and

$$\mathbb{E}\left(\log^+|A_1|\right) < \infty, \quad \mathbb{E}\left(\log^+|B_1|\right) < \infty.$$

Then the infinite random series

$$\sum_{n=1}^{\infty}\Big(\prod_{j=1}^{n-1} A_j\Big)B_n$$

converges a.s. to a finite limit if and only if $\mathbb{E}\log|A_1| < 0$.

Sketch of the proof

- The case $\nu \geq 2$ is very simple: then $\mathrm{Bes}^{\nu}(x)$ never hits 0, so $\sigma_1 = \infty$, a.s., so $\tau_\infty = \infty$, a.s.
- For $\nu < 2$ we apply the above Theorem with $A_n = \alpha_n^2$ and $B_n = \sigma_n$.
- After some calculations we conclude that $\mathbb{E} \log^+ |B_1| = \mathbb{E} \log^+ \sigma_1 < \infty$ and

$$\mathbb{E}(\log |A_1|) = \mathbb{E} \log(\alpha_1^2) = \frac{1}{2} \mathbb{E} \log \frac{2|X|}{\sqrt{2-\nu}}$$

  where X is a random variable with t-distribution with $(2-\nu)$-degrees of freedom
- Therefore

$$\mathbb{E}(\log A_1) = \frac{1}{4} \left(\psi(1) - \psi\left(\frac{2-\nu}{2} \right) \right),$$

  where ψ is the digamma function
- Therefore $\mathbb{E}(\log A_1) < 0$ iff $\frac{2-\nu}{2} > 1$ iff $\nu < 0$.

Sketch of the proof

- The case $\nu \geq 2$ is very simple: then $\mathrm{Bes}^{\nu}(x)$ never hits 0, so $\sigma_1 = \infty$, a.s., so $\tau_\infty = \infty$, a.s.
- For $\nu < 2$ we apply the above Theorem with $A_n = \alpha_n^2$ and $B_n = \sigma_n$.
- After some calculations we conclude that $\mathbb{E}\log^+|B_1| = \mathbb{E}\log^+\sigma_1 < \infty$ and

$$\mathbb{E}(\log|A_1|) = \mathbb{E}\log(\alpha_1^2) = \frac{1}{2}\mathbb{E}\log\frac{2|X|}{\sqrt{2-\nu}}$$

  where X is a random variable with t-distribution with $(2-\nu)$-degrees of freedom
- Therefore

$$\mathbb{E}(\log A_1) = \frac{1}{4}\left(\psi(1) - \psi\left(\frac{2-\nu}{2}\right)\right),$$

  where ψ is the digamma function
- Therefore $\mathbb{E}(\log A_1) < 0$ iff $\frac{2-\nu}{2} > 1$ iff $\nu < 0$.

- The case $\nu \geq 2$ is very simple: then $\mathrm{Bes}^{\nu}(x)$ never hits 0, so $\sigma_1 = \infty$, a.s., so $\tau_\infty = \infty$, a.s.
- For $\nu < 2$ we apply the above Theorem with $A_n = \alpha_n^2$ and $B_n = \sigma_n$.
- After some calculations we conclude that $\mathbb{E}\log^+|B_1| = \mathbb{E}\log^+\sigma_1 < \infty$ and

$$\mathbb{E}(\log|A_1|) = \mathbb{E}\log(\alpha_1^2) = \frac{1}{2}\mathbb{E}\log\frac{2|X|}{\sqrt{2-\nu}}$$

where X is a random variable with t-distribution with $(2-\nu)$-degrees of freedom
- Therefore

$$\mathbb{E}(\log A_1) = \frac{1}{4}\left(\psi(1) - \psi\left(\frac{2-\nu}{2}\right)\right),$$

where ψ is the digamma function
- Therefore $\mathbb{E}(\log A_1) < 0$ iff $\frac{2-\nu}{2} > 1$ iff $\nu < 0$.

Sketch of the proof

- The case $\nu \geq 2$ is very simple: then $\text{Bes}^{\nu}(x)$ never hits 0, so $\sigma_1 = \infty$, a.s., so $\tau_\infty = \infty$, a.s.
- For $\nu < 2$ we apply the above Theorem with $A_n = \alpha_n^2$ and $B_n = \sigma_n$.
- After some calculations we conclude that $\mathbb{E}\log^+|B_1| = \mathbb{E}\log^+\sigma_1 < \infty$ and

$$\mathbb{E}(\log|A_1|) = \mathbb{E}\log(\alpha_1^2) = \frac{1}{2}\mathbb{E}\log\frac{2|X|}{\sqrt{2-\nu}}$$

  where X is a random variable with t-distribution with $(2-\nu)$-degrees of freedom
- Therefore

$$\mathbb{E}(\log A_1) = \frac{1}{4}\left(\psi(1) - \psi\left(\frac{2-\nu}{2}\right)\right),$$

  where ψ is the digamma function
- Therefore $\mathbb{E}(\log A_1) < 0$ iff $\frac{2-\nu}{2} > 1$ iff $\nu < 0$.

- The case $\nu \geq 2$ is very simple: then $\text{Bes}^{\nu}(x)$ never hits 0, so $\sigma_1 = \infty$, a.s., so $\tau_\infty = \infty$, a.s.
- For $\nu < 2$ we apply the above Theorem with $A_n = \alpha_n^2$ and $B_n = \sigma_n$.
- After some calculations we conclude that $\mathbb{E} \log^+ |B_1| = \mathbb{E} \log^+ \sigma_1 < \infty$ and

$$\mathbb{E}(\log |A_1|) = \mathbb{E} \log(\alpha_1^2) = \frac{1}{2} \mathbb{E} \log \frac{2\,|X|}{\sqrt{2-\nu}}$$

  where X is a random variable with t-distribution with $(2-\nu)$-degrees of freedom
- Therefore

$$\mathbb{E}(\log A_1) = \frac{1}{4}\left(\psi(1) - \psi\left(\frac{2-\nu}{2}\right)\right),$$

  where ψ is the digamma function
- Therefore $\mathbb{E}(\log A_1) < 0$ iff $\frac{2-\nu}{2} > 1$ iff $\nu < 0$.

Outline of the talk

1. Fleming-Viot type process
2. Two-particle F-V driven by Bessel process on $(0, \infty)$
3. N-particle F-V driven by a diffusion with strong drift

N particles with "strong" drift

- Intuitively it may seem that $\tau_\infty = \infty$ for sufficiently large N.
- Our next result shows that this claim is false: once the drift of the diffusion is slightly stronger than the drift of any Bessel process, then $\tau_\infty < \infty$ for the Fleming-Viot process driven by this diffusion and *every* N.

N particles with “strong” drift

- Intuitively it may seem that $\tau_\infty = \infty$ for sufficiently large N.
- Our next result shows that this claim is false: once the drift of the diffusion is slightly stronger than the drift of any Bessel process, then $\tau_\infty < \infty$ for the Fleming-Viot process driven by this diffusion and *every* N.

N particles with "strong" drift

- Consider the following SDE for a diffusion on $(0, 2]$,

$$X_t = x_0 + W_t - \int_0^t \frac{1}{\beta X_t^{\beta-1}} \, ds - L_t, \quad t \leq T_0, \tag{1}$$

where $x_0 \in (0, 2]$, $\beta > 2$, W is Brownian motion, T_0 is the first hitting time of 0 by X, and L_t is the local time of X at 2

- We consider a Fleming-Viot process on $D = (0, 2]$ driven by this diffusion. The role of the boundary is played only by the point 0, since X is reflected at 2.

N particles with "strong" drift

- Consider the following SDE for a diffusion on $(0,2]$,

$$X_t = x_0 + W_t - \int_0^t \frac{1}{\beta X_t^{\beta-1}}\,ds - L_t, \quad t \le T_0, \tag{1}$$

  where $x_0 \in (0,2]$, $\beta > 2$, W is Brownian motion, T_0 is the first hitting time of 0 by X, and L_t is the local time of X at 2
- We consider a Fleming-Viot process on $D = (0,2]$ driven by this diffusion. The role of the boundary is played only by the point 0, since X is reflected at 2.

Theorem

Theorem

Fix any $\beta > 2$. For every $N \geq 2$, the N-particle Fleming-Viot process on $(0, 2]$ driven by diffusion X has the property that $\tau_\infty < \infty$, a.s. Moreover,

$$\mathbb{P}^{\mathbf{x}}(\tau_\infty > t) \leq c_1 e^{-c_2 t}, \qquad t \geq 0, \ \mathbf{x} \in (0, 2]^N,$$

where c_1 and c_2 depend only on N and β, and satisfy $0 < c_1, c_2 < \infty$.

Theorem

Theorem

Fix any $\beta > 2$. For every $N \geq 2$, the N-particle Fleming-Viot process on $(0, 2]$ driven by diffusion X has the property that $\tau_\infty < \infty$, a.s. Moreover,

$$\mathbb{P}^{\mathbf{x}}(\tau_\infty > t) \leq c_1 \mathrm{e}^{-c_2 t}, \qquad t \geq 0,\ \mathbf{x} \in (0, 2]^N,$$

where c_1 and c_2 depend only on N and β, and satisfy $0 < c_1, c_2 < \infty$.

Freidlin-Wentzell inequality

- Consider a diffusion $X_t, t \in [s, u]$, satisfying SDE

$$dX_t = dW_t + b\left(X_t\right)\, dt, \quad X_s = a.$$

- Let y_t be the solution to the ordinary differential equation

$$\frac{d}{dt} y_t = b(y_t), \quad y_s = a.$$

- **If b is a Lipschitz function** on $[s, u]$ then for every $\delta > 0$

$$\mathbb{P}\left(\sup_{s \le t \le u} |X_t - y_t| > \delta\right) \le c_0 \exp\left(-\frac{\delta^2}{2(u-s)} \mathrm{e}^{-2L(u-s)}\right)$$

where L is a Lipschitz constant of b and c_0 is an absolute costant.

Freidlin-Wentzell inequality

- Consider a diffusion $X_t, t \in [s, u]$, satisfying SDE

$$dX_t = dW_t + b\left(X_t\right) dt, \quad X_s = a.$$

- Let y_t be the solution to the ordinary differential equation

$$\frac{d}{dt} y_t = b(y_t), \quad y_s = a.$$

- **If b is a Lipschitz function** on $[s, u]$ then for every $\delta > 0$

$$\mathbb{P}\left(\sup_{s \le t \le u} |X_t - y_t| > \delta\right) \le c_0 \exp\left(-\frac{\delta^2}{2(u-s)} \mathrm{e}^{-2L(u-s)}\right)$$

where L is a Lipschitz constant of b and c_0 is an absolute costant.

Freidlin-Wentzell inequality

- Consider a diffusion $X_t, t \in [s,u]$, satisfying SDE

$$dX_t = dW_t + b\left(X_t\right)\, dt, \quad X_s = a.$$

- Let y_t be the solution to the ordinary differential equation

$$\frac{d}{dt} y_t = b(y_t), \quad y_s = a.$$

- **If b is a Lipschitz function** on $[s,u]$ then for every $\delta > 0$

$$\mathbb{P}\left(\sup_{s \le t \le u} |X_t - y_t| > \delta\right) \le c_0 \exp\left(-\frac{\delta^2}{2(u-s)}\, \mathrm{e}^{-2L(u-s)}\right)$$

where L is a Lipschitz constant of b and c_0 is an absolute costant.

- In our case
$$b(x) = -\frac{1}{\beta x^{\beta-1}}, \quad x > 0.$$
with $\beta > 2$.
- If $y' = b(y)$, $y_s = a$, then
$$y(t) := \left(a^\beta + s - t\right)^{1/\beta}, \quad s \le t \le s + a^\beta,$$
- So b is not Lipschitz on $[s, s + a^\beta]$
- This causes a lot of complicated and tedious calculations and considerations

Sketch of the proof

- In our case
$$b(x) = -\frac{1}{\beta x^{\beta-1}}, \quad x > 0.$$
with $\beta > 2$.
- If $y' = b(y)$, $y_s = a$, then
$$y(t) := \left(a^\beta + s - t\right)^{1/\beta}, \quad s \le t \le s + a^\beta,$$
- So b is not Lipschitz on $[s, s + a^\beta]$
- This causes a lot of complicated and tedious calculations and considerations

Sketch of the proof

- In our case

$$b(x) = -\frac{1}{\beta x^{\beta-1}}, \quad x > 0.$$

  with $\beta > 2$.
- If $y' = b(y)$, $y_s = a$, then

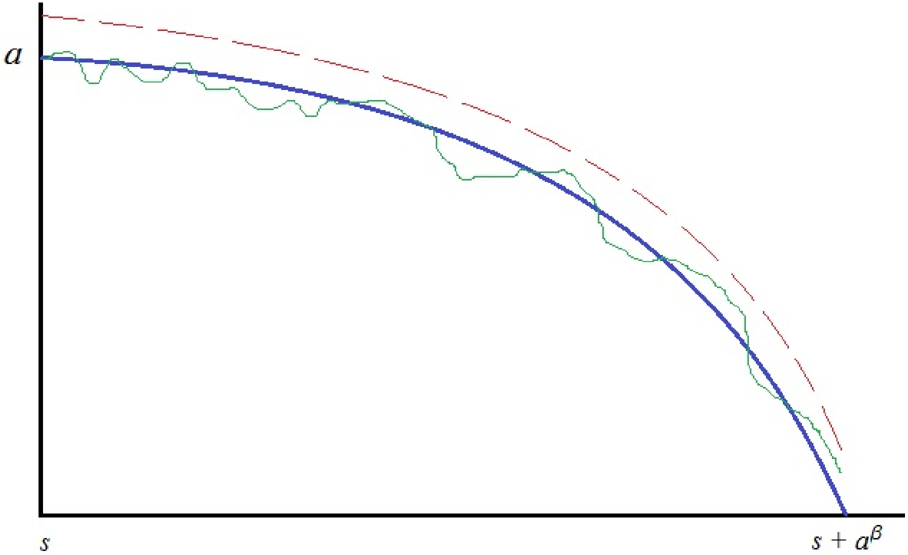
$$y(t) := \left(a^\beta + s - t\right)^{1/\beta}, \quad s \le t \le s + a^\beta,$$

- So b is not Lipschitz on $[s, s + a^\beta]$
- This causes a lot of complicated and tedious calculations and considerations

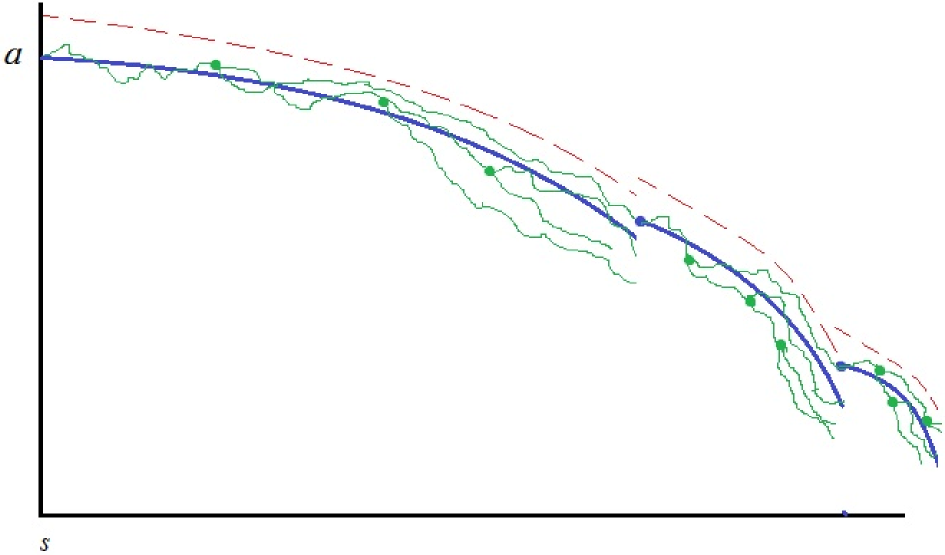
Sketch of the proof

- In our case
$$b(x) = -\frac{1}{\beta x^{\beta-1}}, \quad x > 0.$$
with $\beta > 2$.
- If $y' = b(y)$, $y_s = a$, then
$$y(t) := \left(a^\beta + s - t\right)^{1/\beta}, \quad s \le t \le s + a^\beta,$$
- So b is not Lipschitz on $[s, s + a^\beta]$
- This causes a lot of complicated and tedious calculations and considerations

Sketch of the proof



Sketch of the proof



Relation to original Fleming-Viot superprocess

- Consider N independent Brownian motions in $\mathbb{R}^d$ (all starting from a fixed point)
- Every ε units of time, two particles are chosen uniformly
- The first particle jumps to the location of the second one
- Between the jumps the particles are independent Brownian motions
- Now assume that $N \to \infty$ and $\varepsilon \to 0$ (at some rate related to N)
- Then for each $t \geq 0$, the empirical distributions of the particles converge to a random measure (carried by a set of fractal nature)

Relation to original Fleming-Viot superprocess

- Consider N independent Brownian motions in $\mathbb{R}^d$ (all starting from a fixed point)
- Every ε units of time, two particles are chosen uniformly
- The first particle jumps to the location of the second one
- Between the jumps the particles are independent Brownian motions
- Now assume that $N \to \infty$ and $\varepsilon \to 0$ (at some rate related to N)
- Then for each $t \geq 0$, the empirical distributions of the particles converge to a random measure (carried by a set of fractal nature)

Relation to original Fleming-Viot superprocess

- Consider N independent Brownian motions in $\mathbb{R}^d$ (all starting from a fixed point)
- Every ε units of time, two particles are chosen uniformly
- The first particle jumps to the location of the second one
- Between the jumps the particles are independent Brownian motions
- Now assume that $N \to \infty$ and $\varepsilon \to 0$ (at some rate related to N)
- Then for each $t \geq 0$, the empirical distributions of the particles converge to a random measure (carried by a set of fractal nature)

Relation to original Fleming-Viot superprocess

- Consider N independent Brownian motions in $\mathbb{R}^d$ (all starting from a fixed point)
- Every ε units of time, two particles are chosen uniformly
- The first particle jumps to the location of the second one
- Between the jumps the particles are independent Brownian motions
- Now assume that $N \to \infty$ and $\varepsilon \to 0$ (at some rate related to N)
- Then for each $t \geq 0$, the empirical distributions of the particles converge to a random measure (carried by a set of fractal nature)

Relation to original Fleming-Viot superprocess

- Consider N independent Brownian motions in $\mathbb{R}^d$ (all starting from a fixed point)
- Every ε units of time, two particles are chosen uniformly
- The first particle jumps to the location of the second one
- Between the jumps the particles are independent Brownian motions
- Now assume that $N \to \infty$ and $\varepsilon \to 0$ (at some rate related to N)
- Then for each $t \geq 0$, the empirical distributions of the particles converge to a random measure (carried by a set of fractal nature)

Relation to original Fleming-Viot superprocess

- Consider N independent Brownian motions in $\mathbb{R}^d$ (all starting from a fixed point)
- Every ε units of time, two particles are chosen uniformly
- The first particle jumps to the location of the second one
- Between the jumps the particles are independent Brownian motions
- Now assume that $N \to \infty$ and $\varepsilon \to 0$ (at some rate related to N)
- Then for each $t \geq 0$, the empirical distributions of the particles converge to a random measure (carried by a set of fractal nature)